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USAF TEST PILOT SCHOOL

FLYING QUALITIES TEXTBOOK

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CHAPTER 1

INTRODUCTION TO FLYING QUALITIES

1.1 TERMINOLOGY

Flying Qualities is that discipline in the aeronautical sciences that is concerned with basic aircraft stability and pilot-in-the-loop controllability. With the advent of sophisticated flight control systems, vectored thrust, forward-swept wings, and negative static margins, the concept of flying qualities takes on added dimensions.

In aeronautical literature there are three terms bandied about which are generally considered synonymous. These terms are "flying qualities," "stability and control," and "handling qualities." Strictly speaking, they are synonymous ~~(1.1)~~. An early publication by Phillips in 1949 defines flying qualities of an aircraft as those stability and control characteristics that have an important bearing on the safety of flight and on the pilots' impressions of the ease of flying an aircraft in steady flight and in maneuvers (1.2). Strangely enough, the current document specifying military flying qualities requirements, MIL-F-8785C, Flying Qualities of Piloted Airplanes, does not explicitly define the term "flying qualities," ~~but~~ the specification's stated purpose of application is "...to assure flying qualities that provide adequate mission performance and flight safety regardless of design implementation or flight control system mechanization" ~~(1.3)~~. Successful execution of the military mission then is the key to flying quality adequacy. A definition of flying qualities which can be agreed upon by both the USAF and the US Navy is: "Flying qualities are those stability and control characteristics which influence the ease of safely flying an aircraft during steady and maneuvering flight in the execution of the total mission" ~~(1.4)~~.

The academic treatment of "stability and control" is usually limited to the interaction of the aerodynamic controls with the external forces and moments on the aircraft. Etkin defines "stability" as "...the tendency or lack of it, of an airplane to fly straight with wings level" and "control" as "...steering an airplane on an arbitrary flightpath" (1.5). This academic treatment sometimes excludes the forces felt and, especially, exerted on the cockpit controls by the pilot. "Handling qualities" is the term generally used to define this aspect of the problem. "Handling qualities" are defined by Cooper and Harper as "...those qualities or characteristics of an aircraft

that govern the ease and precision with which a pilot is able to perform the tasks required in support of an aircraft role" (1.1, 1.6). Handling qualities are definitely pilot-in-the-loop characteristics which affect mission accomplishment.

Figure 1.1 shows that the terms "stability" and "control" do not include the pilot-in-the-loop or man/machine interface concepts suggested by the term "handling qualities." In the terminology relationship shown in Figure 1.1, the pilot is considered to be in the "handling qualities" block.

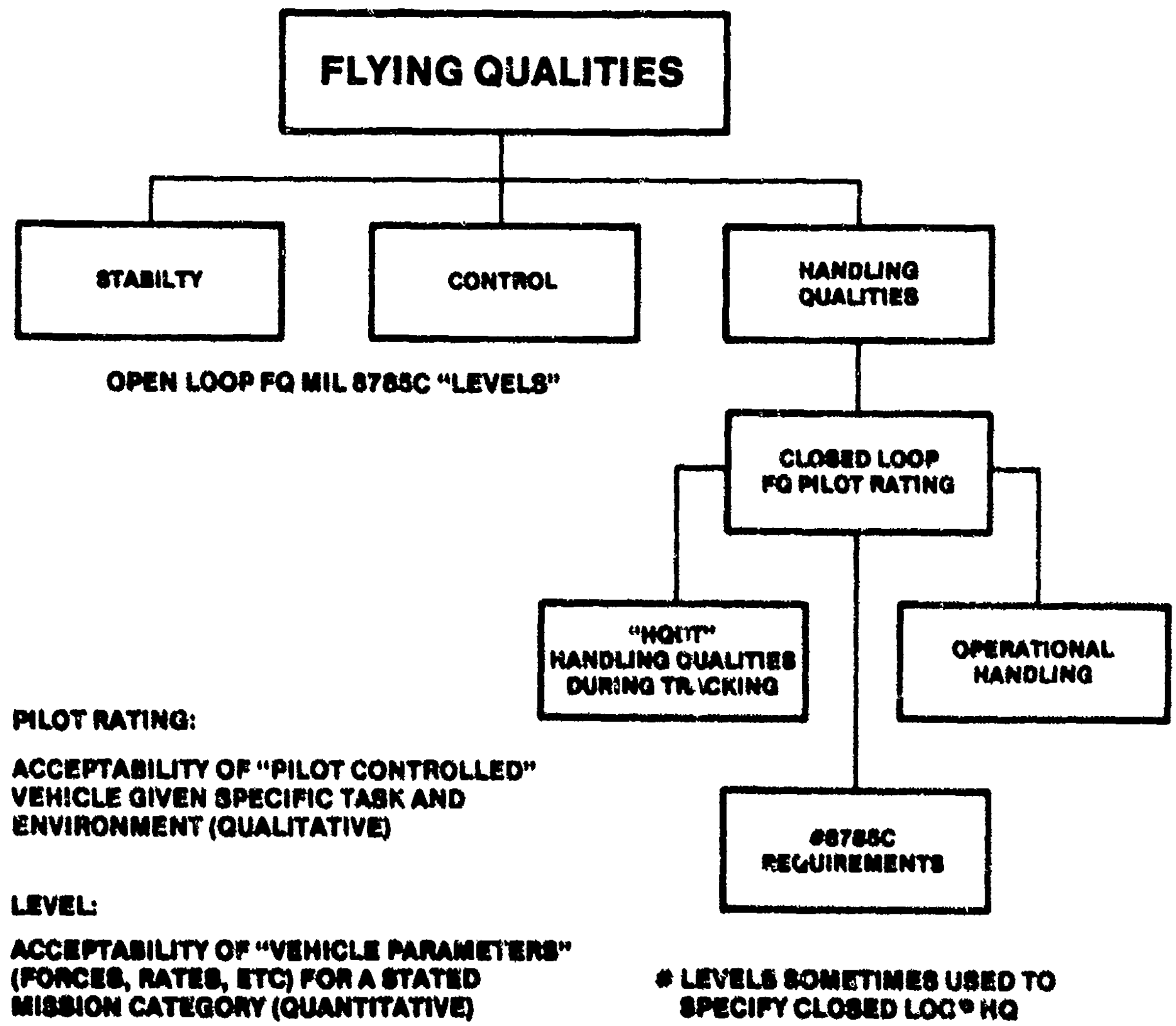


FIGURE 1.1. FLYING QUALITIES BREAKDOWN

Stability and control parameters are generally derived from "open loop" testing, that is, from testing where an aircraft executes specific maneuvers under the control of an assumed "ideal programmed controller" or exhibits free response resulting from a more or less "mechanical" pilot input. The quantitative results are thus independent of pilot evaluation. Many parameters derived in this fashion such as damping and frequency of aircraft oscillation are assigned a "Level" of acceptability as defined by the contents of Reference 1.3, MIL-F-8785C. The intent is to ensure adequate flying qualities for the design mission of an aircraft. Other parameters such as aircraft stability and control derivatives are obtained using parameter identification techniques such as the Modified Maximum Likelihood Estimator (MMLE). These flight test determined derivatives are used as analysis tools for flying quality optimization which occurs during developmental flight testing (1.7).

Handling qualities, on the other hand, are generally determined by performing specifically defined operationally oriented tasks where pilot evaluations of both system task accomplishment and workload are crucial. Pilot ratings defined by the Cooper-Harper Pilot Rating Scale (Figure 1.2) are frequently the results in this "closed loop" type of testing, although a few tasks (such as landing) are assigned a "Level" by MIL-F-8785C. Handling qualities are currently evaluated at the AFFTC using precise pilot-in-the-loop tracking tasks. Two of these test methods are known as Handling Qualities During Tracking (HQDT) and Systems Identification From Tracking (SIFT) (1.6, 1.8, 1.9).

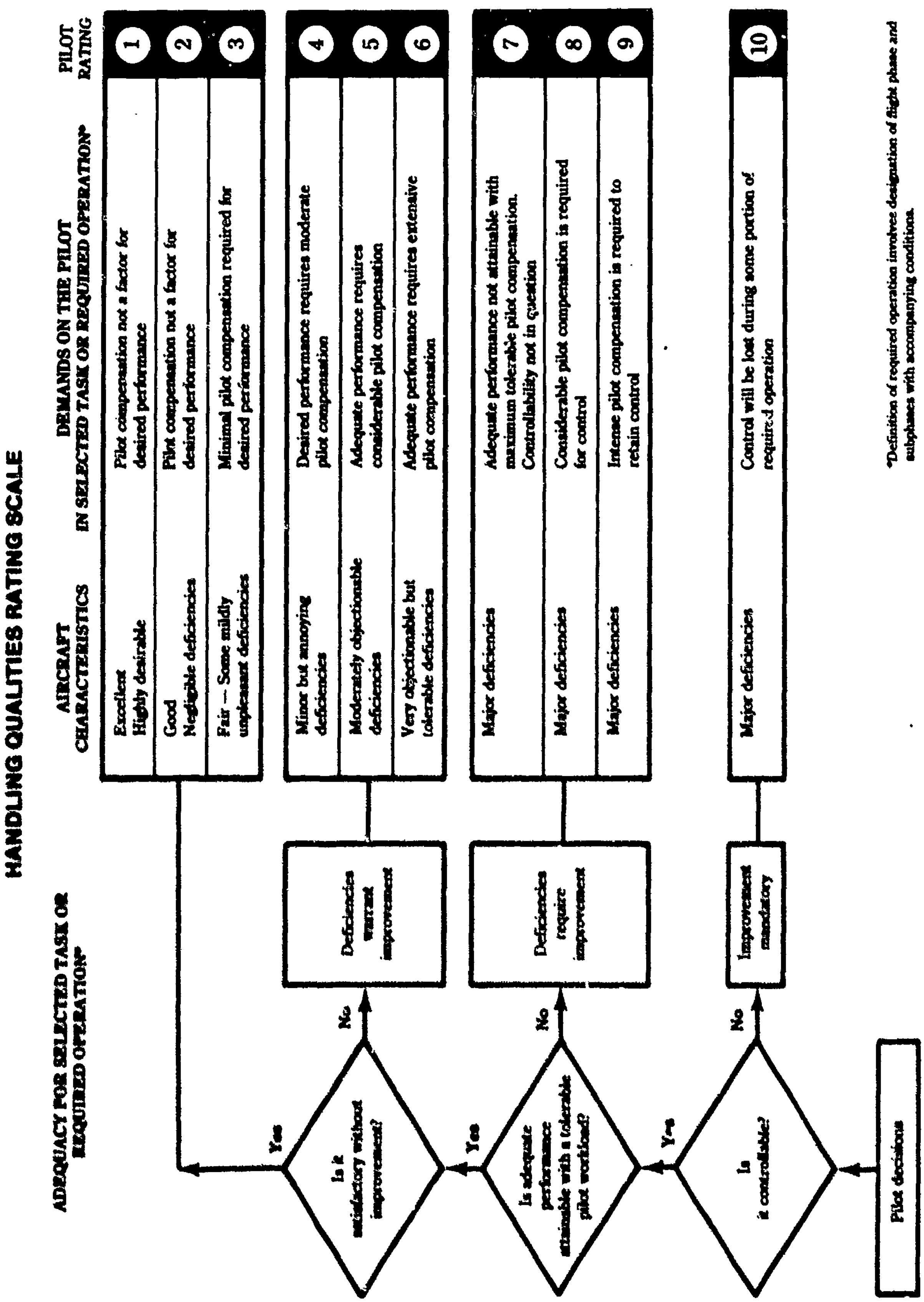


FIGURE 1.2. COOPER-HARPER PILOT RATING SCALE (1.6)

To complicate matters, sometimes "closed loop" tasks are required to gather stability and control data. An example is maneuvering stick force gradient, which requires stabilizing on aim airspeed at high load factor. This is a "closed loop" task for the pilot, particularly if the aircraft has a low level of stability. Because of such complications and interactions between "stability," "control," and "handling qualities," the term "flying qualities" is considered the more inclusive term and is customarily used at the AFFTC to the maximum extent possible (1.1). Unfortunately, current Air Force Flight Dynamics Laboratory practice is to use the terms "flying qualities" and "handling qualities" synonymously without defining either one (1.3).

1.2 PHILOSOPHY OF FLYING QUALITIES TESTING

The flying qualities of a particular aircraft cannot be assessed unless the total mission of the aircraft and the multitude of individual tasks associated with that mission are defined. The mission is initially defined when the concept for a new aircraft is developed; however, missions can be completely changed during the service life of an aircraft. In the formulation of a test and evaluation program for any aircraft, the mission must be defined and clearly understood by all test pilot and flight test engineer members of the test team (1.4).

The individual tasks associated with the accomplishment of a total mission must also be determined before the test and evaluation program can be planned. Although individual tasks may be further subdivided, a military mission will normally require the pilot (crew) to perform the following tasks:

1. Preflight and ground operations
2. Takeoff and Climbs
3. Navigation
4. Mission Maneuvering/Employment
5. Approach and Landing
6. Postflight and Ground Operation

The tasks for which the "best" levels of flying qualities are required are the essential or critical tasks required by the total mission. For a fighter or attack aircraft performing air-to-air or air-to-ground maneuvers (and training for those maneuvers), the greatest emphasis must be placed on the flying qualities exhibited while performing these critical tasks. For a bomber or tanker aircraft, low level terrain following, or air-to-air refueling might be critical maneuvers. These tasks vary with aircraft mission. In any case, adequate flying qualities must be provided so that takeoff, approach, and landing maneuvers can be consistently accomplished safely and precisely (1.4).

The primary reason for conducting flying quality investigations then is to determine if the pilot-aircraft combination can safely and precisely perform the various tasks and maneuvers required by the total aircraft mission. This determination can often be made by a purely qualitative approach; however, this is usually only part of the complete test program. Quantitative flight testing must also be performed to:

1. Substantiate pilot qualitative opinion.
2. Document aircraft characteristics which particularly enhance or degrade some flying quality.
3. Provide data for comparing aircraft characteristics and for improving aircraft and simulator design criteria.
4. Provide baseline data for future expansion in terms of flight or center of gravity envelopes or change in aircraft mission.
5. Determine compliance or noncompliance with flying qualities guarantees, appropriate military specifications, and federal airworthiness regulations, as applicable.

A balance between qualitative and quantitative testing is normally achieved in any flying quality flight test evaluation program (1.4).

1.3 FLYING QUALITY REQUIREMENTS

In December 1907, the United States Army Signal Corps issued Signal Corps Specification 486 for procurement of a heavier-than-air flying machine. The specification stated, "During this trial flight of one hour it must be steered

in all directions without difficulty and at all times be under perfect control and equilibrium." This was clearly a flight demonstration requirement (1.10). The Air Force Lightweight Fighter Request for Proposal in 1972, in addressing stability and control, specified only that the aircraft should have no handling qualities deficiencies which would degrade the accomplishment of its air superiority mission (1.11, 1.12). In response, the contractor predicted that the handling qualities of the prototype would "...permit the pilot to maneuver with complete abandon" (1.12, 1.13). The requirements placed on the Wright Flyer and the Lightweight Fighter contractor's flying quality predictions were remarkably similar. From these examples, one might assume that the art or science of specifying flying quality requirements has not progressed since 1907.

In the late 1930's, flying quality requirements appeared in a single but all encompassing statement appearing in the Army Air Corps designers handbook: "The stability and control characteristics should be satisfactory" (1.10).

In 1940, the National Advisory Committee for Aeronautics (NACA) concentrated on a sophisticated program to correlate aircraft stability and control characteristics with pilots' opinions on the aircraft's flying qualities. They determined parameters that could be measured in-flight which could be used to quantitatively define the flying qualities of aircraft. The NACA also started accumulating data on the flying qualities of existing aircraft to use in developing design requirements (1.14).

Probably the first effort to set down an actual specification for flying qualities was performed by Warner for the airlines during the Douglas DC-4 development (1.14). During World War II, research branches of both the Army Air Corps and Navy became involved in flying quality development and started to build their own capabilities in this area. An important study headed by Gilruth, published in 1943, was the culmination of all of this work up to that time (1.15). This study was supplemented by additional stability and control tests conducted at Wright Field under the auspices of Perkins (1.16). Shortly thereafter, the first set of Air Corps requirements was issued as a result of joint effort between the Army Air Corps, the Navy, and NACA. At the same time, the Navy issued a similar specification. These specifications were superseded and revised in 1945 (1.10, 1.17). Perkins also published a manual which presented methods for conducting flight tests and reducing data to

demonstrate compliance with the stability and control specification. This manual, published in 1945, is remarkably similar to, and is the forerunner of, the present USAF Test Pilot School Flying Qualities Flight Test Handbook (1.16).

Progress in the development of military flying quality specifications is well documented in References 1.10 and 1.18. Work on MIL-F-8785 was started in 1966 and first published in August 1969 as MIL-F-8785B(ASG). It was revised in 1974 and again in 1980 when it was republished as MIL-F- 8785C. The Background Information and User Guide for MIL-F-8785B(ASG) (1.18) explains the concept and arguments upon which the current requirements were based. Data reduction techniques recommended to determine military specification compliance are essentially those presently in use at the USAF Test Pilot School. The School was actively involved in the development of MIL-F-8785B(ASG), and was first used by students evaluating their data group aircraft (T-33A, T-38A, and B-57) against specification requirements. MIL-F-8785C was first used by Class 81-A evaluating the KC-135, T-38, F-4, and A-7 aircraft.

The School also participated in development of MIL-F-83300 which places flying quality requirements on piloted V/STOL aircraft (1.19). Reference 1.20 is a companion background document for this specification. No effort was made to evaluate the School's H-13 helicopters against specification requirements (1.21).

Formal discussions of aircraft flying qualities almost always revolve about the formal military document MIL-F-8785C. As mentioned earlier, this specification focuses almost entirely on open loop vehicle characteristics in attempting to ensure that piloted flight tasks can be performed with sufficient ease and precision; that is, the aircraft has satisfactory handling qualities. This approach is quite different from that used to specify the acceptability of automatic flight control systems, where desired closed loop performance and reliability are specified. This occurs despite the fact that most flying quality deficiencies appear only when the pilot is in the loop acting as a high-gain feedback element (1.22).

This task-related nature of handling qualities is now popularly recognized. However, for modern flight control systems concepts, it is not quite so clear just what the critical pilot tasks will be; therefore, a problem exists in developing design criteria for fly-by-wire and higher order

control systems. A complicating factor is the changing nature of air warfare tactics as a result of the changing threat, improving avionics capability, and the increasing functional integration of hardware and aircraft subsystems (1.23).

Military flying quality specifications have been failures as "requirements." That is, they have not recently (at least since 1970) been used as procurement compliance documents. The search for an alternative approach to the specification of aircraft flying qualities has been going on for some time. The difficulty is in developing a physically sound approach which is acceptable to the military services and to those contractors who must design to stated requirements (1.23). The current attempt to define an approach is an Air Force Flight Dynamics Laboratory funded effort by Systems Technology Incorporated to develop a military standard for flying qualities to replace the present MIL-F-8785C.

It is generally true that developing engineering specifications or standards for something so elusive as handling qualities is an art form; however, there is no basis for believing that Cooper-Harper ratings--properly obtained--are not adequate measures of handling qualities. Pilot opinion rating is the only acceptable, available method for handling qualities quantification. In fact, in current literature pilot opinion rating is considered to be synonymous with handling qualities evaluation (1.22, 1.23).

1.4 CONCEPTS OF STABILITY AND CONTROL

In order to exhibit satisfactory flying qualities, an aircraft must be both stable and controllable. The optimum "blend" depends on the total mission of the aircraft. A certain stability is necessary if the aircraft is to be easily controlled by a human pilot. However, too much stability can severely degrade the pilot's ability to perform maneuvering tasks. The optimum blend of stability and control should be the aircraft designer's goal. Flying qualities greatly enhance the ability of the pilot to perform the intended mission when the optimum blend is attained (1.4).

1.4.1 Stability

An aircraft is a dynamic system, i.e., it is a body in motion under the influence of forces and moments producing or changing that motion. In order to investigate aircraft motion, it is first necessary to establish that it can be brought into a condition of equilibrium, i.e., a condition of balance between opposing forces and moments. Then the stability characteristics can be determined. The aircraft is statically stable if restoring forces and moments tend to restore it to equilibrium when disturbed. Thus, static stability characteristics must be investigated from equilibrium flight conditions, in which all forces and moments are in balance. The direct in-flight measurement of some static stability parameters is not feasible in many instances. Therefore, the flight test team must be content with measuring parameters which only give indications of static stability. However, these indications are usually adequate to establish the mission effectiveness of the aircraft conclusively and are more meaningful to the pilot than the numerical value of stability derivatives (1.4).

The pilot makes changes from one equilibrium flight condition to another through one or more of the aircraft modes of motion. These changes are initiated by exciting the modes by the pilot and terminated by suppressing the modes by the pilot. This describes the classic pilot-in-the-loop flight task. These modes of motion may also be excited by external perturbations. The study of the characteristics of these modes of motion is the study of dynamic stability. Dynamic stability may be classically defined as the time history of the aircraft as it eventually regains equilibrium flight conditions after being disturbed. Dynamic stability characteristics are measured during nonequilibrium flight conditions when the forces and moments acting on the aircraft are not in balance (1.4).

Static and dynamic stability determine the pilot's ability to control the aircraft. While static instability about any axis is generally undesirable, excessively strong static stability about any axis degrades controllability to an unacceptable degree. For some pilot tasks, neutral static stability may actually be desirable because of increased controllability which results. Obviously, the optimum level of static stability depends on the aircraft mission (1.4).

The modes of motion of the aircraft determine its dynamic stability characteristics. The most important characteristics are the frequency, damping, and time constant of the motion. Frequency is defined as the "number of cycles per unit time" and is a measure of the "quickness" of the motion. Damping is a progressive diminishing of amplitude and is a measure of the subsidence of the motion. Damping of the aircraft modes of motion has a profound effect on flying qualities. If it is too low, the aircraft motion is too easily excited by inadvertent pilot inputs or by atmospheric turbulence. If it is too high, the aircraft motion following a control input is slow to develop, and the pilot may describe the aircraft as "sluggish." The aircraft mission again determines the optimum dynamic stability characteristics. However, the pilot desires some damping of aircraft modes of motion. The time constant of the motion is a measure of the overall quickness with which an aircraft, once disturbed from equilibrium, returns to the equilibrium condition (1.4).

Static and dynamic stability prevent unintentional excursions into dangerous flight regimes (with regard to aircraft strength) of dynamic pressure, normal acceleration, and sideforce. The stable aircraft is resistant to deviations in angle of attack, sideslip, and bank angle without action by the pilot. These characteristics not only improve flight safety, but allow the pilot to perform maneuvering tasks with smoothness, precision, and a minimum of effort (1.4).

1.4.2 Control

Controllability is the capability of the aircraft to perform any maneuvering required in total mission accomplishment at the pilot's command. The aircraft characteristics should be such that these maneuvers can be performed precisely and simply with minimum pilot effort (1.4).

1.5 AIRCRAFT CONTROL SYSTEMS

The aircraft flight control system consists of all the mechanical, electrical, and hydraulic elements which convert cockpit control inputs into aerodynamic control surface deflections, or action of other control

devices which in turn change the orientation of the vehicle. The flight control system together with the powerplant control system enables the pilot to "fly" his aircraft, that is, to place it at any desired flight condition within its capability.

The powerplant control system acts as a thrust metering device, while the flight control system varies the moments about the aircraft center of gravity. Through these control systems, the pilot varies the velocity, normal acceleration, sideslip, roll rate, and other parameters within the aircraft's envelope. How easily and effectively he accomplishes his task is a measure of the suitability of his control systems. An aircraft with exceptional performance characteristics is virtually worthless if it is not equipped with at least an acceptable flight control system. Since the pilot-control system acts on an aircraft with specific static and dynamic stability properties, it follows that the characteristics of the closed-loop system must be satisfactory.

The control system must meet two conditions if the pilot is to be given suitable command over his aircraft.

1. It must be capable of actuating the control surface.
2. It must provide the pilot with a "feel" that bears a satisfactory relationship to the aircraft's reaction.

There are numerous aircraft control systems designs. However, these systems may be rather simply classified. Aerodynamic controls can be broken down into "reversible" and "irreversible" systems. These systems can be simple mechanical controls in which the pilot supplies all of the force required to move the control surface. This type system is called "reversible" since all of the forces required to overcome the hinge moments at the control surface are transmitted to the cockpit controls. The system may have a mechanical, hydraulic, or some other type of boosting device, which supplies some specific proportion of the control force. Systems of this nature are called "boosted control systems." However, they are still considered "reversible." Even though the force required of the pilot is less than the control surface hinge moments, the force required is proportional to these moments. In other words, the pilot furnishes a fraction of the force required

to overcome the hinge moments throughout the aircraft's envelope. The control system is said to be "irreversible" if the pilot actuates a hydraulic or electronic device which in turn moves the control surface. In this system, the aerodynamic hinge moments at the control surface are no longer transmitted to the control wheel or stick. Without artificial feel devices, the pilot would feel only the force required to actuate the valves or sensing devices of his powered control system. Because of this, artificial feel is added which approximates the feel that the pilot senses with the "reversible" system.

A thorough knowledge of the aircraft control system is necessary before a flying quality evaluation can be planned and executed. The flying qualities test team must be intimately familiar with the control system of the test aircraft. Is the system reversible or irreversible; what type of control surfaces does it have; is a stability augmentation system incorporated, if so, how does it work; is an autopilot included, if so how does it work; are there interconnects between control surfaces (e.g., rudder deflection limited with landing gear up or ailerons limited when the aircraft exceeds a certain airspeed); and what malfunctions effect flying qualities? Total understanding of the test aircraft is necessary in order to get the most information out of a limited flight test program (1.1).

Aircraft control systems will be checked against several paragraphs of MIL-F-8785C here at the USAF Test Pilot School during student evaluation of data group aircraft.

1.6 SUMMARY

An aircraft's flying qualities evaluation incorporates all aspects of the aircraft's stability and control characteristics, control system design, and pilot-in-the-loop handling qualities. The interaction of all these elements determines the ability of a pilot/aircraft/flight control combination to safely and successfully accomplish a mission.

1.7 USAF TEST PILOT SCHOOL CURRICULUM APPROACH

1. Review the mechanical tools, vectors, matrices, and differential equations required for flying quality analysis.

2. Develop the aircraft equations of motion.

3. Examine static longitudinal and lateral-directional aircraft characteristics and steady state maneuvers.

4. Analyze the aircraft longitudinal and lateral-directional dynamics modes of motion.

5. Study specialized flying quality topics such as stall/post-stall/spin and departure, engine-out, and qualitative and operational aircraft evaluations.

6. Discuss advanced flying quality topics. These include the using systems identification techniques for closed loop handling qualities evaluations and extracting stability derivatives from flight test data. Discuss effects of higher order control systems on aircraft flying qualities.

BIBLIOGRAPHY

1.1 Anon., Engineer's Handbook for Aircraft Performance and Flying Qualities Flight Testing, Performance and Flying Qaulities Branch, Flight Test Engineering Division, 6510th Test Wing, Air Force Flight Test Center, Edwards AFB, CA, May 1971, UNCLASSIFIED.

1.2 Phillips, W.H., Appreciation and Prediction of Flying Qualities, NACA Report 927, National Advisory Committee for Aeronautics, Washington DC, 1949, UNCLASSIFIED.

1.3 Anon., Military Specification, Flying Qualities of Piloted Airplanes, MIL-F-8785C, 5 Nov 80, UNCLASSIFIED.

1.4 Langdon, S.D., et al., Fixed Wing Stability and Control Theory and Flight Test Techniques, USNTPS-FTM-No. 103, US Naval Test Pilot School, Naval Air Test Center, Patuxent River, MD, 1 Jan 75, UNCLASSIFIED.

1.5 Etkin, B., Dynamics of Flight, New York: John Wiley & Sons, Inc., 1959.

1.6 Cooper, G.E., and Harper, R.P., Jr., The Use of Pilot Rating in the Evaluation of Aircraft Handling Qualities, NASA TN D-5153, National Aeronautics and Space Agency, Washington DC, April 1959, UNCLASSIFIED.

1.7 Nagy, C.J., A New Method for Test and Analysis of Dynamic Stability and Control, AFFTC-TD-75-4, AFFTC, Edwards AFB, CA 93523, May 1976, UNCLASSIFIED.

1.8 Twisdale, T.R., and Franklin, D.T., Tracking Techniques for Handling Qualities Evaluation, AFFTC-TD-75-1, AFFTC, Edwards AFB, CA 93523, April 1975, UNCLASSIFIED.

1.9 Twisdale, T.R., and Ashurst, T.A., Jr., Systems Identification from Tracking (SIFT), A New Technique for Handling Qualities Test and Evaluation, AFFTC TR-77-27, AFFTC, Edwards AFB, CA 93523, Nov 1977, UNCLASSIFIED.

1.10 Westbrook, C.B., The Status and Future of Flying Quality Requirements, AFFDL-FDCC-TM-65-29, Air Force Flight Dynamics Laboratory, Research and Technology Division, Wright-Patterson AFB, OH, June 1955, UNCLASSIFIED.

1.11 Anon., Request for Proposal: Lightweight Fighter Prototype Aircraft, F33657-72-R-0235, Aeronautical Systems Division, Wright-Patterson AFB, OH, 6 Jan 72, CONFIDENTIAL.

1.12 Eggers, J.A., and Bryant, W.F., Flying Qualities Evaluation of the YF-16 Prototype Lightweight Fighter, AFFTC-TR-75-15, Air Force Flight Test Center, Edwards AFB, CA 93523, July 75, DISTRIBUTION LIMITED.

1.13 Anon., Proposed Lightweight Fighter Prototype, FZP-1401, General Dynamics Convair Aerospace Division, Fort Worth, TX, 18 Feb 72, CONFIDENTIAL.

1.14 Perkins, C.D., Development of Airplane Stability and Control Technology, AIAA Journal of Aircraft, Vol. 7, No. 4, Apr 70, pp. 290-310, UNCLASSIFIED.

1.15 Gilruth, R.R., Requirements for Satisfactory Flying Qualities of Airplanes, NACA TR No. 755, National Advisory Committee for Aeronautics, Washington DC, 1943, UNCLASSIFIED.

1.16 Perkins, C.D., and Walkowicz, A.C., Stability and Control Flight Test Methods, AAT Technical Report No. 5242, Army Air Forces Air Technical Services Command, Dayton, OH, 14 Jul 45, UNCLASSIFIED.

1.17 Anon., Stability and Control Requirements for Airplanes, Army Air Forces Specifications C-1815, 1943, and R-1815A, 1945, UNCLASSIFIED.

1.18 Chalk, C.R., et al., Background Information and User Guide for MIL-F-8785B(ASG), "Military Specifications - Flying Qualities of Piloted Airplanes," AFFDL-TR-69-72, Air Force Flight Dynamics Laboratory, Wright-Patterson AFB OH, Aug 69, UNCLASSIFIED.

1.19 Anon., Military Specification, Flying Qualities of Piloted V/STOL Aircraft, MIL-F-83300, 31 Dec 70, UNCLASSIFIED.

1.20 Chalk, C.R., et al., Background Information and User Guide for MIL-F-83300, "Military Specification - Flying Qualities of Piloted V/STOL Aircraft," AFFDL-TR-70-88, Air Force Flight Dynamics Laboratory, Wright-Patterson AFB, OH, Mar 71, UNCLASSIFIED.

1.21 Jones, R.L., Major, USAF, Introduction to VSTOL Technology, Lecture notes used at the USAF Aerospace Research Pilot School, Edwards AFB, CA 93523, Dec 70, UNCLASSIFIED.

1.22 Hess, R.A., A Pilot Modeling Technique for Handling Qualities Research, Paper No. 80-1624, presented at a workshop on Pilot Induced Oscillations held at NASA Dryden Flight Research Center, Edwards AFB, CA 93523, Nov 18-19, 1980, UNCLASSIFIED.

1.23 Smith, R.H., and Geddes, N.D., Handling Quality Requirements for Advanced Aircraft Design: Longitudinal Mode, AFFDL-TR-78-154, Air Force Flight Dynamics Laboratory, Wright-Patterson AFB, OH, 45433, Aug 79, UNCLASSIFIED.

CHAPTER 2

VECTORS AND MATRICES

2.1 INTRODUCTION

This chapter studies the algebra and calculus of vectors and matrices, as specifically applied to the USAF Test Pilot School curriculum. The course is a prerequisite for courses in Equations of Motion, Dynamics, Linear Control Systems, Flight Control Systems, and Inertial Navigation Systems. The course deals only with applied mathematics; therefore, the theoretical scope of the subject is limited.

The text begins with the definition of determinants as a prerequisite to the remainder of the text. Vector analysis follows with rigid body kinematics introduced as an application. The last section deals with matrices.

2.2 DETERMINANTS

A determinant is a function which associates a number (real, imaginary, or vector) with every square array (n columns and n rows) of numbers. The determinant is denoted by vertical bars on either side of the array of numbers. Thus, if A is an (n x n) array of numbers where i designates rows and j designates columns, the determinant of A is written

$$|A| = |a_{ij}| = \begin{vmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \cdot & \cdot \cdots \cdot \\ \cdot & \cdot \cdots \cdot \\ a_{n1} & a_{n2} \cdots a_{nn} \end{vmatrix}$$

2.2.1 First Minors and Cofactors

When the elements of the i^{th} row and j^{th} column are removed from a (n x n) square array, the determinant of the remaining (n-1) x (n-1) square array is called a first minor of A and is denoted by M_{ij}. It is also called the minor of a_{ij}. The signed minor, with the sign determined by the sum of

the row and column, is called the <u>cofactor</u> of a_{ij} and is denoted by

$$A_{ij} = (-1)^{i+j} M_{ij}$$

Example:

$$\text{If } |A| = |a_{ij}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

then,

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \qquad M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

Also, $A_{11} = (-1)^{1+1} M_{11} = (+1)\ M_{11}$ and $A_{32} = (-1)^{3+2} M_{32} = (-1)\ M_{32}$

2.2.2 <u>Determinant Expansion</u>

The determinant is equal to the sum of the products of the elements of any single row or column and their respective cofactors; i.e.,

$$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots + a_{in}A_{in} = \sum_{j=1}^{n} a_{ij}A_{ij}, \text{ for any single } i^{th} \text{ row.}$$

or

$$= a_{1j}A_{1j} + a_{2j}A_{2j} + \ldots + a_{nj}A_{nj} = \sum_{i=1}^{n} a_{ij}A_{ij}, \text{ for any single } j^{th}$$

column.

2.2.2.1 <u>Expanding a 2 x 2 Determinant</u>. Expanding a 2 x 2 determinant about the first row is the easiest. The sign of the cofactor of an element can be determined quickly by observing that the sums of the subscripts alternate from even to odd when advancing across rows or down columns, meaning the signs

alternate also. For example,

$$\text{if} \quad |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

the signs of the associated cofactors alternate as shown,

$$\begin{vmatrix} + & - \\ - & + \end{vmatrix}$$

By deleting the row and column of a_{11}, we find its cofactor is just the element a_{22} [actually $(+1) \times a_{22}$] for a 2 x 2 array, and likewise the cofactor for a_{12} is $(-a_{21})$ [or $(-1) \times a_{21}$]. The sum of these two products gives us the expansion or value of the determinant.

$$|A| = a_{11} A_{11} + a_{12} A_{12} = a_{11}\, a_{22} + a_{12}\, (-a_{21}) = a_{11}\, a_{22} - a_{12}\, a_{21}$$

This simple example has been shown for clarity. Actual calculation of a 2 x 2 determinant is easy if we just remember it as the subtraction of the cross multiplication of the elements. For example,

$$|R| = \begin{array}{c} \begin{array}{cc} (+) & (-) \end{array} \\ \begin{vmatrix} 8 & 3 \\ 6 & 5 \end{vmatrix} \end{array} = (8)(5) - (3)(6) = 22$$

2.2.2.2 <u>Expanding a 3 x 3 Determinant.</u>

$$|A| = |a_{ij}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expanding $|A|$ about the first row gives

$$|A| = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13} =$$

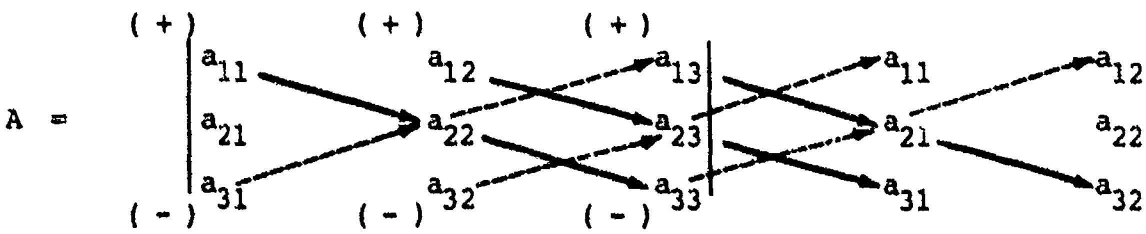
$$a_{11}\ (+1) \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}\ (-1) \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\ (+1) \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} =$$

$$a_{11} (a_{22} a_{33} - a_{23} a_{32}) - a_{12} (a_{21} a_{33} - a_{23} a_{31}) + a_{13} (a_{21} a_{32} - a_{22} a_{31})$$

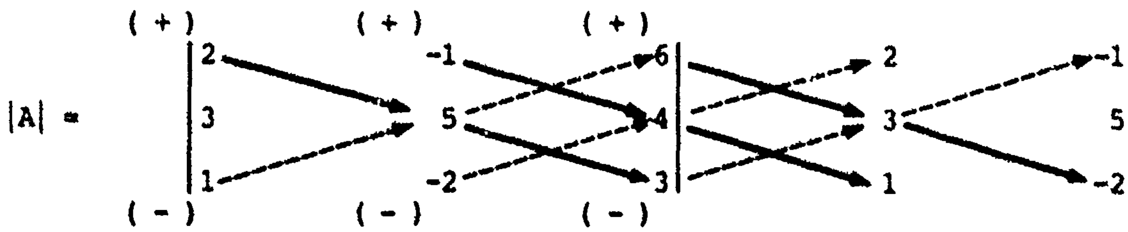
Expanding and grouping like signs,

$$= a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32}$$
$$- a_{13} a_{22} a_{31} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33}$$

Close inspection of the last equation shows a quicker method for 3 x 3 determinants using diagonal multiplication. If the first two columns are appended to the determinant, six sets of diagonals are used to find the six terms above. The signs are determined by the direction of the diagonal as shown in the illustration.



For example,



$$= (2)(5)(3) + (-1)(4)(1) + (6)(3)(-2) - (6)(5)(1) - (2)(4)(-2) - (-1)(3)(3)$$

$$= 30 + (-4) + (-36) - 30 - (-16) - (-9) = -40 + 25 = \underline{-15}$$

The quicker methods of calculating determinants are useful for the two simple cases here. The row expansion method will be more useful for calculating vector cross products. The use of determinants for solving sets of linear equations will be discussed later in this chapter in the matrix section. Determinants will also be used in solving sets of linear differential equations in Chapter 3, Differential Equations.

While the general tool for evaluating determinants by hand calculation is simple, for determinants of greater size the calculations are lengthy. A 5 x 5 determinant would contain 120 terms of 5 factors each. Evaluating larger determinants is an ideal task for the computer, and standard programs are available for this task.

2.3 VECTOR AND SCALAR DEFINITIONS

In general, a vector can be defined as an ordered set of "n" quantities such as $\langle a_1, a_2, a_3, \ldots, a_n\rangle$. In TPS, vector analysis will be limited to two-and three-dimensional space. Thus, $x\bar{i} + y\bar{j}$ and $x\bar{i} + y\bar{j} + z\bar{k}$ are representations of vectors in each space, while $x\bar{i}$, $y\bar{j}$, and $z\bar{k}$ are referred to as components of the vector.

Physically, a vector is an entity such as force, velocity, or acceleration, which possesses both magnitude and direction. This is the usual approach in applied physics and engineering, and the results can be directly applied to courses here at the School.

Almost any physical discussion will involve, in addition to vectors, entities such as volume, mass, and work, which possess only magnitude and are known as scalars. To distinguish vectors from scalars, a vector quantity will be indicated by putting a line above the symbol; thus $\bar{F}$, $\bar{v}$, and $\bar{a}$ will be used to represent force, velocity, and acceleration, respectively.

The magnitude of vector $\bar{F}$ is indicated by enclosing the symbol for the vector between absolute value bars, $|\bar{F}|$. Graphically, a scalar quantity can be adequately represented by a mark on a fixed scale. To represent a vector quantity requires a directed line segment whose direction is the same as the direction of the vector and whose measured length is equal to the magnitude of the vector.

The direction of a vector is determined by a single angle in two dimensions and two angles in three dimensions, angles whose cosines are called direction cosines. This text will not deal directly with direction cosines, so no example is necessary.

2.3.1 Vector Equality

Two vectors whose magnitude and direction are equal are said to be equal. If two vectors have the same length but the opposite direction, either is the negative of the other. This is true even when graphically two vectors are not physically drawn from the same starting point.

A vector that may be drawn from any starting point is called a free vector. However, when applied in a problem, the position of a vector may be important. For instance, in Figure 2.1, the distance of the line of application of a force from the center of gravity of a rigid body is critical if calculating moments, although the actual point of application along the line isn't critical.

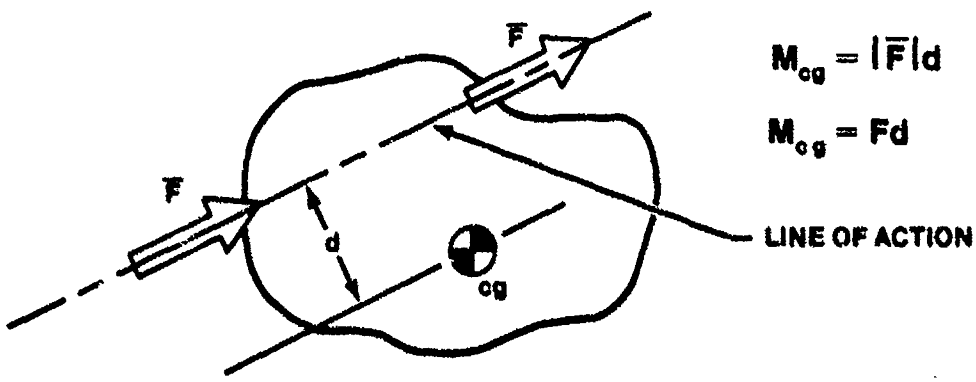


FIGURE 2.1. MOMENT CALCULATION

For other applications, the point of action as well as the line of action must be fixed. Such a vector is usually referred to as bound. The velocity of the satellite in the orbital mechanics problem shown in Figure 2.2 is an example of a bound vector.

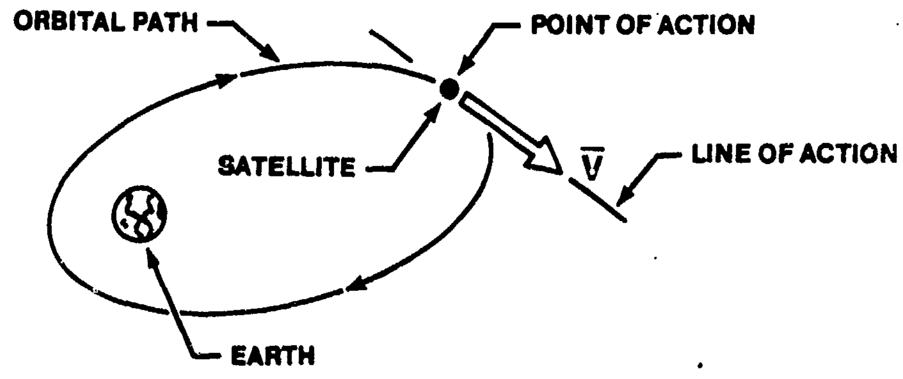


FIGURE 2.2. EXAMPLE OF A BOUND VECTOR

2.3.2 Vector Addition

Graphically, the sum of two vectors $\overline{A}$ and $\overline{B}$ is defined by the familiar parallelogram law; i.e., if $\overline{A}$ and $\overline{B}$ are drawn from the same point or origin, and if the parallelogram having $\overline{A}$ and $\overline{B}$ as adjacent sides is constructed, then the sum $\overline{A} + \overline{B}$ can be defined as the vector represented by the diagonal of this parallelogram which passes through the common origin of $\overline{A}$ and $\overline{B}$. Vectors can also be added by drawing them "nose-to-tail." See Figure 2.3.

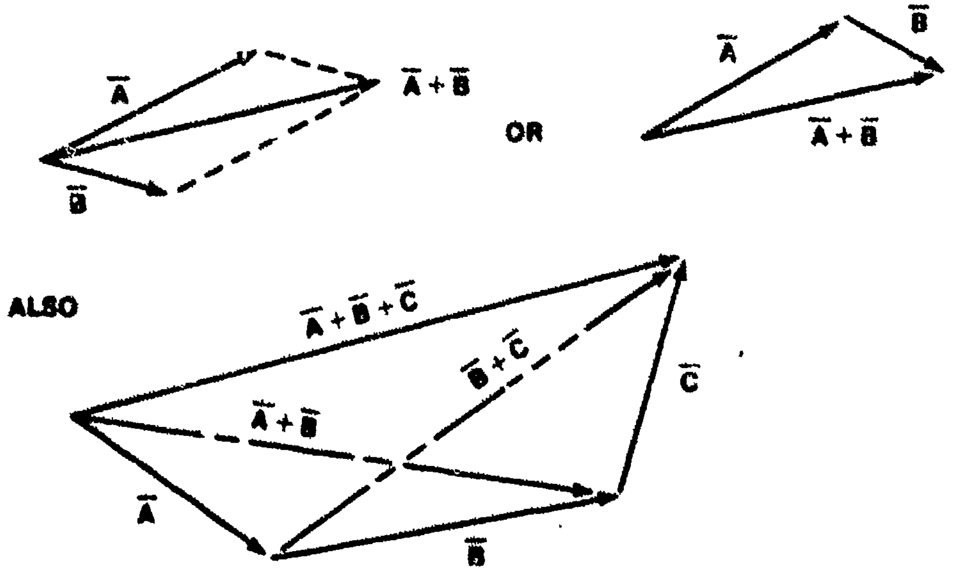


FIGURE 2.3. ADDITION OF VECTORS

Graphically from Figure 2.3, it is evident that vector addition is commutative and associative, respectively,

$$\bar{A} + \bar{B} = \bar{B} + \bar{A} \quad \text{and} \quad \bar{A} + (\bar{B} + \bar{C}) = (\bar{A} + \bar{B}) + \bar{C}$$

2.3.3 Vector Subtraction

Vector subtraction is defined as the difference of two vectors $\bar{A}$ and $\bar{B}$,

$$\bar{A} - \bar{B} = \bar{A} + (-\bar{B}),$$

where

$$(-1)(\bar{B}) = (-\bar{B})$$

and is defined as a vector with the same magnitude but opposite direction. See Figure 2.4. This introduces the necessity for vector-scalar multiplication.

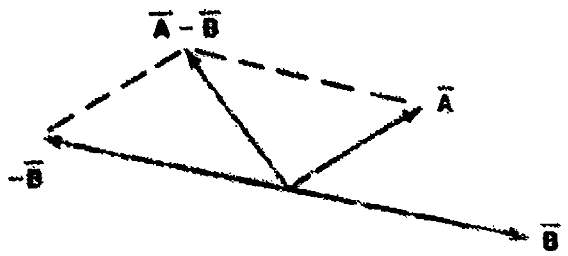


FIGURE 2.4. VECTOR SUBTRACTION

2.3.4 Vector-Scalar Multiplication

The product of a vector and a scalar follows algebraic rules. The product of a scalar m and a vector $\bar{A}$ is the vector $m\bar{A}$, whose length is the product of the absolute value of m and magnitude of A, and whose direction is the same as the direction of A, if m is positive, and opposite to it, if m is negative.

2.3.5 Unit and Zero Vectors

Regardless of its direction, a vector whose length is one (unity) is called a unit vector. If $\bar{a}$ is a vector with magnitude other than zero, then unit vector $\hat{a}$ is defined as $\bar{a}/|\bar{a}|$, where $\hat{a}$ is a unit vector having the same direction as $\bar{a}$ and magnitude of one. It happens that the components of a unit vector are also the cosines of the angles necessary to define the direction. Unit vectors in the body axis coordinate system will retain the bar symbol; i.e., $\bar{i}$, $\bar{j}$ and $\bar{k}$.

The zero vector has zero magnitude and in this text has any direction. It is notationally correct to designate the zero vector with a bar, $\bar{0}$.

2.4 LAWS OF VECTOR - SCALAR ALGEBRA

If $\bar{A}$, $\bar{B}$, and $\bar{C}$ are vectors and m and n are scalars, then

1.	$m\bar{A} = \bar{A}m$	Commutative Multiplication
2.	$m(n\bar{A}) = (mn)\bar{A}$	Associative Multiplication
3.	$(m + n)\bar{A} = m\bar{A} + n\bar{A}$	Distributive
4.	$m(\bar{A} + \bar{B}) = m\bar{A} + m\bar{B}$	Distributive

These laws involve multiplication of a vector by one or more scalars. Products of vectors will be defined later.

These laws, along with the vector addition laws already introduced, enable vector equations to be treated the same way as ordinary scalar algebraic equations. For example,

$$\text{if } \bar{A} + \bar{B} = \bar{C}$$

then by algebra

$$\bar{A} = \bar{C} - \bar{B}$$

2.4.1 Vectors in Coordinate Systems

The right-handed rectangular coordinate system is used unless otherwise stated. Such a system derives its name from the fact that a right threaded screw rotated through 90° in the direction from the positive x-axis to the positive y-axis will advance in the positive z direction, as shown in Figure 2.5. Practically, curl the fingers of the right hand in a direction from the positive x-axis to the positive y-axis, and the thumb will point in the positive z-axis direction.

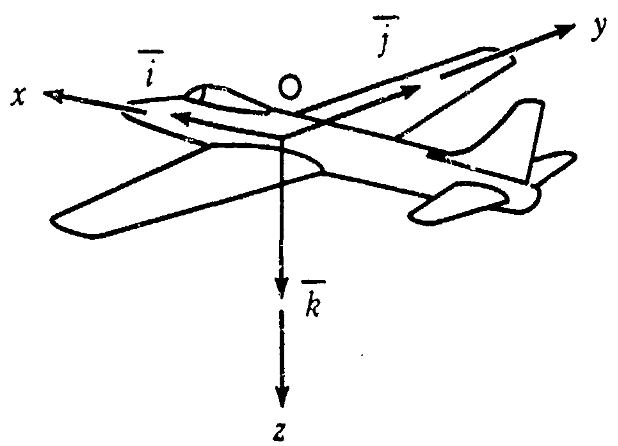


FIGURE 2.5. RIGHT-HANDED COORDINATE SYSTEM

An important set of unit vectors are those having the directions of the positive x, y, and z axes of a three-dimensional rectangular coordinate system and are denoted $\bar{i}$, $\bar{j}$, and $\bar{k}$, respectively, as shown in Figure 2.5.

Any vector in three dimensions can be represented with initial point at the origin of a rectangular coordinate system as shown in Figure 2.6. The perpendicular projection of the vector on the axes gives the vector's components on the axes. Multiplying the scalar magnitude of the projection by the appropriate unit vector in the direction of the axis gives a component vector of the original vector. Note that summing the component vectors graphically gives the original vector as a resultant.

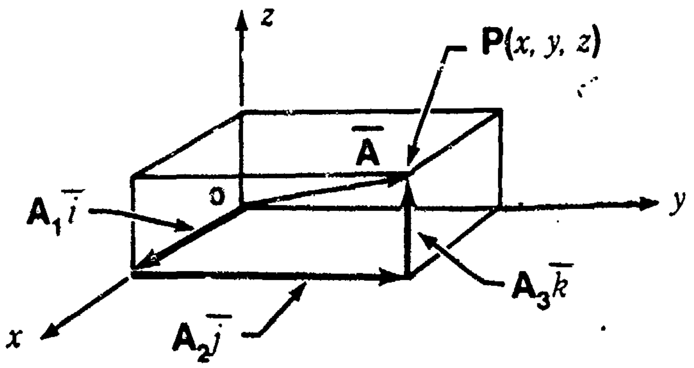


FIGURE 2.6. COMPONENTS OF A VECTOR

In Figure 2.6 the component vectors are $A_1\overline{i}$, $A_2\overline{j}$, and $A_3\overline{k}$. The sum or resultant of the components gives a new notation for a vector in terms of its components.

$$\overline{A} = A_1\overline{i} + A_2\overline{j} + A_3\overline{k}$$

After noticing that the coordinates of the end-point of a vector $\overline{A}$ whose tail is at the origin are equal to the components of the vector itself ($A_1 = x$, $A_2 = y$, and $A_3 = z$), the vector may be more easily written as

$$\overline{A} = x\overline{i} + y\overline{j} + z\overline{k}$$

The vector from the origin to a point in a coordinate system is called a <u>position vector</u>, so the vector notation above is also the position vector for the point P. The same definitions for notation, components, and position hold for a two-dimensional system with the third component eliminated.

The magnitude is easily calculated as,

$$A = \sqrt{A_1^2 + A_2^2 + A_3^2} \quad \text{or} \quad A = \sqrt{x^2 + y^2 + z^2}$$

An arbitrary vector from initial point $P(x_1, y_1, z_1)$ and terminal point $Q(x_2, y_2, z_2)$ such as shown in Figure 2.7 can be represented in terms of unit vectors, also.

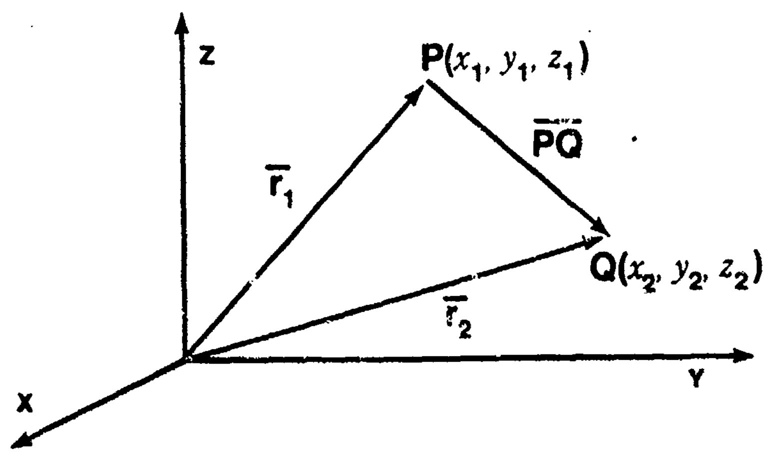


FIGURE 2.7. ARBITRARY VECTOR REPRESENTATION

First write the position vectors for the two points P and Q.

$$\bar{r}_1 = x_1\bar{i} + y_1\bar{j} + z_1\bar{k}$$

and

$$\bar{r}_2 = x_2\bar{i} + y_2\bar{j} + z_2\bar{k}$$

Then using addition,

$$\bar{r}_1 + \overline{PQ} = \bar{r}_2$$

or

$$\overline{PQ} = \bar{r}_2 - \bar{r}_1 = (x_2 - x_1)\bar{i} + (y_2 - y_1)\bar{j} + (z_2 - z_1)\bar{k}$$

2.4.2 Dot Product

In addition to the product of a scalar and a vector, two other types of products are defined in vector analysis. The first of these is the dot, or scalar product, denoted by a dot between the two vectors. The dot product is an operation between two vectors, and results in a scalar (thus the name scalar product). Analytically, it is calculated by adding the products of like components. This is, if

$$\bar{A} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$$

and

$$\bar{B} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$$

then

$$\bar{A}\cdot\bar{B} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

which is a real number or scalar.

Geometrically, it is equal to the product of the magnitudes of two vectors and the cosine of the angle between them (the angle is measured in the plane formed by the two vectors, if they had the same origin). The dot product is written

$$\bar{A} \cdot \bar{B} = |\bar{A}|\ |\bar{B}| \cos \theta$$

Example

$$(2\bar{i} - 1\bar{j} + 4\bar{k}) \cdot (-\bar{i} + 3\bar{j} + 5\bar{k}) = (2)(-1) + (-1)(3) + (4)(5) = 15$$

The magnitudes are

$$\sqrt{4 + 1 + 16} = 4.6$$

and

$$\sqrt{1 + 9 + 25} = 5.9$$

Therefore, solving for

$$\mathrm{Cos}\ \theta = 15 / (4.6)(5.9) = 15/27.1 = 0.553$$

so

$$\theta = 56.1^{\circ}$$

Some interesting applications of the dot product are the geometric implications. For instance, the geometric, scalar projection of one vector on another is shown on Figure 2.8.

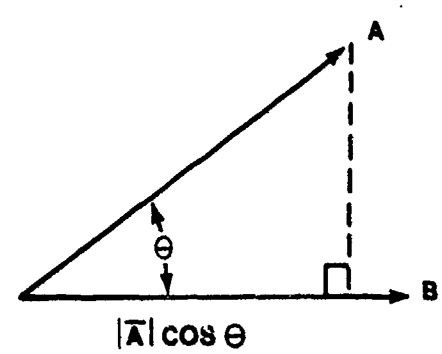


FIGURE 2.8. GEOMETRIC PROJECTION OF VECTORS

Using trigonometry, the projection of $\bar{A}$ on $\bar{B}$ is seen to be equal to $|\bar{A}| \cos \theta$. A quick method to calculate such a projection without knowing the angle is to calculate the dot product and divide by the magnitude of the vector projected on to. That is, the projection of $\bar{A}$ on to $\bar{B}$ is equal to $\bar{A} \cdot \bar{B}/|B| = |A| \cos \theta$.

Several particular dot products are worth mentioning. If one of the vectors is a unit vector, the dot product becomes

$$\bar{i} \cdot \bar{B} = |\bar{i}|\ |\bar{B}| \cos \theta = (1)\ |\bar{B}| \cos \theta = |\bar{B}| \cos \theta,$$

which is the projection of $\bar{B}$ on $\bar{i}$ or more importantly the component of $\bar{B}$ in the direction of $\bar{i}$. Also note the dot product of a vector with itself is just equal to the magnitude squared, since the angle is zero and $\cos \theta = 1$. More useful is the situation where two non-zero vectors are perpendicular (orthogonal). The dot product is zero because the cosine of 90 degrees is zero. Thus, for non-zero vectors the dot product may be a test of orthogonality. Examples of these properties using standard unit vectors are

$$\bar{i} \cdot \bar{i} = \bar{j} \cdot \bar{j} = \bar{k} \cdot \bar{k} = 1$$

and

$$\bar{i} \cdot \bar{j} = \bar{j} \cdot \bar{k} = \bar{k} \cdot \bar{i} = 0$$

2.4.3 Dot Product Laws

If A, B, and C are vectors and m is scalar, then

1. $\bar{A} \cdot \bar{B} = \bar{B} \cdot \bar{A}$ — Commutative Product
2. $\bar{A} \cdot (\bar{B} + \bar{C}) = \bar{A} \cdot \bar{B} + \bar{A} \cdot \bar{C}$ — Distributive Product
3. $m(\bar{A} \cdot \bar{B}) = (m\bar{A}) \cdot \bar{B} = \bar{A} \cdot (m\bar{B})$ — Associative Product

2.4.4 Cross Product

The third type of product involving vector operations is the cross, or vector product, denoted by placing an "X" between two vectors. By definition the cross product is an operation between two vectors which results in another vector (thus, vector product). Again both analytic and geometric definitions are given.

Analytically, the cross product is calculated for three-dimensional vectors (without using memory) by a top row expansion of a determinant.

$$\bar{A} \times \bar{B} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \bar{i} + (-1) \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \bar{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \bar{k}$$

$$= (a_2 b_3 - a_3 b_2)\bar{i} + (a_3 b_1 - a_1 b_3)\bar{j} + (a_1 b_2 - a_2 b_1)\bar{k}$$

For example,

$$(2\bar{i} + 4\bar{j} + 5\bar{k}) \times (3\bar{i} + \bar{j} + 6\bar{k}) =$$

$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 4 & 5 \\ 3 & 1 & 6 \end{vmatrix}$$

$$= [(4)(6)-(5)(1)]\bar{i} - [(2)(6)-(3)(5)]\bar{j} + [(2)(1)-(3)(4)]\bar{k}$$

$$= \underline{19\bar{i} + 3\bar{j} - 10\bar{k}}$$

The geometrical definition has to be approached carefully because it must be remembered that the geometrical definition is not a vector. The magnitude (a scalar) of the cross product is equal to the product of the two magnitudes and the sine of the angle between the two vectors. Thus

$$|\bar{A} \times \bar{B}| = |\bar{A}|\ |\bar{B}| \sin \theta$$

While the magnitude is determined as above, the direction of the resultant cross product vector is always orthogonal to the plane of the crossed vectors. The sense is such that when the fingers of the right hand are curled from the first vector to the second, through the smaller of the angles between the vectors, the thumb points in the direction of the cross product as shown in Figure 2.9. Note the importance in the order of writing $\bar{A} \times \bar{B}$ since $\bar{A} \times \bar{B} \neq \bar{B} \times \bar{A}$. That $\bar{A} \times \bar{B} = -\bar{B} \times \bar{A}$ is easily seen using the right-hand rule.

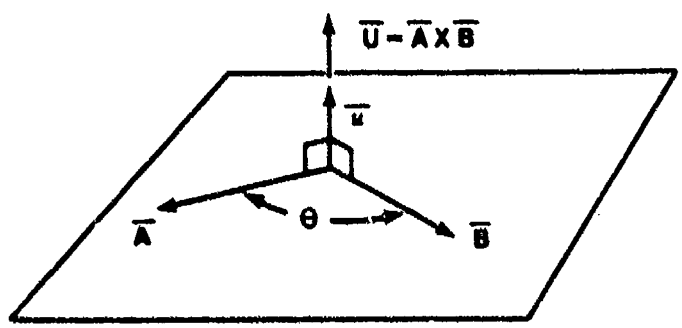


FIGURE 2.9. GEOMETRIC DEFINITION OF THE CROSS PRODUCT

The cross product vector $\bar{U}$ can be represented as

$$\bar{U} = \bar{A} \times \bar{B} = |\bar{A}|\ |\bar{B}| \sin \theta\ \bar{u}$$

where $\bar{u}$ is a unit vector in the direction of $\bar{U}$, which is perpendicular to the plane containing $\bar{A}$ and $\bar{B}$.

Some practical applications of the above definitions using the sine of zero and 90° are shown for unit vectors of a rectangular coordinate system.

and its direction becomes tangential to the trajectory. The $\Delta s/\Delta t$ portion gives the magnitude of the derivative which should be noted as the speed of the particle or a change of distance per time. In summary, the first derivative of a vector function is tangential to the trajectory and has a magnitude that is the speed of the particle.

Using differentiation law four to take the derivative of the vector written in the form of magnitude times a unit vector,

$\bar{r}(t) = r(t)\hat{r}$, as follows,

$$\frac{d\,\bar{r}(t)}{dt} = \frac{d\,[r(t)\,\hat{r}]}{dt} = \frac{d\,r(t)}{dt}\,\hat{r} + r(t)\,\frac{d\,\hat{r}}{dt}$$

note that the linear velocity using this form of a vector has two components, the first is the rate of change of the scalar function with direction the same as the original vector itself. The second component is the scalar function itself with the rate of change of the unit vector as its direction. We know that the unit vector doesn't change magnitude, but it may change direction giving a non-zero derivative. In the development of the derivative earlier, this was overlooked since the rate of change of the $\bar{i}$, $\bar{j}$, and $\bar{k}$ vectors that are fixed in a coordinate system do not change direction or magnitude.

2.6 REFERENCE SYSTEMS

Linear velocity and acceleration have meaning only if expressed (or implied) in reference to another point and only if relative to a particular frame of reference. In this text for discussions of single reference systems, the linear velocity and acceleration will always be relative to the origin of the reference frame in which the problem is given and will be denoted by single letters, $\bar{V}$ and $\bar{a}$. If there are two reference systems in the problem, the notation will be changed to read

$$\bar{V}_{A/B}$$

which means the velocity of point or reference A relative to reference B. To take a time derivative of a vector relative to reference system "A," the notation will be

2.5 LINEAR VELOCITY AND ACCELERATION

The time derivative of a position vector relative to some reference system is the linear velocity. Note in particular that the velocity of a particle is a vector that has a direction and a magnitude. The magnitude of the velocity is referred to as speed. The second derivative is the linear acceleration.

Graphically, the derivative of a vector is illustrated as shown in Figure 2.10.

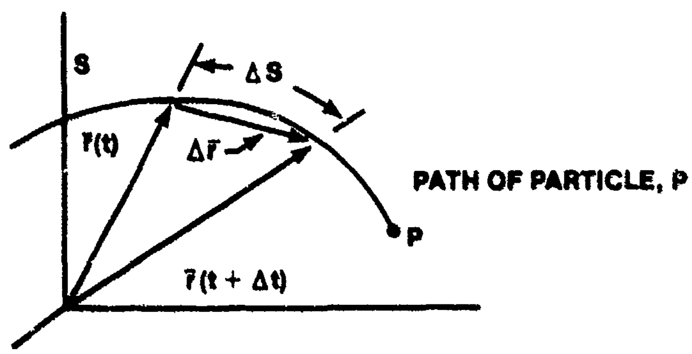


FIGURE 2.10. ILLUSTRATION OF THE DERIVATIVE OF A POSITION VECTOR

The difference between position vectors $\bar{r}(t + \Delta t)$ and $\bar{r}(t)$ is the numerator of the definition of the derivative. The arc length of the trajectory for some Δt is Δs. If we neglect the division by Δt and are concerned only with direction of the derivative, the difference of the two vectors is just Δr which would have the direction as shown in Figure 2.10. The derivative for a vector $r(t)$ can be expanded by multiplying by the quantity $\Delta s/\Delta s = 1$, as follows,

$$\frac{d\bar{r}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\bar{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta\bar{r}}{\Delta t}\frac{\Delta s}{\Delta s} = \lim_{\Delta t \to 0} \frac{\Delta\bar{r}}{\Delta s}\frac{\Delta s}{\Delta t}$$

but as $\Delta t \to 0$, $|\Delta\bar{r}| = \Delta s$, therefore $\lim_{\Delta t \to 0} \Delta\bar{r}/\Delta s = \hat{e}_t$, since its magnitude is one

and its direction becomes tangential to the trajectory. The $\Delta s/\Delta t$ portion gives the magnitude of the derivative which should be noted as the speed of the particle or a change of distance per time. In summary, the first derivative of a vector function is tangential to the trajectory and has a magnitude that is the speed of the particle.

Using differentiation law four to take the derivative of the vector written in the form of magnitude times a unit vector,

$$\bar{r}(t) = r(t)\hat{r}, \text{ as follows,}$$

$$\frac{d\,\bar{r}(t)}{dt} = \frac{d\,[r(t)\,\hat{r}]}{dt} = \frac{d\,r(t)}{dt}\,\hat{r} + r(t)\,\frac{d\,\hat{r}}{dt}$$

note that the linear velocity using this form of a vector has two components, the first is the rate of change of the scalar function with direction the same as the original vector itself. The second component is the scalar function itself with the rate of change of the unit vector as its direction. We know that the unit vector doesn't change magnitude, but it may change direction giving a non-zero derivative. In the development of the derivative earlier, this was overlooked since the rate of change of the $\bar{i}$, $\bar{j}$, and $\bar{k}$ vectors that are fixed in a coordinate system do not change direction or magnitude.

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$$\bar{V}_{A/B}$$

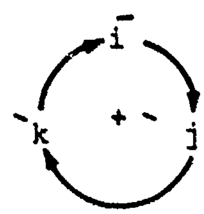
which means the velocity of point or reference A relative to reference B. To take a time derivative of a vector relative to reference system "A," the notation will be

$$\bar{i} \times \bar{i} = \bar{j} \times \bar{j} = \bar{k} \times \bar{k} = \bar{0} \text{ (note the zero vector has any direction)}$$

$$\bar{i} \times \bar{j} = \bar{k} \text{ and } \bar{j} \times \bar{k} = \bar{i} \text{ and } \bar{k} \times \bar{i} = \bar{j} \text{ and}$$

$$\bar{j} \times \bar{i} = -\bar{k} \text{ and } \bar{k} \times \bar{j} = -\bar{i} \text{ and } \bar{i} \times \bar{k} = -\bar{j}$$

These cross products are used often, and an easy way to remember them is to use the aid



where the cross product in the positive direction from $\bar{i}$ to $\bar{j}$ gives a positive $\bar{k}$, and to reverse the direction gives a negative answer.

2.4.5 Cross Product Laws

If $\bar{A}$, $\bar{B}$, and $\bar{C}$ are vectors and m is a scalar, then

1.	$\bar{A} \times \bar{B} = -\bar{B} \times \bar{A}$	Anti-Commutative Product
2.	$\bar{A} \times (\bar{B} + \bar{C}) = \bar{A} \times \bar{B} + \bar{A} \times \bar{C}$	Distributive Product
3.	$m(\bar{A} \times \bar{B}) = (m\bar{A}) \times \bar{B} = \bar{A} \times (m\bar{B})$	Associative Product

2.4.6 Vector Differentiation

The following treatment of vector differentiation has notation consistent with later courses and has been highly specialized for the USAF Test Pilot School curriculum. The scalar definition of the time derivative of a scalar function of the variable t is defined as,

$$\frac{d\,f(t)}{dt} = \lim_{\Delta t \to 0} \left[\frac{f(t + \Delta t) - f(t)}{\Delta t} \right]$$

Before proceeding, a vector function is defined as

$$\bar{F}(t) = f_x(t)\bar{i} + f_y(t)\bar{j} + f_z(t)\bar{k},$$

where f_x, f_y, and f_z are scalar functions of time and $\bar{i}$, $\bar{j}$, and $\bar{k}$ are unit vectors parallel to the x, y, and z axes, respectively. A vector function is a vector that changes magnitude and direction as a function of time and is referred to as a position vector. It gives the position of a particle in space at time t. The trace of the end points of the position vector gives the trajectory of the particle. The time derivative of a vector function with respect to some reference frame is defined as,

$$\frac{d\,\bar{F}(t)}{dt} = \lim_{\Delta t \to 0} \left[\frac{\bar{F}(t+\Delta t) - \bar{F}(t)}{\Delta t} \right] =$$

$$\frac{d\,f_x(t)}{dt}\bar{i} + \frac{d\,f_y(t)}{dt}\bar{j} + \frac{d\,f_z(t)}{dt}\bar{k} = \frac{df_x}{dt}\bar{i} + \frac{df_y}{dt}\bar{j} + \frac{df_z}{dt}\bar{k} =$$

$$\dot{f}_x\bar{i} + \dot{f}_y\bar{j} + \dot{f}_z\bar{k}$$

where the lack of a function variable indicates the function has the same variable as the differentiation variable, and the dot denotes time differentiation.

2.4.7 Vector Differentiation Laws

For vector functions $\bar{A}(t)$ and $\bar{B}(t)$, and scalar function $f(t)$

1. $\frac{d(\bar{A} + \bar{B})}{dt} = \frac{d\bar{A}}{dt} + \frac{d\bar{B}}{dt}$ Distributive Derivative
2. $\frac{d(\bar{A} \cdot \bar{B})}{dt} = \bar{A} \cdot \frac{d\bar{B}}{dt} + \frac{d\bar{A}}{dt} \cdot \bar{B}$ Dot Product Derivative
3. $\frac{d(\bar{A} \times \bar{B})}{dt} = \bar{A} \times \frac{d\bar{B}}{dt} + \frac{d\bar{A}}{dt} \times \bar{B}$ Cross Product Derivative
4. $\frac{d}{dt}(f(t)\,\bar{B}) = f(t)\frac{d\bar{B}}{dt} + \frac{d\,f(t)}{dt}\bar{B}$ Scalar, Vector Product Derivative

$$\frac{{}^{A}d\bar{F}}{dt}$$

There should be less confusion in multiple reference system problems concerning which reference frame the derivative is taken by using this notation.

By introducing the concept of multiple reference systems, it is appropriate to discuss the chain rule. For two reference systems, the chain rule is simply stated. For point A in reference system B, which in turn is in another reference system C, the velocity of A relative to C is equal to

$$\bar{V}_{A/C} = \bar{V}_{A/B} + \bar{V}_{B/C} \tag{2.1}$$

While calculating derivatives when given the time function of the trajectory is seemingly simple, at times the derivatives may be difficult. Also, if the function is not known, the measurements available to determine the trajectory may be in terms of translational or rotational parameters which don't always lend themselves directly to a time function. Another method of determining velocities and accelerations will be determined using pure translation and rotation. Simplification will consist of very specific problems with convenient alignment of reference systems at specific instances in time. So it will appear that the time element has disappeared in the following analysis since the vectors will be constants at the instant we observe them.

2.7 DIFFERENTIATION OF A VECTOR IN A RIGID BODY

The two basic motions, translation and rotation, will be applied to a rigid body which is assumed not to bend or twist (every point in the body remains an equidistance from all others). It will become important to determine not only the velocity and acceleration of a point in a rigid body, but also that of a vector which lies in a rigid body.

2.7.1 Translation

If a body moves so that all the particles have the same velocity relative to some reference at any instant of time, the body is said

to be in pure translation. A vector in pure translation changes neither its magnitude nor direction while translating, so its first derivative would be zero. An example would be a vector from the center of gravity to the wingtip of an airplane in straight and level, unaccelerated flight with respect to a reference system attached to the earth's surface. From the ground it changes neither magnitude nor direction, although every point on the aircraft is traveling at the same velocity. See Figure 2.11.

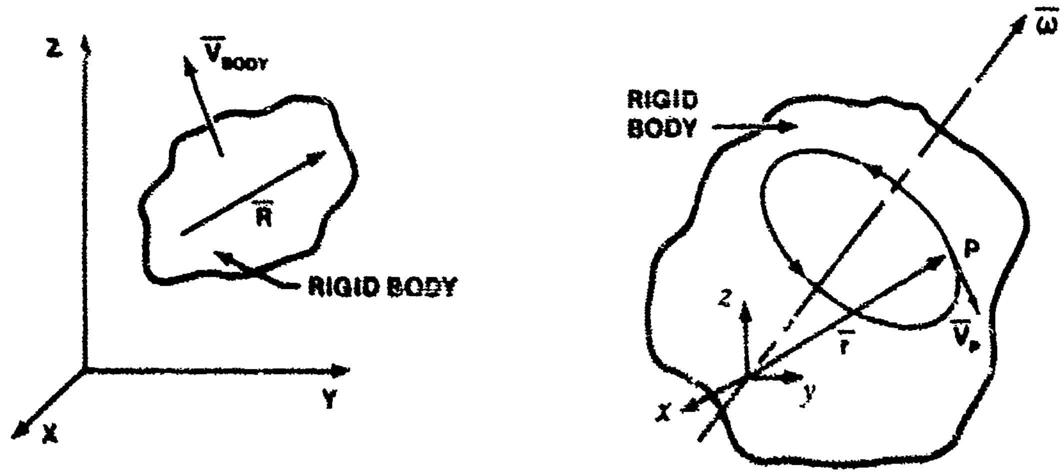


FIGURE 2.11. TRANSLATION AND ROTATION OF VECTORS IN RIGID BODIES

2.7.2 Rotation

If a body moves so that the particles along some line in the body have a zero velocity relative to some reference, the body is said to be in pure rotation relative to this reference. The line of stationary particles shown in Figure 2.11 is called the axis of rotation. A free vector that describes the rotation is called the angular velocity, $\bar{\omega}$, and has direction determined by the axis of rotation, using the right-hand rule to determine the sense. The chain rule as described for linear velocity applies to the angular velocity, as does a definition of its magnitude being angular speed. The first derivative of the angular velocity is the angular acceleration.

It can be proven that the linear velocity $\bar{V}$ of any point in a rigid body described by position vector $\bar{r}$ whose origin is along the axis of rotation can be written

$$\dot{\bar{r}} = \bar{V} = \bar{\omega} \times \bar{r} \qquad (2.2)$$

Note the conventions using the right-hand rule apply, and $\bar{V}$ is perpendicular to the plane of $\bar{r}$ and $\bar{\omega}$.

The pure rotation of one reference system with respect to another would require a transformation of unit vectors from one system to another, unless the reference systems were conveniently aligned at the instant in question. Such transformations are considered beyond the scope of this course.

Equation 2.2 can be generalized to include any vector in a rigid body with pure rotation. Refer to Figure 2.12.

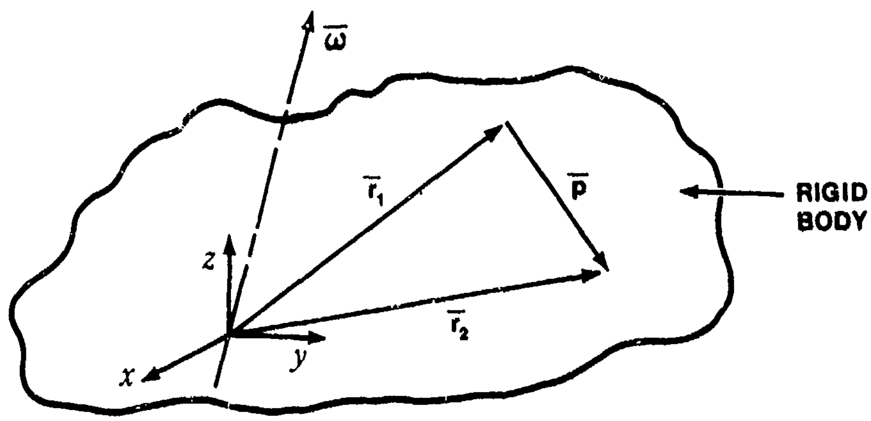


FIGURE 2.12. DIFFERENTIATION OF A FIXED VECTOR

Let $\bar{p}$ be a vector fixed anywhere in the rotating rigid body shown in Figure 2.12. The problem is to find the time rate of change of the vector. Two position vectors, $\bar{r}_1$ and $\bar{r}_2$, from the origin to the end points of the vector p are drawn. From vector addition

$$\bar{r}_1 + \bar{p} = \bar{r}_2$$

or solving

$$\bar{p} = \bar{r}_2 - \bar{r}_1$$

Differentiating

$$\dot{\bar{p}} = \dot{\bar{r}}_2 - \dot{\bar{r}}_1$$

From Equation 2.2,

$$\dot{\overline{r}}_2 = \overline{\omega} \times \overline{r}_2$$

and

$$\dot{\overline{r}}_1 = \overline{\omega} \times \overline{r}_1$$

so

$$\dot{\overline{p}} = \overline{\omega} \times \overline{r}_2 - \overline{\omega} \times \overline{r}_1 .$$

Since the cross product is distributive, this equation becomes

$$\dot{\overline{p}} = \overline{\omega} \times (\overline{r}_2 - \overline{r}_1) = \overline{\omega} \times \overline{p} . \qquad (2.3)$$

Therefore, the derivative of any fixed vector in a purely rotating rigid body is represented by the cross product of the angular velocity of the rotating body and the fixed vector.

2.7.3 Combination of Translation and Rotation in One Reference System

It is possible to combine the two types of velocity. An important point to notice here is that the velocities and accelerations are arrived at directly without the use of position vectors.

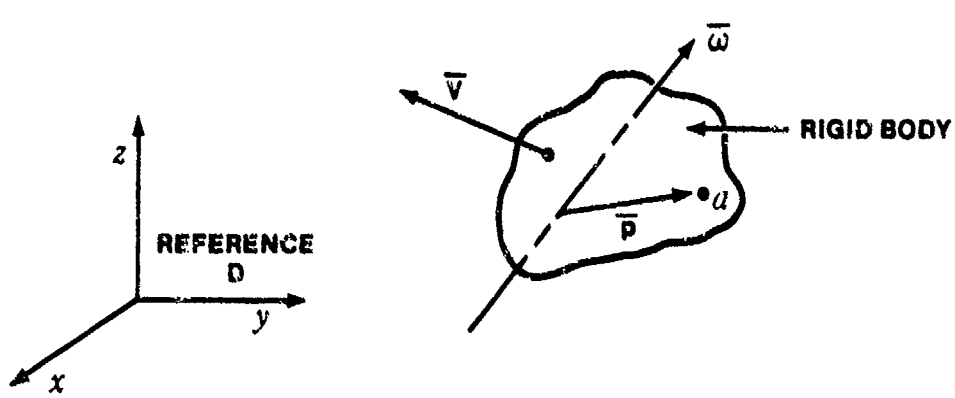


FIGURE 2.13. RIGID BODY IN TRANSLATION AND ROTATION

The velocity of point a in reference system D, Figure 2.13, will be calculated. The rigid body has a pure angular velocity, $\bar{\omega}$, and a pure translation, v, in reference frame D. The requested velocity is just the sum,

$$\bar{V} = \bar{V}_{rotation} + \bar{V}_{translation}$$

$\bar{V}_{rotation}$ is equal to $\bar{\omega} X \bar{p}$. $\bar{V}_{translation}$ is given as $\dot{\bar{v}}$, so

$$\bar{V} = \bar{\omega} X \bar{p} + \bar{v}.$$

When working in one reference system, the acceleration may be calculated by taking the derivative of the velocity.

$$\bar{A} = \frac{d\bar{V}}{dt} = \frac{d(\bar{\omega} X \bar{p})}{dt} + \frac{d\bar{v}}{dt} = \bar{\omega} X \dot{\bar{p}} + \dot{\bar{\omega}} X \bar{p} + \dot{\bar{v}}$$

Here, the $\dot{\bar{p}}$ is equal to $\bar{\omega} X \bar{p}$ as was shown in Equation 2.3 and $\dot{\bar{v}}$ is the translational acceleration $\bar{a}$. The angular acceleration $\dot{\bar{\omega}}$ will not receive any special notation in this text.

So, the acceleration in a single reference system can be written

$$\bar{A} = \bar{\omega} X (\bar{\omega} X \bar{p}) + \dot{\bar{\omega}} X \bar{p} + \bar{a}.$$

2.7.4 Vector Derivatives in Different Reference Systems

The more general problem of relative motion between a point and a reference system that is itself moving relative to another reference system will be approached. More than one reference system is often used in order to simplify the analysis of general problems. As a first step, it is necessary to examine the procedure of differentiation with respect to time in the presence of two references moving relative to each other.

A reference system is a non-deformable system and may be considered a rigid body. So, the work done so far applies here. Figure 2.14 gives the vectors used in the following analysis.

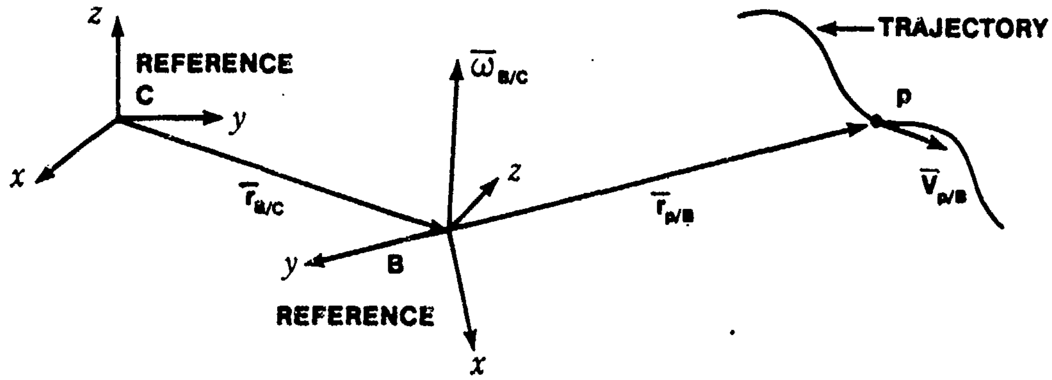


FIGURE 2.14. MOTION WITH TWO REFERENCE SYSTEMS

The problem above shows point p with position vector $\bar{r}_{p/B}$, moving with respect to the reference B, and the origin of B with position vector $\bar{r}_{B/C}$, moving with respect to reference system C. The reference system B also has an angular velocity with respect to C of $\bar{\omega}_{B/C}$. The goal of the following development will be to find the time rate of change of the position vector in the B frame as seen from the C frame or notationally

$$\frac{{}^{C}d}{dt}\,\bar{r}_{p/B}$$

It is very important to realize that this is not the same as the velocity of the point as seen from the C frame. Rather it is the rate of change of a position vector in one frame as seen from another frame. So the derivative sought is not $\bar{V}_{p/C}$. This velocity would be obtained by using the chain rule as given in Equation 2.1.

A representative example is the motion of a point on an aircraft with a body axis system at the center of gravity and the aircraft moving along some path relative to the ground. The second reference system is attached to the ground. It will be assumed in this analysis that the two reference systems have the same unit vectors. Careful attention will be given to circumstances resulting from axes that may not be conveniently aligned during the analysis.

Beginning with the position vector in the frame B,

$$\bar{r}_{p/B} = x\bar{i} + y\bar{j} + z\bar{k}$$

Differentiating this vector with respect to time relative to the C reference frame presents a problem since the unit vectors of the B system are rotating as seen in the C system.

So, the derivative must be done in two parts using the fourth law given earlier.

$$\frac{{}^{C}d}{dt}\bar{r}_{p/B} = \dot{x}\bar{i} + \dot{y}\bar{j} + \dot{z}\bar{k} + x\dot{\bar{i}} + y\dot{\bar{j}} + z\dot{\bar{k}}$$

But the unit vector's derivatives can be written as vector in a single reference system with derivatives as seen in Equation 2.3. Thus,

$$\dot{\bar{i}} = \bar{\omega}_{B/C} \text{ X } \bar{i}, \text{ etc.}$$

So

$$\begin{aligned}\frac{{}^{C}d}{dt}r_{p/B} &= \dot{x}\bar{i} + \dot{y}\bar{j} + \dot{z}\bar{k} + x\dot{\bar{i}} + y\dot{\bar{j}} + z\dot{\bar{k}} \\ &= \dot{x}\bar{i} + \dot{y}\bar{j} + \dot{z}\bar{k} + x(\bar{\omega}_{B/C} \text{ X } \bar{i}) + y(\bar{\omega}_{B/C} \text{ X } \bar{j}) + z(\bar{\omega}_{B/C} \text{ X } \bar{k}) \\ &= \dot{x}\bar{i} + \dot{y}\bar{j} + \dot{z}\bar{k} + \bar{\omega}_{B/C} \text{ X } (x\bar{i}) + \bar{\omega}_{B/C} \text{ X } (y\bar{j}) + \bar{\omega}_{B/C} \text{ X } (z\bar{k}) \\ &= \dot{x}\bar{i} + \dot{y}\bar{j} + \dot{z}\bar{k} + \bar{\omega}_{B/C} \text{ X } (x\bar{i} + y\bar{j} + z\bar{k})\end{aligned}$$

The first three terms are recognized as the velocity of p in the B system and the next term is the cross product of the angular velocity of the B system with respect to the C system and the position vector in the B system.

$$\frac{{}^{C}d}{dt}\bar{r}_{p/B} = \dot{\bar{r}}_{p/B} + \bar{\omega}_{B/C} \text{ X } \bar{r}_{p/B} = \bar{V}_{p/B} + \bar{\omega}_{B/C} \text{ X } \bar{r}_{p/B} \qquad (2.4)$$

This equation may be generalized to any vector in one reference system relative to another. This is a very important relationship and will be used in Chapter 4, in the derivation of the aircraft equations of motion.

The acceleration of a particle at point p would be handled using the definition of acceleration.

$$\bar{A}_{p/C} = \frac{{}^{C}d}{dt}\bar{V}_{p/C} \tag{2.5}$$

Note Equation 2.5 does not address $\frac{{}^{C}d}{dt}\bar{V}_{p/B}$

Hopefully, the velocity would be written in a simple form allowing simple differentiation to obtain the acceleration. If not, a simple exchange of notation with Equation 2.4 would be necessary.

COMMENT The material presented thus far is sufficient to enable solution of any linear or angular velocity or acceleration in a kinematics problem. However, another analysis follows which may clarify multi-reference problems and will provide definition of some terms that will be of value in later courses.

2.7.4.1 Transport Velocity. In this analysis of motion relative to two reference systems, a different approach is taken to the problem. Figure 2.14 is expanded as shown in Figure 2.15 to include the position vector directly from reference C to the point p.

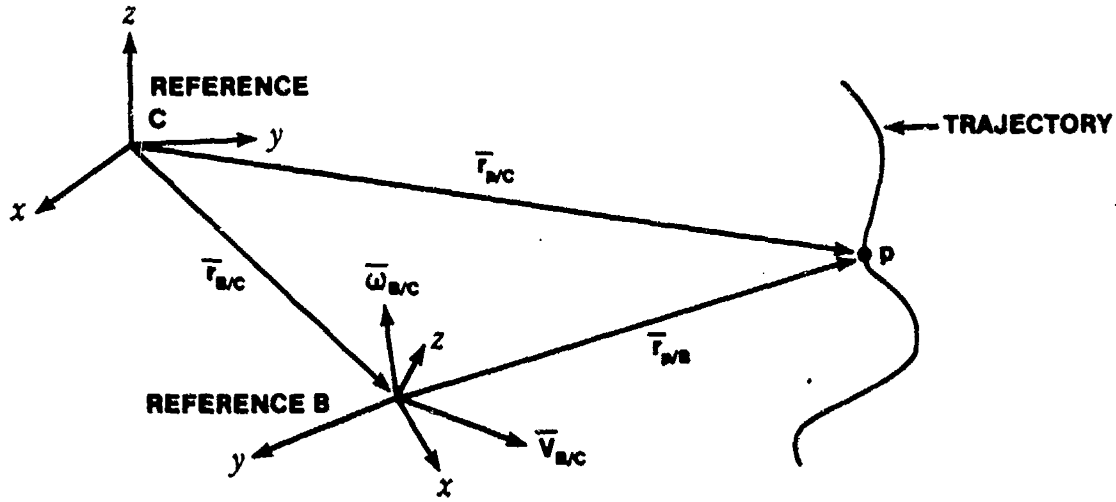


FIGURE 2.15. TWO REFERENCE SYSTEM VECTORS

Thus

$$\bar{r}_{p/C} = \bar{r}_{p/B} + \bar{r}_{B/C}$$

and

$$\bar{V}_{p/C} = \dot{\bar{r}}_{p/C} = \frac{{}^{C}d}{dt}\bar{r}_{p/C} = \frac{{}^{C}d}{dt}\bar{r}_{p/B} + \frac{{}^{C}d}{dt}\bar{r}_{B/C}$$

where the first term is Equation 2.4 and the second is $\bar{V}_{B/C}$. Substituting these terms,

$$\bar{V}_{p/C} = \dot{\bar{r}}_{p/C} = \bar{V}_{p/B} + \bar{\omega}_{B/C} \times \bar{r}_{p/B} + \bar{V}_{B/C} \qquad (2.6)$$

or

$$\bar{V}_{p/C} = \bar{V}_{p/B} + \bar{V}_{pTC}$$

where

$$\bar{V}_{pTC} = \bar{\omega}_{B/C} \times \bar{r}_{p/B} + \bar{V}_{B/C}$$

This term is called the <u>transport velocity</u>. The interpretation of transport velocity defined in this equation is such that $\bar{V}_{p/B}$ is still the velocity of p relative to B and $\bar{V}_{pTC}$ is the velocity in C that p would have,

if p were fixed in B. Note this is just the sum of the translation and rotation of frame B relative to frame C if the point p is considered fixed.

2.7.4.2 Special Acceleration. By taking the derivative of the velocity, as in Equation 2.5, and applying the distributive law to the cross product,

$$\bar{A}_{p/C} = \frac{{}^C d}{dt}\bar{V}_{p/C} = \frac{{}^C d}{dt}\bar{V}_{p/B} + \frac{{}^C d}{dt}\bar{\omega}_{B/C} \times \bar{r}_{p/B} + \bar{\omega}_{B/C} \times \frac{{}^C d}{dt}\bar{r}_{p/B} + \frac{{}^C d}{dt}\bar{V}_{B/C}$$

Now, substituting with the notation for acceleration where possible,

$$\bar{A}_{p/C} = \frac{{}^C d}{dt}\bar{V}_{p/B} + \dot{\bar{\omega}}_{B/C} \times \bar{r}_{p/B} + \bar{\omega}_{B/C} \times \frac{{}^C d}{dt}\bar{r}_{p/B} + \bar{A}_{B/C}$$

The two remaining terms with derivative notation should be recognized as applications of Equation 2.4. So, substituting

$$\bar{A}_{p/C} = (\dot{\bar{V}}_{p/B} + \bar{\omega}_{B/C} \times \bar{V}_{p/B}) + \dot{\bar{\omega}}_{B/C} \times \bar{r}_{p/B} + \bar{\omega}_{B/C} \times (\bar{V}_{p/B} + \bar{\omega}_{B/C} \times \bar{r}_{p/B}) + \bar{A}_{B/C}$$

Expanding and noting

$$\dot{\bar{V}}_{p/B} = \bar{A}_{p/B}$$

$$\bar{A}_{p/C} = \bar{A}_{p/B} + \bar{\omega}_{B/C} \times \bar{V}_{p/B} + \dot{\bar{\omega}}_{B/C} \times \bar{r}_{p/B} + \bar{\omega}_{B/C} \times \bar{V}_{p/B} + \bar{\omega}_{B/C} \times (\bar{\omega}_{B/C} \times \bar{r}_{p/B}) + \bar{A}_{B/C}$$

Rearranging and combining the two like terms

$$\bar{A}_{p/C} = \bar{A}_{p/B} + \bar{A}_{B/C} + \dot{\bar{\omega}}_{B/C} \times \bar{r}_{p/B} + 2\bar{\omega}_{B/C} \times \bar{V}_{p/B} + \bar{\omega}_{B/C} \times (\bar{\omega}_{B/C} \times \bar{r}_{p/B}) \quad (2.7)$$

Of the five terms remaining in the acceleration equation, the last two have descriptive names.

$2\bar{\omega}_{B/C} \times \bar{V}_{p/B}$ is called the <u>Coriolis acceleration</u>, and

$\bar{\omega}_{B/C} \times \left(\bar{\omega}_{B/C} \times \bar{r}_{p/B}\right)$ is called the <u>Centripetal acceleration</u>.

The terms in Equation 2.7 that are independent of the motion of p relative to frame B are called the <u>transport acceleration</u>. These terms

provide the acceleration in frame C of a point that is fixed at p at the instant in question. Notationally, the transport acceleration is

$$\bar{A}_{pTC} = \bar{A}_{B/C} + \dot{\bar{\omega}}_{B/C} \times \bar{r}_{p/B} + \bar{\omega}_{B/C} \times (\bar{\omega}_{B/C} \times \bar{r}_{p/B})$$

These concepts are difficult to realize until a few problems are attempted.

2.7.4.3 Example Two Reference System Problem. The angular velocity of the

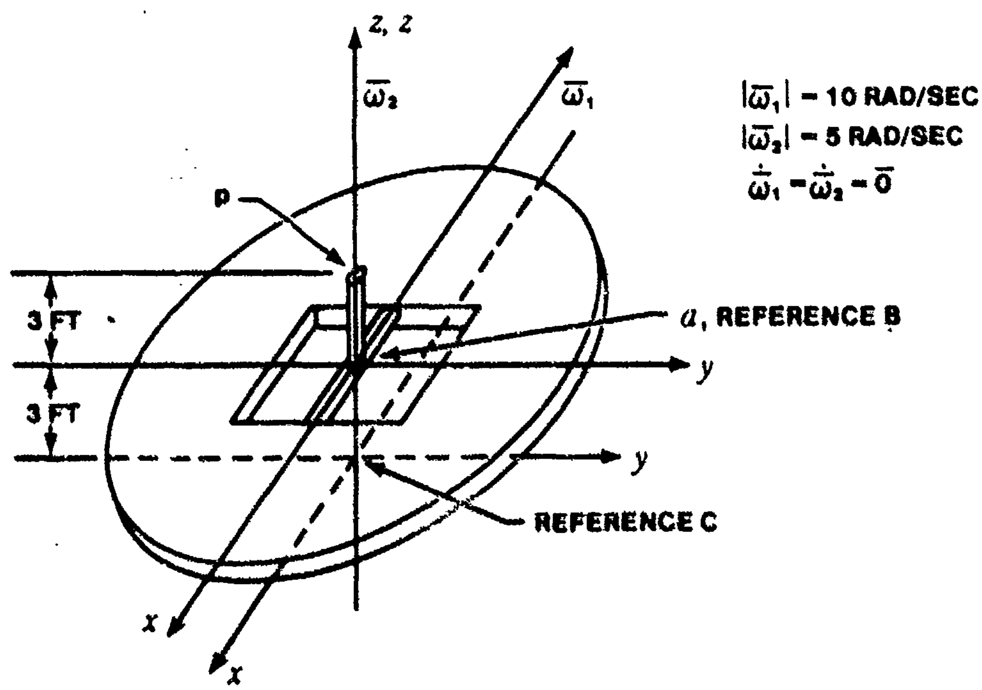


FIGURE 2.16. TWO REFERENCE SYSTEM PROBLEM

arm ap relative to the disk in Figure 2.16 is 10 rad/sec, shown vectorally in the diagram as ω_1, while the angular velocity of the disk relative to the ground is 5 rad/sec, shown vectorally as ω_2. The angular accelerations are zero. Reference B is attached to the platform, while frame C is fixed to the ground, three feet below the disk. At the instant in question, the arm ap is in the vertical position, and the reference axes directions coincide, although displaced.

Find the velocity and acceleration of point p relative to the fixed reference frame C.

Using Equation 2.6

$$\bar{V}_{p/C} = \dot{\bar{r}}_{p/C} = \bar{V}_{p/B} + \bar{\omega}_{B/C} \times \bar{r}_{p/B} + \bar{V}_{B/C} \tag{2.6}$$

we know the last term, $\bar{V}_{B/C} = \bar{0}$, since the B frame is only rotating relative to C.

$$\bar{\omega}_{B/C} = \bar{\omega}_2 = 5\bar{k} \text{ rad/sec and } \bar{r}_{p/B} = 3\bar{k} \text{ feet, by observation}$$

This leaves $\bar{V}_{p/B}$ which involves angular velocity $\bar{\omega}_1 = -10\bar{i}$, relative to B.

$$\bar{V}_{p/B} = \bar{\omega}_1 \times \bar{r}_{p/B} = (-10\bar{i} \times 3\bar{k}) = (-30)(-\bar{j}) = 30\bar{j} \text{ ft/sec}$$

Substituting all the parts into Equation 2.6

$$\bar{V}_{p/C} = 30\bar{j} + 5\bar{k} \times 3\bar{k} + \bar{0} = 30\bar{j} + 15(\bar{k} \times \bar{k}) = \underline{30\bar{j} \text{ ft/sec}}$$

For the acceleration, the general expression is Equation 2.7

$$\bar{A}_{p/C} = \bar{A}_{p/B} + \bar{A}_{B/C} + \dot{\bar{\omega}}_{B/C} \times \bar{r}_{p/B} + 2\bar{\omega}_{B/C} \times \bar{V}_{p/B} + \bar{\omega}_{B/C} \times (\bar{\omega}_{B/C} \times \bar{r}_{p/B})$$

The only unknown terms are $\bar{A}_{B/C} = \bar{0}$ and $\bar{A}_{p/B}$. The latter is a centripetal acceleration due to the rotation of the arm. The centripetal acceleration may be arrived at in several different ways,

$$\bar{A}_{p/B} = \frac{^B d}{dt}V_{p/B} = \frac{d}{dt}(\bar{\omega}_1 \times \bar{r}_{p/B}) = \dot{\bar{\omega}}_1 \times \bar{r}_{p/B} + \bar{\omega}_1 \times \dot{\bar{r}}_{p/B}$$

$$= \bar{0} + \bar{\omega}_1 \times (\bar{\omega}_1 \times \bar{r}_{p/B}) = (-10\bar{i}) \times (30\bar{j})$$

$$= \underline{-300\bar{k} \text{ ft/sec}^2}$$

Substituting this value and the others already calculated

$$\bar{A}_{p/C} = -300\bar{k} + \bar{0} + \bar{0} \times 3\bar{k} + 2(5\bar{k} \times 30\bar{j}) + 5\bar{k} \times (5\bar{k} \times 3\bar{k}) = \underline{-300\bar{i} - 300\bar{k}}$$

While working problems where there is a choice of axes, be careful to choose so that as many parameters as possible are equal to zero, and most importantly so that the axes are aligned at the instant in question. Also, whether a reference system is fixed in a body or not will have profound effects on the velocities as seen from that origin. Try to place yourself at the origin of a system and visualize the velocity and acceleration seen to help avoid confusion. Also check your answers to see if they are logical, both in magnitude and direction. The right-hand rule is essential.

When working with large systems, with many variables it becomes necessary to develop a shorthand method of writing systems of equations. The development of matrix algebra is the solution.

2.8 MATRICES

An m x n matrix is a rectangular array of quantities arranged in m rows and n columns. When there is no possibility of confusion, matrices are often represented by single capital letters. More commonly, however, they are represented by displaying the quantities between brackets; thus,

$$A = [A]_{mxn} = ||a_{ij}||_{mxn} = [a_{ij}]_{mxn} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Note that a_{ij} refers to the element in the ith row and jth column of [A]. Thus, a_{23} is the element in the second row and third column. Matrices having only one column (or one row) are called column (or row) vectors. The matrix [X] below is a column vector, and the matrix [Y] is a row vector.

$$[X] = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \qquad [Y] = [y_1 \; y_2 \; \cdots \; y_n]$$

A matrix, unlike the determinant, is not assigned any "value"; it is simply an array of quantities. Matrices may be considered as single algebraic entities and combined (added, subtracted, multiplied) in a manner similar to the combination of ordinary numbers. It is necessary, however, to observe specialized algebraic rules for combining matrices. These rules are somewhat more complicated than for "ordinary" algebra. The effort required to learn the rules of matrix algebra is well justified, however, by the simplification and organization which matrices bring to problems in linear algebra.

2.8.1 Matrix Equality

Two matrices $[A] = [a_{ij}]$ and $[B] = [b_{ij}]$ are equal if and only if they are identical; i.e., if and only if they contain the same number of rows and the same number of columns, and $a_{ij} = b_{ij}$ for all values of i and j. Thus, the statement

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} 2 & 4 & 3 \\ 1 & 0 & 19 \end{bmatrix}$$

is equivalent to the statements

$$\begin{aligned} a_{11} &= 2 \\ a_{12} &= 4 \\ &\vdots \\ &\text{etc.} \end{aligned}$$

2.8.2 Matrix Addition

Two matrices having the same number of rows and the same number of columns are defined as being conformable for addition and may be added by adding corresponding elements; i.e.,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & & \text{etc.} \\ \vdots & & \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots \\ b_{21} & & \text{etc.} \\ \vdots & & \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots \\ a_{21} + b_{21} & & \text{etc.} \\ \vdots & & \end{bmatrix}$$

Thus

$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \\ 0 & -5 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 1 & 4 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 4 & 6 \\ -3 & 0 \end{bmatrix}$$

2.8.3 Matrix Multiplication by a Scalar

A scalar is a single number (it may be thought of as 1 x 1 matrix). A matrix of any shape may be multiplied by a scalar by multiplying each element of the matrix by the scalar. That is:

$$k[A] = k \begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & & \\ \vdots & & \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} & \cdots \\ ka_{21} & & \\ \vdots & & \end{bmatrix}$$

For example,

$$3 \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ 9 & 0 \end{bmatrix}$$

2.8.4 Matrix Multiplication

Matrix multiplication can be defined for any two matrices when the number of columns of the first is equal to the number of rows of the second matrix. This can be stated mathematically as:

$$[A]_{i\times m}\ [B]_{m\times j} = [C]_{i\times j}$$

where

$$c_{ij} = \sum_{k=1}^{m} a_{ik}\ b_{kj}$$

Multiplication is not defined for other matrices.

Equation 2.8 demonstrates the product of two, 2 x 2 matrices.

$$[A]_{2\times 2}\ [B]_{2\times 2} = [C]_{2\times 2}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

or using the definition of multiplication,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}\ b_{11} + a_{12}\ b_{21} & a_{11}\ b_{12} + a_{12}\ b_{22} \\ a_{21}\ b_{11} + a_{22}\ b_{21} & a_{21}\ b_{12} + a_{22}\ b_{22} \end{bmatrix} \quad (2.8)$$

This situation is sufficiently general to point the way to an orderly multiplication process for matrices of any order.

In the indicated product,

$$[A]\ [B] = [C]$$

the left-hand factor may be treated as a bundle of row-vectors,

$$[A] = \begin{bmatrix} [a_{11} & a_{12}] \\ [a_{21} & a_{22}] \end{bmatrix}$$

and the right-hand factor as a bundle of column vectors,

$$[B] = \left[\begin{bmatrix} b_{11} \\ \\ b_{21} \end{bmatrix} \quad \begin{bmatrix} b_{12} \\ \\ b_{22} \end{bmatrix} \right]$$

Then,

$$\begin{bmatrix} [a_{11} & a_{12}] \\ \\ [a_{21} & a_{22}] \end{bmatrix} \times \left[\begin{bmatrix} b_{11} \\ \\ b_{21} \end{bmatrix} \quad \begin{bmatrix} b_{12} \\ \\ b_{22} \end{bmatrix} \right] = \begin{bmatrix} c_{11} & c_{12} \\ \\ c_{21} & c_{22} \end{bmatrix} \qquad (2.9)$$

and by comparison with Equation 2.8

$$\begin{matrix} [a_{11} & a_{12}] \\ [a_{21} & a_{22}] \end{matrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} [a_{11} \quad a_{12}] \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} & [a_{11} \quad a_{12}] \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} \\ [a_{21} \quad a_{22}] \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} & [a_{21} \quad a_{22}] \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} \end{bmatrix} \qquad (2.10)$$

where, by definition,

$$[a_{11} \quad a_{12}] \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} = \left[a_{11}\ b_{11} + a_{12}\ b_{21}\right]$$

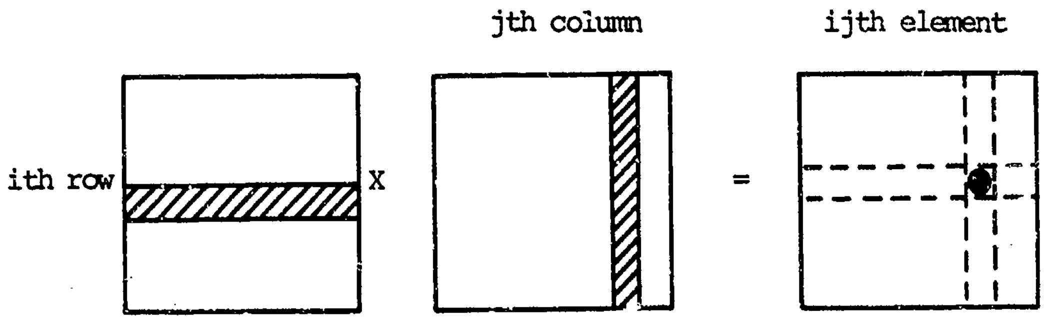
$$[a_{11} \quad a_{12}] \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} = \left[a_{11}\ b_{12} + a_{12}\ b_{22}\right]$$

etc.

A comparison of Equations 2.9 and 2.10 shows that if the rows of [A] and the columns of [B] are treated as vectors, then C_{ij} in the product [C] = [A] [B] is the dot product of the ith row of [A] and the jth column of [B]. This rule holds for matrices of any size, i.e.,

$$c_{ij} = [a_{i1}\ a_{i2} \cdots a_{in}] \begin{bmatrix} b_{ij} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = \left[a_{i1}\, b_{1j} + a_{i2}\, b_{j2} + \ldots + a_{in}\, b_{nj}\right]$$

Matrix multiplication is therefore a "row-on-column" process:



$$\begin{bmatrix} 3 & 2 \\ -1 & 4 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} [3 \quad 2] \begin{bmatrix} 1 \\ -1 \end{bmatrix} & [3 \quad 2] \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\ [-1 \quad 4] \begin{bmatrix} 1 \\ -1 \end{bmatrix} & [-1 \quad 4] \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\ [0 \quad 2] \begin{bmatrix} 1 \\ -1 \end{bmatrix} & [0 \quad 2] \begin{bmatrix} 2 \\ 0 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 6 \\ -5 & -2 \\ -2 & 0 \end{bmatrix}$$

The indicated product [A] [B] can be carried out only if [A] and [B] are conformable; that is, for conformability in multiplication, the number of columns in [A] must equal the number of rows in [B]. For example, the expression

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

is meaningless (as an attempt to carry out the multiplication will show) because the number of columns in [A] is two and the number of rows in [B] is three. A convenient rule is this: if [A] is an m x n matrix (m rows, n columns) and [B] is an n x p matrix, then [C] = [A] [B] is an m x p matrix. That is,

$$[A]_{mxn} \; [B]_{nxp} \; = \; [C]_{mxp}$$

Matrix algebra differs significantly from "ordinary" algebra in that multiplication is not commutative. In general, that is,

$$[A] \; [B] \; = \; [B] \; [A]$$

For example, if

$$[A] = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 1 & -3 \\ 2 & 0 \end{bmatrix}$$

then

$$[A] \; [B] = \begin{bmatrix} 4 & -6 \\ 4 & 0 \end{bmatrix}$$

$$[B] \; [A] = \begin{bmatrix} 2 & -5 \\ 4 & 2 \end{bmatrix}$$

Because multiplication is non-commutative, care must be taken in describing the product

$$[C] = [A]\,[B]$$

to say that [A] "premultiplies" [B], or, equivalently, that [B] "post-multiplies" [A].

2.8.5 The Identity Matrix

The identity (or unit) matrix [I] occupies the same position in matrix algebra that the number one does in ordinary algebra. That is, for any matrix [A],

$$[I]\,[A] = [A]\,[I] = [A]$$

The identity [I] is a square matrix consisting of ones on the principal ($\longleftrightarrow$) diagonal and zeros everywhere else; i.e.,

$$[I] = \begin{bmatrix} 1 & 0 & 0 \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ 0 & 0 & 1 \cdots & 0 \\ \vdots & & & \\ 0 & 0 & 0 \cdots & 1 \end{bmatrix}$$

The order (the number of rows and columns) of an identity matrix depends entirely on the requirement of conformability with adjacent matrices. For example, if

$$[A] = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$$

then

$$[I]\,[A] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$$

$$[A]\,[I] = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$$

Thus, the "left" identity for [A] is 2 x 2 and the "right" identity for [A] is 3 x 3; however, they both leave [A] unaltered.

2.8.6 The Transposed Matrix

The transpose of [A], labeled $[A]^T$, is formed by interchanging the rows and columns of [A]. That is,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

The jth row vector becomes the jth column vector, and vice versa. For example,

$$\begin{bmatrix} 3 & 1 \\ 2 & 4 \\ 0 & 2 \end{bmatrix}^T = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5+j4 \\ -10 & 3 \end{bmatrix}^T = \begin{bmatrix} 2 & -10 \\ 5+j4 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

2.8.7 The Inverse Matrix

Matrix multiplication has been defined; it is natural to inquire next if there is some way to divide matrices. There is not, properly speaking, a division operation in matrix algebra; however, an equivalent result is obtained through the use of the inverse matrix.

In ordinary algebra, every number a (except zero) has a multiplicative inverse, a^{-1} defined as follows: A quantity a^{-1} is the inverse of a if

$$a \cdot a^{-1} = a^{-1} \cdot a = 1$$

In the same way, the matrix $[A]^{-1}$ is called the inverse matrix of [A] if

$$[A]\,[A]^{-1} = [A]^{-1}\,[A] = [I]$$

The symbol 1/a is normally used to signify a^{-1}. Since ordinary multiplication is commutative,

$$(1/a) \cdot (b) = (b) \cdot (1/a) = b \div a$$

for any number b. The use of the division symbol (÷) in this instance is useful and unambiguous. In matrix algebra, however, multiplication is not commutative. Therefore,

$$[A]^{-1}\,[B] \;=\; [B]\,[A]^{-1}$$

and the expression

$$[B] \div [A]$$

cannot be used since it may have either of the (unequal) meanings in the previous equation. Instead of saying "divide [B] by [A]," one must say either "premultiply [B] by $[A]^{-1}$" or "postmultiply [B] by $[A]^{-1}$." The results, in general, are different.

2.8.8 Singular Matrices

Matrices which cannot be inverted are called singular. For inversion to be possible, a matrix must possess a determinant not equal to zero. For example, the matrix

$$\begin{bmatrix} 2 & 1 \\ 3 & 0 \\ 4 & 5 \end{bmatrix}$$

is singular because it is not square, and a determinant cannot be computed. The matrix

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

is singular because its determinant vanishes.

Matrices which do possess an inverse are called nonsingular.

2.9 SOLUTION OF LINEAR SYSTEMS

Consider the set of equations

$$\left.\begin{array}{ccccccccc} a_{11}\,x_1 & + & a_{12}\,x_2 & + & \cdots & + & a_{1n}\,x_n & = & y_1 \\ a_{21}\,x_1 & + & a_{22}\,x_2 & + & \cdots & + & a_{2n}\,x_n & = & y_2 \\ \vdots & & \vdots & & \vdots\;\vdots\;\vdots & & \vdots & & \vdots \\ a_{n1}\,x_1 & + & a_{n2}\,x_2 & + & \cdots & + & a_{nn}\,x_n & = & y_n \end{array}\right\} \qquad (2.11)$$

That is,

$$[A]\ [X] = [Y]$$

Assuming that the inverse of [A] has been computed, both sides of this equation may be premultiplied by $[A]^{-1}$, giving

$$[A]^{-1}\ [A]\ [X] = [A]^{-1}\ [Y]$$

From the definition of the inverse matrix,

$$[I]\ [X] = [A]^{-1}\ [Y]$$

from which, finally,

$$[X] = [A]^{-1}\ [Y]$$

Thus, the system of Equation 2.11 may be solved for $x_1, x_2, \ldots x_n$ by computing the inverse of [A].

2.9.1 Computing the Inverse

There is a straightforward four step method for computing the inverse of a given matrix [A]:

Step 1. Compute the determinant of [A]. This determinant is written as $|A|$. If the determinant is zero or does not exist, the matrix [A] is defined as singular and an inverse cannot be found.

Step 2. Transpose matrix [A]. The resulting matrix is written $[A]^T$.

Step 3. Replace each element a_{ij} of the transposed matrix by its cofactor A_{ij}. This resulting matrix is defined as the adjoint of matrix [A] and is written: Adj [A].

Step 4. Divide the adjoint matrix by the scalar value of the determinant of [A] which was computed in Step 1. The resulting matrix is the inverse and is written: $[A]^{-1}$.

This procedure can be summarized as follows: To calculate the inverse of [A] calculate the Adjoint of [A] and divide by the determinant of [A] or

$$[A]^{-1} = \frac{\text{Adj } [A]}{[A]}$$

Example: Find $[A]^{-1}$, if

$$[A] = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 5 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Step 1. Compute the determinant of [A]. Expanding about the first row

$$|A| = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 5 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$|A| = 3(-5 -2) -2(-1 + 0) + 0(2 - 0)$$

$$|A| = -21 + 2 + 0 = -19$$

The determinant has the value -19; therefore an inverse can be computed.

Step 2. Transpose [A]

$$[A]^T = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 5 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

Step 3: Replace each element a_{ij} of $[A]^T$ by its cofactor A_{ij} to determine the adjoint matrix. Note that signs alternate from a positive A_{11}

$$
\text{Adj } [A] = \begin{bmatrix}
\begin{vmatrix} 5 & 2 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 5 \\ 0 & 1 \end{vmatrix} \\[2ex]
-\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 3 & 0 \\ 0 & -1 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} \\[2ex]
\begin{vmatrix} 1 & 0 \\ 5 & 2 \end{vmatrix} & -\begin{vmatrix} 3 & 0 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 2 & 5 \end{vmatrix}
\end{bmatrix} = \begin{bmatrix} -7 & 2 & 2 \\ 1 & -3 & -3 \\ 2 & -6 & 13 \end{bmatrix}
$$

Step 4: Divide by the scalar value of the determinant of $[A]$ which was computed as -19 in Step 1.

$$
[A]^{-1} = \frac{1}{-19} \begin{bmatrix} -7 & 2 & 2 \\ 1 & -3 & -3 \\ 2 & -6 & 13 \end{bmatrix}
$$

2.9.2 Product Check

From the definition of the inverse matrix

$$
[A]^{-1} [A] = [I]
$$

This fact may be used to check a computed inverse. In the case just completed

$$[A]^{-1}[A] = \frac{1}{-19}\begin{bmatrix}-7 & 2 & -2\\ 1 & -3 & -3\\ 2 & -6 & 13\end{bmatrix}\begin{bmatrix}3 & 2 & 0\\ 1 & 5 & 1\\ 0 & 2 & -1\end{bmatrix}$$

$$[A]^{-1}[A] = \frac{1}{-19}\begin{bmatrix}-19 & 0 & 0\\ 0 & -19 & 0\\ 0 & 0 & -19\end{bmatrix}$$

$$[A]^{-1}[A] = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$[A]^{-1}[A] = [I]$$

Since the product does come out to be the identity matrix, the computation was correct.

2.9.3 Example Linear System Solution

Given the following set of simultaneous equations, solve for x_1, x_2, and x_3.

$$\left.\begin{aligned}3x_1 + 2x_2 - 2x_3 &= y_1\\ -x_1 + x_2 + 4x_3 &= y_2\\ 2x_1 - 3x_2 + 4x_3 &= y_3\end{aligned}\right\} \qquad (2.12)$$

This set of equations can be written as

$$[A]\,[X] = [Y]$$

or

$$[X] = [A]^{-1}\,[Y]$$

Thus, the system of Equations 2.12 can be solved for the values of x_1, x_2, and x_3 by computing the inverse of [A].

$$[A]\ [X] = [Y]$$

$$\begin{bmatrix} 3 & 2 & -2 \\ -1 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Step 1: Compute the determinant of [A]. Expanding about the first row

$$|A| = 3\ (4 + 12)\ -2\ (-4\ -8)\ -2\ (3 - 2)$$

$$|A| = 48 + 24 - 2 = 70$$

Step 2: Transpose [A]

$$[A]^T = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & -3 \\ -2 & 4 & 4 \end{bmatrix}$$

Step 3: Determine the adjoint matrix by replacing each element in $[A]^T$ by its cofactor

$$\text{Adj }[A] = \begin{bmatrix} \begin{vmatrix} 1 & -3 \\ 4 & 4 \end{vmatrix} & -\begin{vmatrix} 2 & -3 \\ -2 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ -2 & 4 \end{vmatrix} \\ -\begin{vmatrix} -1 & 2 \\ 4 & 4 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ -2 & 4 \end{vmatrix} & -\begin{vmatrix} 3 & -1 \\ -2 & 4 \end{vmatrix} \\ \begin{vmatrix} -1 & 2 \\ 1 & -3 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 2 & -3 \end{vmatrix} & \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} \end{bmatrix}$$

$$\text{Adj }[A] = \begin{bmatrix} 16 & -2 & 10 \\ 12 & 16 & -10 \\ 1 & 13 & 5 \end{bmatrix}$$

Step 4: Divide by the scalar value of the determinant of [A] which was computed as 70 in Step 1.

$$[A]^{-1} = \frac{1}{70}\begin{bmatrix} 16 & -2 & 10 \\ 12 & 16 & -10 \\ 1 & 13 & 5 \end{bmatrix}$$

Product Check

$$[A]^{-1}\,[A] = [I]$$

$$[A]^{-1}\,[A] = \frac{1}{70}\begin{bmatrix} 16 & -2 & 10 \\ 12 & 16 & -10 \\ 1 & 13 & 5 \end{bmatrix}\begin{bmatrix} 3 & 2 & -2 \\ -1 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$$

$$[A]^{-1}\,[A] = \frac{1}{70}\begin{bmatrix} 70 & 0 & 0 \\ 0 & 70 & 0 \\ 0 & 0 & 70 \end{bmatrix} \tag{2.13}$$

$$[A]^{-1}\,[A] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the product in Equation 2.13 is the identity matrix, the computation is correct. The values of x_1, x_2, and x_3 can now be found for any y_1, y_2, and y_3 by premultiplying [Y] by $[A]^{-1}$.

$$[X] = [A]^{-1}\,[Y]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{70}\begin{bmatrix} 16 & -2 & 10 \\ 12 & 16 & -10 \\ 1 & 13 & 5 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

For example, if $y_1 = 1$, $y_2 = 13$, and $y_3 = 8$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{70} \begin{bmatrix} 16 & -2 & 10 \\ 12 & 16 & -10 \\ 1 & 13 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 13 \\ 8 \end{bmatrix}$$

$$x_1 = \frac{1}{70}(16 - 26 + 80) = \frac{70}{70} = 1$$

$$x_2 = \frac{1}{70}(12 + 208 - 80) = \frac{140}{70} = 2$$

$$x_3 = \frac{1}{70}(1 + 169 + 40) = \frac{210}{70} = 3$$

Solution of sets of simultaneous equations using matrix algebra techniques has wide application in a variety of engineering problems.

PROBLEMS

2.1 Is $V = \frac{1}{\sqrt{3}}\bar{i} + \frac{1}{\sqrt{3}}\bar{j} + \frac{1}{\sqrt{3}}\bar{k}$ a unit vector?

2.2 Find a unit vector in the direction of

$$\bar{A} = 2\bar{i} + 3\bar{j} - \bar{k}$$

2.3 Are the following two vectors equal?

$$\bar{A} = 2\bar{i} + 3\bar{j} - \bar{k}$$

$$\bar{B} = 4\bar{i} + 6\bar{j} - 2\bar{k}$$

2.4 The following forces measured in pounds act on a body

$$\bar{F}_1 = 2\bar{i} + 3\bar{j} - 5\bar{k}$$

$$\bar{F}_2 = -5\bar{i} + \bar{j} + 3\bar{k}$$

$$\bar{F}_3 = \bar{i} - 2\bar{j} + 4\bar{k}$$

$$\bar{F}_4 = 4\bar{i} - 3\bar{j} - 2\bar{k}$$

Find the resultant force vector and the magnitude of the resultant force vector.

2.5 If

$$\begin{aligned} \bar{A} &= 3\bar{i} - \bar{j} - 4\bar{k} \\ \bar{B} &= -2\bar{i} + 4\bar{j} - 3\bar{k} \\ \bar{C} &= \bar{i} + 2\bar{j} - \bar{k} \end{aligned}$$

Find

$$\begin{aligned} 2\bar{A} - \bar{B} + 3\bar{C} &= \\ |\bar{A} + \bar{B} + \bar{C}| &= \\ |3\bar{A} - 2\bar{B} + 4\bar{C}| &= \end{aligned}$$

2.6 The position vectors of points P and Q are given by

$$\bar{r}_1 = 2\bar{i} + 3\bar{j} - \bar{k}$$
$$\bar{r}_2 = 4\bar{i} - 3\bar{j} + 2\bar{k}$$

Determine the vector from P to Q ($\overline{PQ}$) and find its magnitude.

2.7 Find $\bar{A} \cdot \bar{B}$ using $\bar{A}$ and $\bar{B}$ from Problem 2.5.

2.8 Given

$$\bar{A} = 2\bar{i} + 3\bar{j} - \bar{k}$$
$$\bar{B} = 4\bar{i} + 6\bar{j} - 2\bar{k}$$

a. Find $\bar{A} \cdot \bar{B}$

b. Find the angle between $\bar{A}$ and $\bar{B}$.

2.9 Evaluate

$$\bar{j}\ (2\bar{i} - 3\bar{j} + \bar{k}) =$$
$$(2\bar{i}\ \bar{j})\ (3\bar{i} + \bar{k}) =$$

2.10 If

$$\bar{A} = 3\bar{i} - \bar{j} - 4\bar{k}$$
$$\bar{B} = -2\bar{i} + 4\bar{j} - 3\bar{k}$$

Find A X B.

2.11 Determine the value of "a" so that A and B below are perpendicular.

$$\bar{A} = 2\bar{i} + a\bar{j} + \bar{k}$$
$$\bar{B} = 4\bar{i} - 2\bar{j} - 2\bar{k}$$

2.12 Determine a unit vector perpendicular to the plane of A and B below.

$$\bar{A} = 2\bar{i} - 6\bar{j} - 3\bar{k}$$
$$\bar{B} = 4\bar{i} + 3\bar{j} - \bar{k}$$

2.13 If

$$\bar{A} = 2\bar{i} - 3\bar{j} - \bar{k}$$
$$\bar{B} = \bar{i} + 4\bar{j} - 2\bar{k}$$

Find

$\bar{A} \times \bar{B}$

$\bar{B} \times \bar{A}$

and

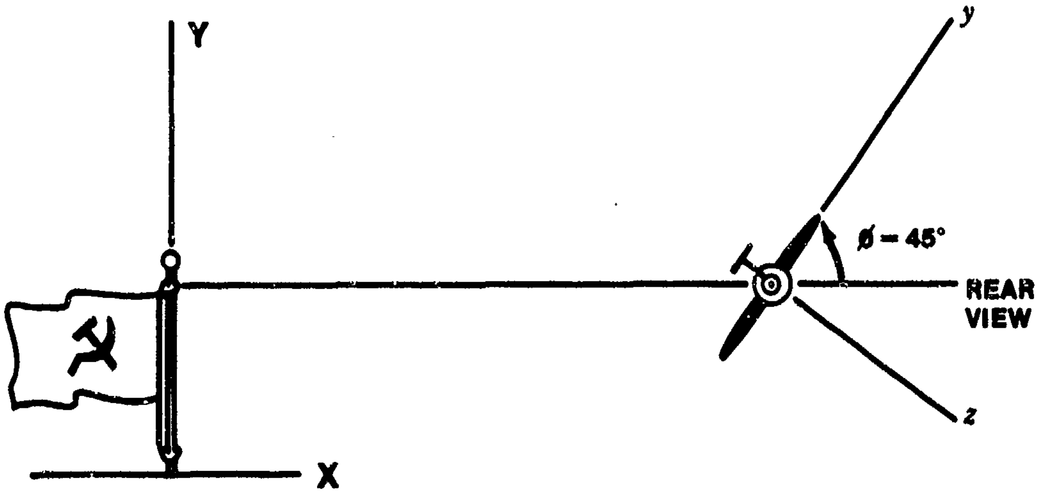
$(\bar{A} + \bar{B}) \times (\bar{A} - \bar{B})$ (the quick way using vector algebra).

2.14 Evaluate

a. $2\bar{j} \times (3\bar{i} - 4\bar{k})$

b. $(\bar{i} + 2\bar{j}) \times \bar{k}$

2.15 The aircraft shown below is flying around the flagpole in a steady state turn at a true velocity of 600 ft/sec. The turn radius is 6,000 ft. What is turn rate ω expressed in unit vectors $(\bar{i}, \bar{j}, \bar{k})$ of the XYZ system shown?



2.16 For the same aircraft and conditions as Problem 2.15, what is turn rate expressed in unit vectors $(\bar{i}, \bar{j}, \bar{k})$ of the xyz system shown?

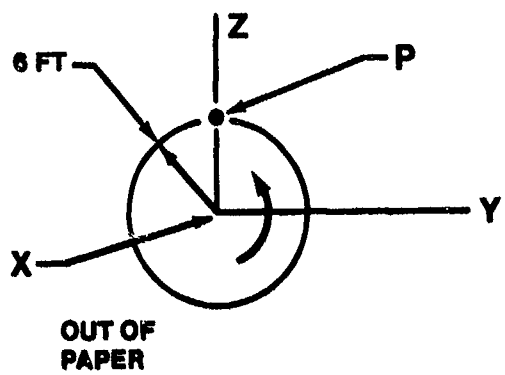
2.17 Given

$$r = t^3\bar{i} - 6t\bar{j} + 6\bar{k}$$

Find $\dot{\bar{r}}$ with respect to the axis system xyz which has $\bar{i}, \bar{j}, \bar{k}$ as its unit vectors. Is $\dot{\bar{r}}$ a velocity?

2.18 If the xyz system in Problem 2.17 is rotating at $3\bar{i} + 2\bar{j} - \bar{k}$ rad/sec with respect to another system XYZ, find $\dot{\bar{r}}$ with respect to XYZ. Is $\dot{\bar{r}}$ the velocity of the point whose radius vector is $\bar{r}$ with respect to XYZ? What system is the answer of this problem referred to?

2.19 A flywheel starts from rest and accelerates counterclockwise at a constant 3 rad/sec^2. After six seconds the point P on the rim of the wheel has reached the position shown in the sketch. What is the velocity of point P with respect to the fixed XYZ system shown?



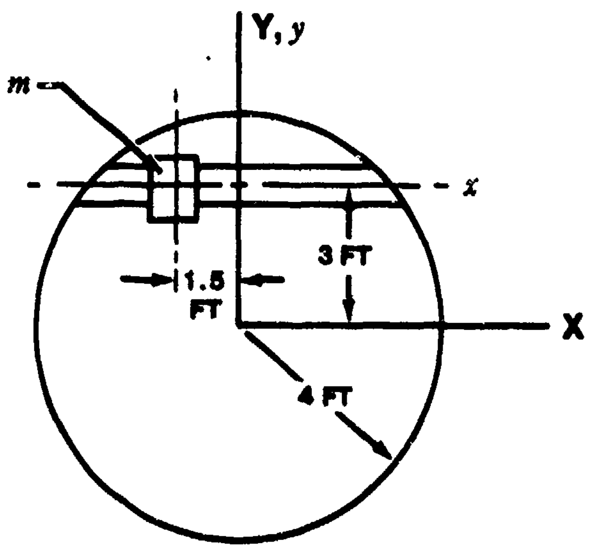
2.20 If

$$\bar{A} = 3t^2\bar{i} - t\bar{j}$$
$$\bar{B} = -6t\bar{i} + t\bar{k}$$

Find $d(\bar{A}\cdot\bar{B})/dt$ relative to the system having $\bar{i}$, $\bar{j}$, and $\bar{k}$ as its unit vectors. Is the answer a vector?

2.21 A small body of mass m slides on a rod which is a chord of a circular wheel as shown below. The wheel rotates about its center with a clockwise velocity 4 rad/sec and a clockwise angular acceleration of 5 rad/sec^2. The body m has a constant velocity on the rod of 6 ft/sec to the right. Relative to the fixed axis system XY shown below, find the absolute velocity and acceleration of m when at the position shown. Hint: Let xy system rotate with the disk as shown.

$\bar{\omega} = -4\bar{k}$
$\dot{\bar{\omega}} = -5\bar{k}$



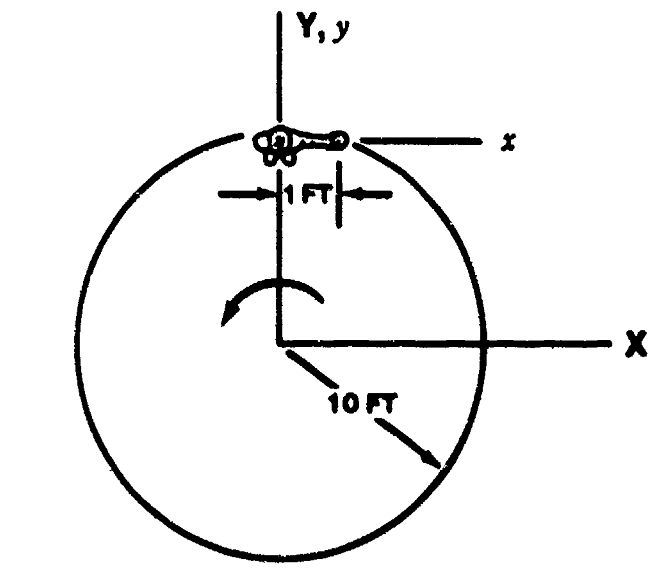
2.22 A small boy holding an ice cream cone in his left hand is standing on the edge of a carousel. The carousel is rotating at 1 rad/sec counterclockwise. As the boy starts walking toward the center of the wheel, what is the velocity and acceleration vector of the ice cream cone relative to the ground XY?

Hint: Let xy be attached to the edge of the carousel.

Boy's velocity = 2 ft/sec toward center

Boy's acceleration = 1 ft/sec^2 toward center

Carousel's acceleration = 1 rad/sec^2 counterclockwise.



2.23 Solve the following equations for x_1, x_2, and x_3 by use of the inverse matrix.

$$x_1 + x_3 = 2$$

$$2x_2 + 2x_3 = 1$$

$$-x_1 + x_2 + 2x_3 = 3$$

2.24 For a - i let

$$[A] = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \qquad [B] = \begin{bmatrix} 4 & 3 & 2 \\ -1 & 1 & 0 \end{bmatrix}$$

$$[C] = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \qquad [D] = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

$$[X] = \begin{bmatrix} x_1 \\ \\ x_2 \end{bmatrix} \qquad [Y] = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Compute

a. $[C] - [A]$
b. $[A]\ [B]$
c. $([A]\ [B])\ [Y]$
d. $[A]\ ([B]\ [Y])$
e. $[A]\ [C]$
f. $[C]\ [A]$
g. $[X]^T\ [B]$
h. $[X]^T\ ([A]\ [X])$
i. $[X]\ [D]^T$

2.25 If $[A] = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$, Find $[A]^2$

2.26 Find x, y, and z

$$\begin{aligned} -4x - 3y - 3z &= 1 \\ x + z &= 1 \\ 4x + 4y + 3z &= 1 \end{aligned}$$

2.27 Read the question and circle the correct answer, True (T) or False (F):

T F A vector is a quantity whose direction and sense are fixed, but whose magnitude is unspecified.

T F A scalar is a quantity with magnitude only.

T F The magnitude of a unit vector is one.

T F Zero vectors have any direction necessary.

T F A free vector can be moved along its line of action, but not parallel to itself.

T F Free vectors may be rotated without change.

T F A 3 x 2 matrix can pre-multiply a 2 x 4 matrix and the result will be a 3 x 4 matrix.

T F A 3 x 2 matrix can post-multiply a 2 x 4 matrix and the result will be a 3 x 4 matrix.

T F Multiplying a matrix by a scalar is the same as multiplying its determinant by the same scalar.

T F Identity matrices are always square.

T F $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

T F Both matrices in the preceding question are identity matrices.

T F Singular matrices can be inverted.

T F The determinant of a non-singular matrix is zero.

T F Inverting a matrix is a straightforward process.

T F $$\bar{A} \quad \bar{B} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

T F The determinant of any matrix can be calculated.

T F The value of a determinant depends upon which row or column it was expanded about.

T F Velocity is the time rate of change of a velocity vector.

T F Acceleration is the time rate of change of a velocity vector.

T F Acceleration has to be expressed in (referred to) unit vectors of an inertial reference system.

T F Bodies moving with pure translation only do not rotate.

T F Reference systems are considered to be non-deformable rigid bodies.

T F [A] [B] = [B] [A], if the two matrices are conformable for multiplication on the left hand side of the equation.

T F $|\bar{V}| = |-\bar{V}|$

T F $|\hat{a}| = |\bar{i}|$

T F The magnitude of $\bar{A}/|\bar{A}|$ is equal to $\bar{B}/|\bar{B}|$

T F $2(3\bar{A}) = 5\bar{A}$

T F $\bar{i}$, $\bar{j}$, and $\bar{k}$ are orthogonal.

T F $|\overline{PQ}|$ is the distance between points P and Q.

T F $\bar{A} \cdot \bar{B} = \bar{B} \cdot \bar{A}$

T F If $\bar{A} \cdot \bar{B}$ is zero and neither $\bar{A}$ nor $\bar{B}$ are zero, then $\bar{A}$ and $\bar{B}$ must be parallel.

T F $\bar{i} \cdot \bar{i} = 1$

T F $\bar{A} \times \bar{B} = \bar{B} \times \bar{A}$

T F $\bar{A} \times \bar{B} = A_1 B_1 + A_2 B_2 + A_3 B_3$

2.28 Define:

Determinant

Vector

Scalar

Free vector

Bound vector

Velocity vector of a particle

Unit vector

Zero vector

Parallel vectors

Position vector

Matrix

Square Matrix

Column Vector

Row Vector

Matrix Equality

Matrix Conformability

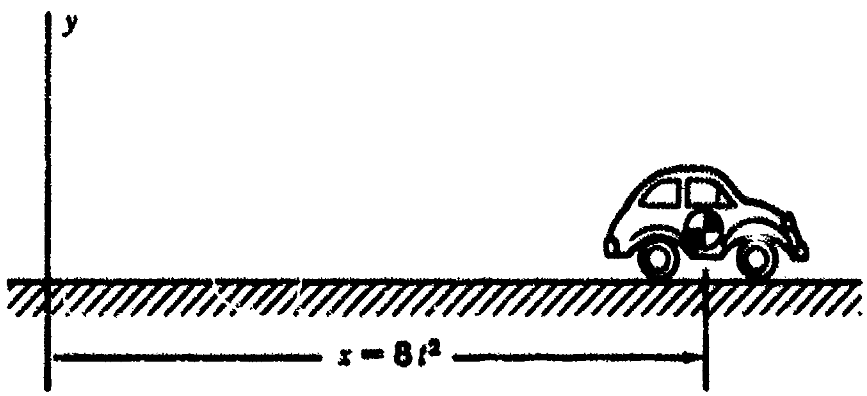
Matrix non-commutativity

Identity Matrix

Transposed Matrix

Singular Matrix

2.29 Find the VW's velocity and acceleration vectors:



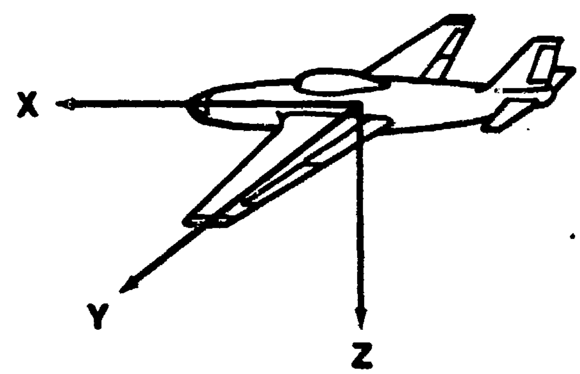
2.30 Find a unit vector parallel to

$$\bar{A} = 2\bar{i} - 3\bar{j} + 6\bar{k}$$

2.31 What is the magnitude of the following vector?

$$\bar{A} = 2\bar{i} + 3\bar{j} + 6\bar{k}$$

2.32 Is the axis shown a "right-handed" axis system?



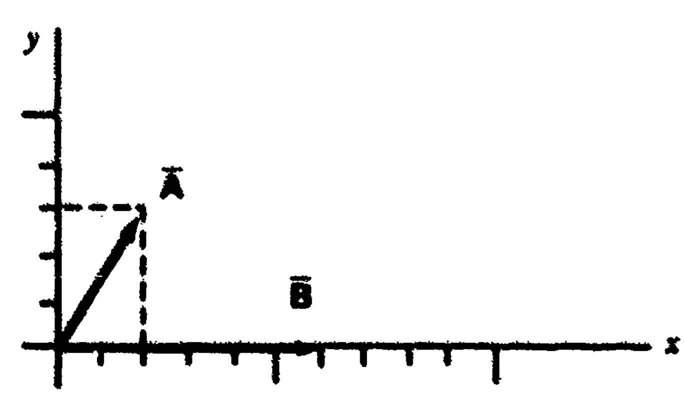
2.33 Given the following position vector, find the acceleration at time t = 0.

$$\bar{r} = 6t^3\bar{i} - 3t\bar{j} + 6t^2\bar{k}$$

2.34 Add the following vectors

$$\bar{A} = 3\bar{i}$$
$$\bar{B} = 4\bar{k}$$
$$\bar{C} = 1/2\bar{j}$$

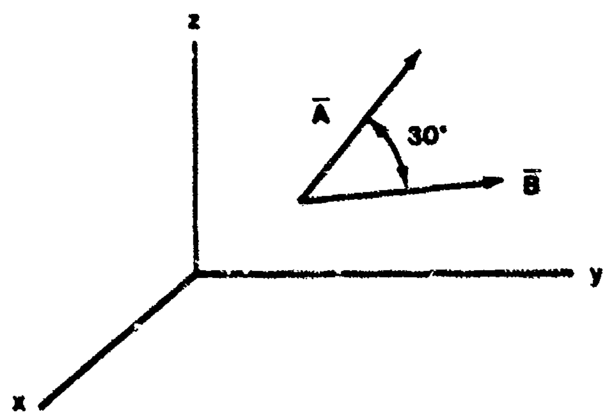
2.35 Find $\bar{A} + \bar{B}$ and the angle it makes with the x axis.



2.36 What is the angle between the two vectors given below?

$$\bar{A} = 2\bar{i} - 7\bar{k}$$
$$\bar{B} = 5\bar{i} + 2\bar{j} - 6\bar{k}$$

2.37



If $|\bar{A}| = 7,$
$|\bar{B}| = 8,$
and both vectors lie in the y - z plane, find
$\bar{A} \times \bar{B}$

2.38 Given

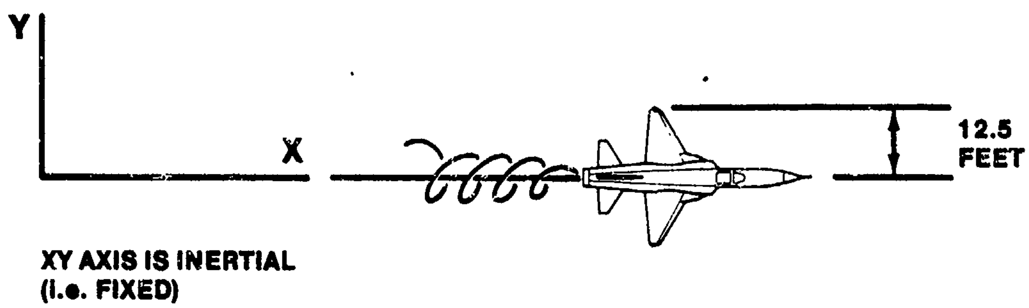
$$\bar{A} = 6\bar{i} - 2\bar{j} + \bar{k}$$
$$\bar{B} = \bar{i} + \bar{k}$$

Find

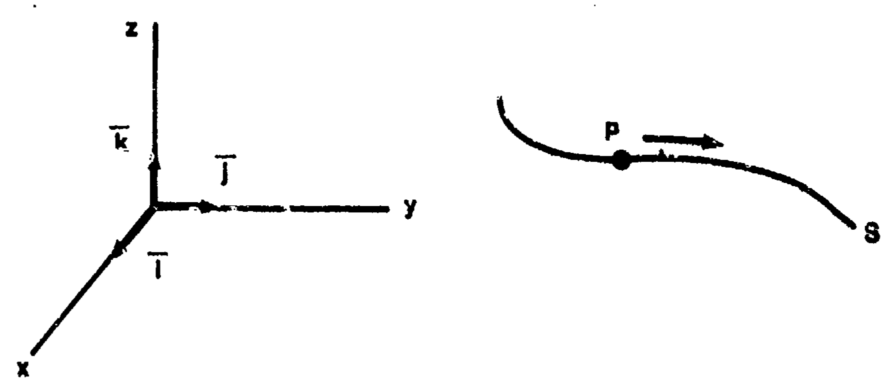
$$\bar{A} \times \bar{B}$$

2.39 The angular velocity of a rotating rigid body about an axis of rotation is given by $\bar{\omega} = 4\bar{i} + 2\bar{j} + \bar{k}$. Find the linear velocity of the Point P on the body whose position vector relative to a point on the axis of rotation is $\bar{i} - 2\bar{j} + 2\bar{k}$.

2.40 The T-38 shown is in a right continuous roll at 2 rad/sec while traveling at 480 ft/sec. Find the velocity of the wingtip light with respect to the axis shown.



2.41 The particle, P, is following a path described by : $x = 6t^2$, $y = t + 1$, $z = t^3$. Find the velocity and acceleration of P, with respect to the axis shown.



2.42 If $\bar{A} = 3\bar{i} - \bar{j} + 2\bar{k}$ and $\bar{B} = -\bar{i} + 3\bar{j} - \bar{k}$, find

a. $|\bar{A}|$

b. $|\bar{B}|$

c. $\bar{A} + \bar{B}$

d. $|\bar{A} + \bar{B}|$

e. $\bar{A} \cdot \bar{B}$

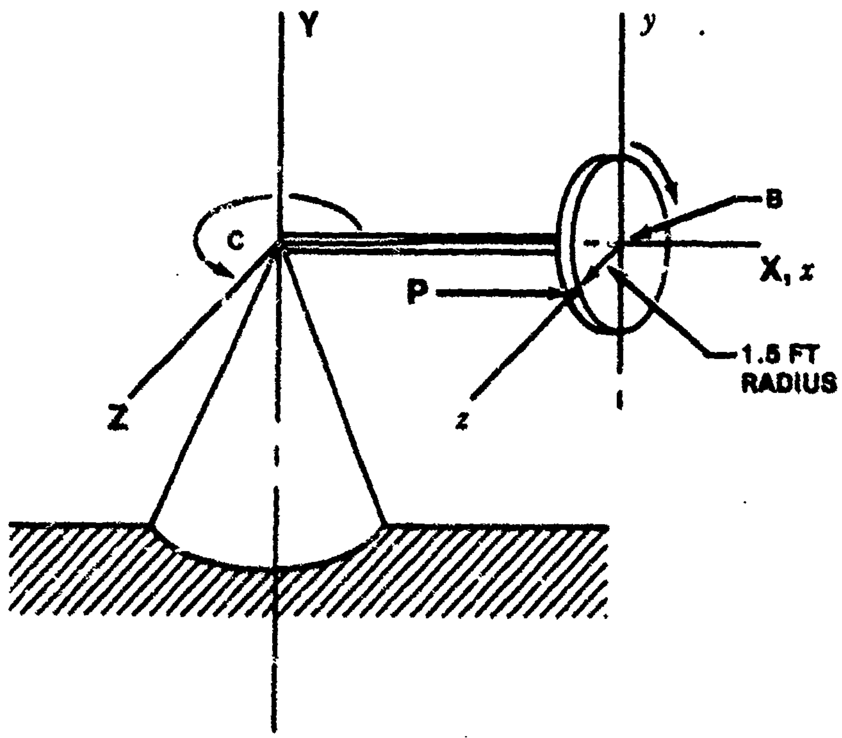
f. $\bar{A} \times \bar{B}$

g. Unit vector, $\hat{a}$, parallel to $\bar{A}$

h. $\hat{a} \cdot \bar{A}$

i. $\hat{a} \cdot \bar{B}$

2.43 The shaft is rotating counterclockwise around the cone in the XZ plane at 5 rad/sec and accelerating at 3 rad/sec^2. The wheel is rotating as shown at 200 rad/sec and decelerating at 50 rad/sec^2. Find the velocity of point P with respect to reference system C at the instant shown. Hint: Let x be fixed in the shaft, and xz and XZ planes remain coplanar.



2.44 If $[A] = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \end{bmatrix}$ and $[B] = \begin{bmatrix} 1 & -1 \\ 0 & 3 \\ 2 & 1 \end{bmatrix}$

Find: [A] [B] and [B] [A]

2.45 If $[A] = \begin{bmatrix} [illegible] & 2 \\ 0 & 1 \end{bmatrix}$ and $[B] = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Find [A] [B] and [B] [A]

2.46 $\begin{bmatrix} 3 & 6 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} =$

2.47 $\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} + (3)\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} =$

2.48 If $\begin{bmatrix} x & 1 \\ 2 & y + z \end{bmatrix} = \begin{bmatrix} 4 & y - z \\ 2 & 5 \end{bmatrix}$

Find x, y, and z.

2.49 If $\begin{bmatrix} 2 \\ -a \end{bmatrix} = k \begin{bmatrix} -4 \\ 2a \end{bmatrix}$

Find k.

2.50 Compute the inverse of

$$\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

2.51 Compute the inverse of

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 5 & 4 \\ 6 & 4 & 2 \end{bmatrix}$$

2.52 Compute the inverse of

$$\begin{bmatrix} 2 & 4 & 1 \\ -1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

2.53 For what value of y is this matrix singular?

$$\begin{bmatrix} 1 & 3 & y \\ 2 & 0 & -1 \\ 1 & 1 & -y \end{bmatrix}$$

2.54 Find the determinant of

$$\begin{bmatrix} 6 & 0 & 0 & 0 & 0 & 0 \\ 8 & x & 0 & 0 & 0 & 0 \\ 12 & 10 & 3 & 0 & 0 & 0 \\ 1 & -1 & 6 & x^{-1} & 0 & 0 \\ 0 & 0 & 2 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

(Look for the easy way; great bar game question.)

2.55 If $[A] = \begin{bmatrix} 2 & 4 & 1 \\ -1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

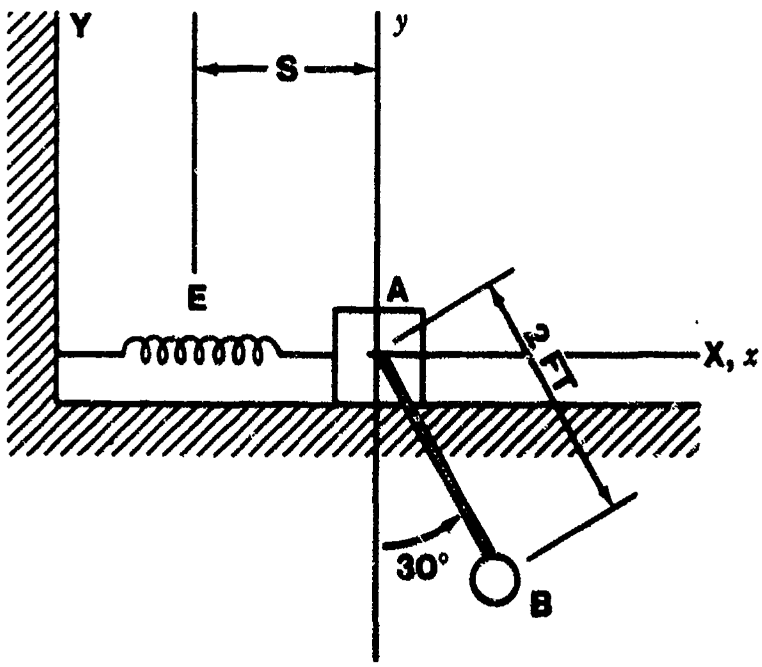
Find $[A]^2$

2.56 If

$$\begin{aligned} x + 2y + 3z &= a_1 \\ 4x + 5y + 6z &= a_2 \\ 7x + 8y + 9z &= a_3 \end{aligned}$$

Find x, y, and z for any value of a_1, a_2, and a_3.
Find x, y, and z, when $a_1 = 1$, $a_2 = 2$, and $a_3 = 3$.

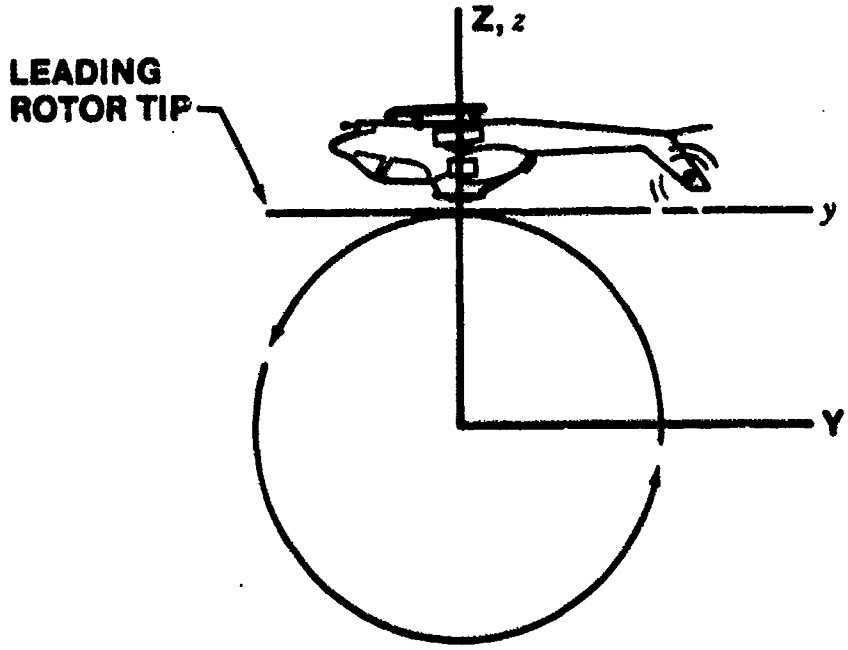
2.57 The mechanism shown below vibrates about its equilibrium position, E. At the instant shown block A has a velocity of 5 ft/sec to the right and is decelerating at 4 ft/sec^2 to the left. The bob B in its counterclockwise motion maintains a constant angular velocity $|\bar{\Omega}|$ of 5 rad/sec. Calculate the velocity and acceleration of the bob relative to the given XY system at the instant shown. Hint: Let the xy axis be fixed to the block A.



2.58 Capt. Marvel, US Army, is performing a loop with an angular velocity $|\overline{\omega}|$ of 1 rad/sec in his Huey Cobra to roll in on a target. At the top of the loop, the leading rotor blade is just parallel with the helicopter's centerline. The rotation of the rotor $|\overline{\Omega}|$ is 3 rad/sec counterclockwise as viewed from the top of the helicopter. At this instant, what is the velocity of the leading rotor blade tip? If Capt. Marvel were to raise the collective and accelerate the rotor speed by 3 rad/sec^2, this would accelerate an angular velocity of his loop by 1 rad/sec^2. What would the acceleration of the leading rotor blade tip be? Hint: Let the xyz system be attached to the helicopter rotor path plane as shown.

radius of loop = 1,000 ft

radius of rotor path plane = 10 ft



Hint: Let the YZ and yz planes remain coplanar.

<u>ANSWERS</u>

2.1 Yes, magnitude of $\overline{V} = 1$

2.2 $\dfrac{2\overline{i} + 3\overline{j} - \overline{k}}{\sqrt{14}}$

2.3 No, $\overline{B} = 2\overline{A}$

2.4 $\overline{F}_y = 2\overline{i} - \overline{j}$

$|\overline{F}_T| = \sqrt{5}$

2.5 $11\overline{i} - 8\overline{k}$; $\sqrt{93}$; $\sqrt{398}$

2.6 $\overline{PQ} = 2\overline{i} - 6\overline{j} + 3\overline{k}$

$|\overline{PQ}| = \sqrt{49}$

2.7 $\overline{A} \cdot \overline{B} = 2$

2.8 $\overline{A} \cdot \overline{B} = 28$; $\phi = 0$

2.9 -3; undefined $\overline{i} \cdot \overline{j} = 0$

2.10 $\overline{A} \times \overline{B} = 19\overline{i} + 17\overline{j} + 10\overline{k}$

2.11 $a = 3$

2.12 $\hat{u} = \frac{3}{7}\overline{i} - \frac{2}{7}\overline{j} + \frac{6}{7}\overline{k}$

2.13 $\overline{A} \times \overline{B} = 10\overline{i} + 3\overline{j} + 11\overline{k}$; $\overline{B} \times \overline{A} = -10\overline{i} - 3\overline{j} - 11\overline{k}$;

$(\overline{A} + \overline{B}) \times (\overline{A} - \overline{B}) = -20\overline{i} - 6\overline{j} - 22\overline{k}$

2.14 $-8\overline{i} - 6\overline{k}$; $+2\overline{i} - \overline{j}$

2.15 $\overline{\omega} = \frac{1}{10}\overline{j}$

2.16 $\bar{\omega} = .07\bar{j} - .07\bar{k}$

2.17 $\dot{\bar{r}} = 3t^2\bar{i} - 6\bar{j}$

2.18 $(\dot{\bar{r}})_{xyz} = (3t^2 - 6t + 12)\bar{i}$
$-(t^3 + 24)\bar{j}$
$-(2t^3 + 18t)\bar{k}$
in XYZ system

2.19 $\bar{V}_{xyz} = -108\bar{j}$ ft/sec

2.20 $\dfrac{d(\bar{A} \cdot \bar{B})}{dt} = -54t^2$

2.21 $\bar{V}_{P/C} = 18\bar{i} + 6\bar{j}$; $\bar{A}_{P/C} = 39\bar{i} - 88.5\bar{j}$

2.22 $\bar{V}_{P/C} = -10\bar{i} - \bar{j}$; $\bar{A}_{P/C} = -10\bar{j} - 7\bar{i}$

2.23 $x_1 = -1/4$; $x_2 = -7/4$; $x_3 = 9/4$

2.25 $$\begin{bmatrix} 5 & -3 & 1 \\ 2 & 1 & 4 \\ 3 & -1 & 2 \end{bmatrix}$$

2.26 $x = -10$; $y = 2$; $z = 11$

2.29 $\bar{V} = 16t\bar{i}$; $a = 16\bar{i}$

2.30 $\hat{a} = \frac{2}{7}\bar{i} - \frac{3}{7}\bar{j} + \frac{6}{7}\bar{k}$

2.31 $|\bar{A}| = 7$

2.33 $\bar{a} = 12\bar{k}$

2.34 $\overline{A} + \overline{B} + \overline{C} = 3\overline{i} + 4\overline{k} + 1/2\overline{j}$

2.35 $\overline{A} + \overline{B} = 8\overline{i} + 3\overline{j}$; $\phi = 20^{\circ}$

2.36 $\phi = 27.6^{\circ}$

2.37 $\overline{A} \times \overline{B} = -28\overline{i}$

2.38 $\overline{A} \times \overline{B} = -2\overline{i} - 5\overline{j} + 2\overline{k}$

2.39 $\overline{V}_p = 6\overline{i} - 7\overline{j} - 10\overline{k}$

2.40 $\overline{V} = 480\overline{i} + 25\overline{k}$

2.41 $\overline{V} = 12t\overline{i} + \overline{j} + 3t^2\overline{k}$; $a = 12\overline{i} + 6t\overline{k}$

2.42
a $\sqrt{14}$
b $\sqrt{11}$
c $2\overline{i} + 2\overline{j} + \overline{k}$
d 3
e -8
f $-5\overline{i} + \overline{j} + 8\overline{k}$
g $(3/\sqrt{14})\overline{i} - (1/\sqrt{14})\overline{j} + (2/\sqrt{14})\overline{k}$
h $\sqrt{14}$
i $-8/\sqrt{14}$

2.43 $\overline{V}_{P/C} = 300\overline{j} + 7.5\overline{i} - 15\overline{k}$
$\overline{A}_{P/C} = -705\overline{i} - 75\overline{j} - 60046.5\overline{k}$

2.44 $[A]\ [B] = \begin{bmatrix} 7 & -4 \\ 2 & 1 \end{bmatrix}$ $\quad [B]\ [A] = \begin{bmatrix} -1 & -3 & 3 \\ 6 & 3 & 0 \\ 4 & -3 & 6 \end{bmatrix}$

2.45 $[A]\ [B] = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$ $\quad [B]\ [A]$ Cannot do

2.46 $\begin{bmatrix} 2 & 9 \\ 2 & 2 \end{bmatrix}$

2.47 $\begin{bmatrix} 4 & 3 \\ 5 & 5 \end{bmatrix}$

2.48 $x = 4; \ y = 3; \ z = 2$

2.49 $k = -1/2$

2.50 $[A]^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$

2.51 No Inverse

2.52 $[A]^{-1} = \frac{1}{7} \begin{bmatrix} -2 & -3 & 8 \\ 3 & 1 & -5 \\ 1 & 2 & 4 \end{bmatrix}$

2.53 $y = 1/4$

2.54 72

2.55 $[A]^2 = \begin{bmatrix} 1 & 9 & 11 \\ 0 & -2 & 1 \\ 2 & 5 & 4 \end{bmatrix}$

2.56 No Solution

2.57 $\vec{V}_{P/C} = 13.66\vec{i} + 5\vec{j}; \ \vec{A}_{P/C} = -29\vec{i} + 43.3\vec{j}$

2.58 $\vec{V}_{P/C} = -30\vec{i} - 1000\vec{j} - 10\vec{k}; \ \vec{A}_{P/C} = -30\vec{i} - 900\vec{j} - 1010\vec{k}$

CHAPTER 3

DIFFERENTIAL EQUATIONS

3.1 INTRODUCTION

This chapter reviews the mathematical tools and techniques required to solve differential equations. Study of these operations is a prerequisite for courses in aircraft flying qualities and linear control systems taught at the USAF Test Pilot School. Only analysis and solution techniques which have direct application for work at the School will be covered.

Many systems of interest can be represented (mathematically modeled) by linear differential equations. For example, the pitching motion of an aircraft in flight displays motion similar to a mass-spring-damper system as shown in Figure 3.1.

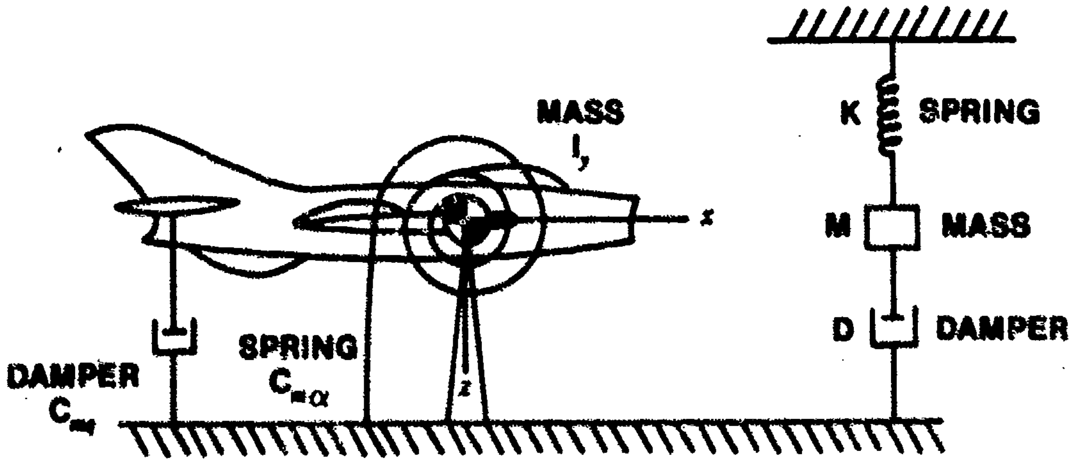


FIGURE 3.1. AIRCRAFT PITCHING MOTION

The static stability of the aircraft is similar to the spring, the moment of inertia about the y-axis is similar to the mass, and the airflow (aerodynamic forces) serves to damp the aircraft motion. Chapter ; shows that stability derivatives can be used to represent the static stability and damping terms. These derivatives are C_{m_α} and C_{m_q}. In this chapter, M, K, and D will D will be used to represent mass, spring, and damper terms respectively.

The following terms will be used extensively:

Differential Equation: An equation relating two or more variables in terms of derivatives.

Independent Variables: Variables that are not dependent on other variables.

Dependent Variables: Variables that are dependent on other variables. In a differential equation, the dependent variables are the variables on the left-hand side of the equation that have their derivatives taken with respect to another variable. The other variable, usually time in our study, is the independent variable.

Solution. Any function without derivatives that satisfies a differential equation.

Ordinary Differential Equation. A differential equation with only one independent variable.

Partial Differential Equation. A differential equation with more than one independent variable.

Order. An n^{th} derivative is a derivative of order n. A differential equation has the order of its highest derivative.

Degree. The exponent of a differential term. The degree of a differential equation is the exponent of its highest order derivative.

Linear Differential Equation. A differential equation in which the dependent variable and all its derivatives are only first degree, and the coefficients are either constants or functions of the independent variable.

Linear System. Any physical system that can be described which satisfies a differential equation of order n which contains n arbitrary constants.

General Solution. Any function without derivatives which satisfies a differential equation of order n which contains n arbitrary constants.

3.2 BASIC DIFFERENTIAL EQUATION SOLUTION

Unfortunately, there is no general method to solve all types of differential equations. The solving of a differential equation involves finding a mathematical expression without derivatives which satisfies the

differential equation. It is usually much easier to determine whether or not a candidate solution to a differential equation is a solution than to determine a likely candidate. For example, given the linear first order differential equation

$$\frac{dy}{dx} - x = 4 \tag{3.1}$$

and a possible candidate solution

$$y = \frac{1}{2}x^2 + 4x + C \tag{3.2}$$

it is easy to differentiate Equation 3.2 and substitute into Equation 3.1 to see if Equation 3.2 is a solution of Equation 3.1. The derivative of Equation 3.2 is

$$\frac{dy}{dx} = x + 4 \tag{3.3}$$

Substituting Equation 3.3 into Equation 3.1,

$$(x + 4) - x = 4 \tag{3.4}$$

$$4 = 4$$

Therefore, Equation 3.2 is a solution of Equation 3.1.

It is interesting that, in general, solutions to linear differential equations are not linear functions. Note that Equation 3.2 is not an equation of the form

$$y = mx + b \tag{3.5}$$

which represents a straight line. As shown in Equation 3.2, y is a function of x and x^2.

There are several methods in use to solve differential equations. The methods to be discussed in this chapter are:

1. Direct Integration
2. Separation of Variables
3. Exact Differential Integration
4. Integrating Factor
5. Special Procedures, to include Operator Techniques and Laplace Transforms.

3.2.1 Direct Integration

Since a differential equation contains derivatives, it is sometimes possible to obtain a solution by anti-differentiation or integration. This process removes the derivatives and provides arbitrary constants in the solution. For example, given

$$\frac{dy}{dx} - x = 4 \tag{3.1}$$

rewriting

$$dy - xdx = 4dx \tag{3.6}$$

integrating

$$\int dy - \int xdx = \int 4dx + C$$

$$y - \frac{x^2}{2} = 4x + C \tag{3.7}$$

or, solving for y

$$y = \frac{x^2}{2} + 4x + C \tag{3.8}$$

where C is an arbitrary constant of integration.

Unfortunately, application of the direct integration process fails to work in many cases.

3.2.2 Separation of Variables

If direct integration fails for a first order differential equation, then the next step is to try to separate the variables. Direct integration may then be possible. When a differential equation can be put in the form

$$f_1(x)\,dx + f_2(y)\,dy = 0 \tag{3.9}$$

where one term contains function of x and dx only, and the other functions of y and dy only, the variables are said to be separated. A solution of Equation 3.9 can then be obtained by direct integration

$$\int f_1(x)\,dx + \int f_2(y)\,dy = C \tag{3.10}$$

where C is an arbitrary constant. Note, that for a differential equation of the first order there is one arbitrary constant. In general, the number of arbitrary constants is equal to the order of the differential equation.

EXAMPLE

$$\frac{dy}{dx} = \frac{x^2 + 3x + 4}{y + 6}$$

$$(y + 6)\,dy = (x^2 + 3x + 4)\,dx$$

$$\int (y + 6)\,dy = \int (x^2 + 3x + 4)\,dx + C$$

$$\frac{y^2}{2} + 6y = \frac{x^3}{3} + \frac{3x^2}{2} + 4x + C$$

Not all first order equations can be separated in this fashion.

3.2.3 Exact Differential Integration

If direct integration, or direct integration after separation is not possible, then it still may be possible to obtain a solution if the

differential equation is an exact differential. Associated with each suitably differentiable function of two variables f (x,y), there is an expression called its differential, namely

$$df \equiv \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0 \tag{3.11}$$

that can be written as

$$df = M(x,y)\ dx + N(x,y)\ dy = 0 \tag{3.12}$$

and is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{3.13}$$

If the differential equation is exact, then for all values of C

$$\int_a^x M(x,y)\ dx + \int_b^y N(x,y)\ dy = C \tag{3.14}$$

is a solution of the equation, where a and b are dummy variables of integration.

EXAMPLE

Show that the equation

$$(2x + 3y - 2)dx + (3x - 4y + 1)dy = 0 \tag{3.15}$$

is exact and find a general solution.

Applying the test in Equation 3.13

$$\frac{\partial M}{\partial y} = \frac{\partial\ (2x + 3y - 2)}{\partial y} = 3$$

$$\frac{\partial N}{\partial x} = \frac{\partial\ (3x - 4y + 1)}{\partial x} = 3$$

Since the two partial derivatives are equal, the equation is exact. Its solution can be found by means of Equation 3.14.

$$\int_a^x (2x + 3y - 2)\,dx + \int_b^y (3x - 4y + 1)\,dy = C$$

The integration is performed assuming y is a constant while integrating the first term.

$$(x^2 + 3xy - 2x)\Big|_a^x + (3xy - 2y^2 + y)\Big|_b^y = C$$

$$(x^2 + 3xy - 2x) - (a^2 + 3ay - 2a) + (3xy - 2y^2 + y) - (3xb - 2b^2 + b) = C$$

$$x^2 + 6xy - 2x - 2y^2 + y + 3ay + 3xb = C + a^2 - 2a - 2b^2 + b = C_1 \qquad (3.16)$$

The same result can be obtained with less algebra and probably less chance of error by comparing Equation 3.15 with the differential form in Equation 3.11.

$$\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 0 \qquad (3.11)$$

$$(2x + 3y - 2)\,dx + (3x - 4y + 1)\,dy = 0 \qquad (3.15)$$

Comparing these two equations,

$$\frac{\partial f}{\partial x} = 2x + 3y - 2 = 0 \qquad (3.17)$$

and

$$\frac{\partial f}{\partial y} = 3x - 4y + 1 = 0 \qquad (3.18)$$

Since Equation 3.15 is an exact differential, then Equations 3.17 and 3.18 can be obtained by taking partial derivatives of the same function f. To find the unknown function f, first integrate Equations 3.17 and 3.18 assuming that y is constant when integrating with respect to x and that x is constant when integrating with respect to y.

$$f = x^2 + 3xy - 2x + f(y) + C = 0 \tag{3.19}$$

$$f = 3xy - 2y^2 + y + f(x) + C = 0 \tag{3.20}$$

Note that if Equation 3.17 had been obtained from Equation 3.19, any term that was a function of y only, f(y), and any constant term, C, would have disappeared. Similarly, obtaining Equation 3.18 from Equation 3.20, the f(x) and C terms would have vanished. By a direct comparison of Equation 3.19 and 3.20 the total function f can be determined.

$$f = x^2 + 6xy - 2x - 2y^2 + y + C = 0 \tag{3.21}$$

Note that the unknown f(y) term in Equation 3.19 is $(-2y^2 + y)$ and the unknown f(x) term in Equation 3.20 is 2x. Redefining the constant of integration, Equation 3.21 can be written as

$$x^2 + 6xy - 2x - 2y^2 + y = C_1 \tag{3.16}$$

and was obtained earlier by integrating using dummy variables of integration.

3.2.4 Integrating Factor

When none of the above procedures or techniques work, it may still be possible to integrate a differential equation using an integrating factor. When some unintegrable differential equation is multiplied by some algebraic factor which permits it to be integrated term by term, then the algebraic factor is called an integrating factor. Determining integrating factors for arbitrary differential equations is beyond the scope of this course; however, two integrating factors will be introduced in later sections of this chapter

when developing operator techniques and Laplace transforms. These two factors will be e^{mx} and e^{-st}.

3.3 FIRST ORDER EQUATIONS

The solution to a first order linear differential equation can be obtained by direct integration. Consider the form

$$\frac{dy}{dx} = R(x)\ y = 0 \qquad (3.22)$$

where R(x) is a function of x only or a constant. To solve, separate variables

$$\frac{dy}{y} + R(x)\ dx = 0 \qquad (3.23)$$

Integrating

$$\int \frac{dy}{y} = - \int R(x)\ dx + C' \qquad (3.24)$$

where

$$C' = \ln C$$

Thus

$$\ln y = - \int R(x)\ dx + \ln C \qquad (3.25)$$

or

$$y = Ce^{-\int R(x)\ dx} \qquad (3.26)$$

If R(x) is a constant, R, then

$$y = Ce^{-Rx} \qquad (3.27)$$

From this result, it can be concluded that a first order linear differential equation in the form of Equation 3.22 can be solved by simply expressing the solution in the form of Equation 3.27.

EXAMPLE

$$\frac{dy}{dx} + 2y = 0 \qquad (3.28)$$

then the solution can be written directly as

$$y = Ce^{-2x} \tag{3.29}$$

EXAMPLE

$$\frac{dy}{dx} + x^3 y = 0 \tag{3.30}$$

is in the form

$$\frac{dy}{dx} + R(x)y = 0 \tag{3.22}$$

which has the solution

$$y = Ce^{-\int R(x)dx} \tag{3.26}$$

Therefore, the solution to Equation 3.30 can be obtained directly

$$y = Ce^{-\int x^3 dx}$$

$$y = Ce^{-\frac{1}{4}x^4}$$

3.4 LINEAR DIFFERENTIAL EQUATIONS AND OPERATOR TECHNIQUES

A form of differential equation that is of particular interest

$$A_n \frac{d^n y}{dx^n} + A_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + A_1 \frac{dy}{dx} + A_0 y = f(x) \tag{3.31}$$

If the coefficient expression A_n, A_{n-1}, . . . , A_0 are all functions of x only, then Equation 3.31 is called a linear differential equation. If the coefficient expressions A_n, . . . , A_0 are all constants, then Equation 3.31 is called a linear differential equation with constant coefficients.

EXAMPLE

$$x^2 \frac{d^2y}{dx^2} + 3\frac{dy}{dx} + xy = \sin x$$

is a linear differential equation.

EXAMPLE

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = e^x$$

is a linear differential equation with constant coefficients. Linear differential equations with constant coefficients occur frequently in the analysis of physical systems. Mathematicians and engineers have developed simple and effective techniques to solve this type of equation by using either "classical" or operational methods. When attempting to solve a linear differential equation of the form

$$A_n \frac{d^ny}{dx^n} + A_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + A_1\frac{dy}{dx} + A_0y = f(x), \qquad (3.32)$$

it is helpful to first examine the equation

$$A_n \frac{d^ny}{dx^n} + A_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + A_1\frac{dy}{dx} + A_0y = 0 \qquad (3.33)$$

Equation 3.33 is the same as Equation 3.32 with the right-hand side set equal to zero. Equation 3.32 is known as the general equation and Equation 3.33 as the complementary or homogeneous equation. Solutions of Equation 3.33 possess a useful property known as superposition, which may be briefly stated as follows: Suppose $y_1(x)$ and $y_2(x)$ are distinct solutions of Equation 3.33. Then any linear combination of $y_1(x)$ and $y_2(x)$ is also a solution of Equation 3.33. A linear combination would be $C_1y_1(x) + C_2y_2(x)$.

EXAMPLE

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

It can be verified that $y_1(x) = e^{3x}$ is a solution, and that $y_2(x) = e^{2x}$ is another solution which is distinct from $y_1(x)$. Using superposition, then, $y(x) = c_1e^{3x} + c_2e^{2x}$ is also a solution.

Equation 3.32 may be interpreted as representing a physical system where the left side of the equation describes the natural or designed state of the system, and where the right side of the equation represents the input or forcing function.

The following line of reasoning is used to find a solution to Equation 3.32:

1. A general solution of Equation 3.32 must contain n arbitrary constants and must satisfy the equation.

2. The following statements are justified by experience:

   a. It is reasonably straightforward to find a solution to the complementary Equation 3.33, containing n arbitrary constants. Such a solution will be called the <u>transient solution</u>. Physically, it represents the response present in the system regardless of input.

   b. There are varied techniques for finding the solution of a differential equation due to a forcing function. Such solutions do not, in general, contain arbitrary constants. This solution will be called the <u>particular</u> or <u>steady state solution</u>.

3. If the transient solution which describes the response already existing in the system is added to the response due to the forcing function, it would appear that a solution so written would blend the two responses and describe the total response of the system represented by Equation 3.32. In fact, the definition of a general solution is satisfied under such an arrangement. This is simply an extension of the principle of superposition. The transient solution contains the correct number of arbitrary constants, and the particular solution guarantees that the combined solutions satisfy the general Equation 3.32. A general solution of Equation 3.32 is then given by

$$y = y_t + y_p \tag{3.34}$$

where y_t is the transient solution and y_p is the particular solution.

3.4.1 Transient Solution

Equation 3.28 is a complementary or homogeneous first order linear differential equation with constant coefficients. A quick and simple method of solving this equation was found. The solution was always of exponential form; hopefully, solutions of higher order equations of the same family take the same form.

$$\frac{dy}{dx} + 2y = 0 \tag{3.28}$$

Next, a second order differential equation with constant coefficients will be examined to determine if the candidate solution

$$y = e^{mx} \tag{3.35}$$

is a solution of the equation

$$ay'' + by' + cy = 0 \tag{3.36}$$

when the prime notation indicates derivatives with respect to x. That is,

$$y' = dy/dx, \quad y'' = d^2y/dx^2$$

Substituting

$$y = e^{mx}$$

$$am^2e^{mx} + bme^{mx} + ce^{mx} = C \tag{3.37}$$

or

$$(am^2 + bm + c)\, e^{mx} = 0. \tag{3.38}$$

Since

$$e^{mx} \neq 0$$

$$am^2 + bm + c = 0 \tag{3.39}$$

and, using the quadratic formula

$$m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3.40}$$

Substituting these values into the assumed candidate solution, it is a solution when m_1 and m_2 are defined by Equation 3.40.

$$y_t = C_1 e^{m_1 x} + C_2 e^{m_2 x} \tag{3.41}$$

Equation 3.41 represents a transient solution since there is no forcing function in Equation 3.36. When working numerical problems, it is not necessary to take the derivatives of e^{mx}. This will be true for any order differential equation with constant coefficients. From the foregoing, it is seen that the method for first order complementary equations has been extended to higher order complementary or homogeneous equations. Again an integration problem has been traded for an algebra problem (solving Equation 3.39 for m's).

There are four possibilities for m_1 and m_2, and each is discussed below.

3.4.1.1 Case 1: Roots Real and Unequal. If m_1 and m_2 are real and unequal, the desired form of solution is just as given by Equation 3.41.

EXAMPLE

Given the homogeneous equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 12y = 0,$$

rewriting in operator form where

$$m = \frac{dy}{dx}$$

and

$$m^2 = \frac{dy}{dx^2}$$

$$(m^2 + 4m - 12)y = 0.$$

Solving for the values of m,

$$m^2 + 4m - 12 = 0$$

gives

$$m = \frac{-4 \pm \sqrt{16 + 48}}{2} = \frac{-4 \pm 8}{2} = -6,2$$

and the required transient solution is

$$y_t = c_1e^{-6x} + c_2e^{2x}.$$

3.4.1.2. <u>Case 2: Roots Real and Equal</u>. If m_1 and m_2 are real and equal, an alternate form of solution is required.

EXAMPLE

Given the homogeneous equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0, \tag{3.42}$$

rewriting in operator form

$$(m^2 - 4m + 4)\ y = 0.$$

Solving for the values of m,

$$m = \frac{4 \pm \sqrt{16 - 16}}{2} = \frac{4}{2} = 2$$

or m = 2. But this gives only one value of m, and two values of m are required to result in a solution of the form of Equation 3.41 which has two arbitrary constants. The operator expression

$$m^2 - 4m + 4 + 0$$

can also be written

$$(m - 2)^2 = 0$$

or

$$(m - 2)\ (m - 2) = 0$$

now a repeated polynomial factor resulting in two (repeated) roots

$$m = 2,2.$$

Writing the solution in the form of Equation 3.41 when the roots are repeated does not give a solution because the two arbitrary constants can be combined into a single arbitrary constant as shown below.

$$y_t = c_1e^{2x} + c_2e^{2x} = (c_1 + c_2)\ e^{2x} = c_3e^{2x}$$

To solve this problem one of the arbitrary constants is multiplied by x. The solution now contains two arbitrary constants which cannot be combined, and it is easily verified that

$$y_t = c_1e^{2x} + c_2xe^{2x}$$

is a transient solution of Equation 3.42.

3.4.1.3 Case 3: Roots Purely Imaginary.

EXAMPLE

Given the homogeneous equation

$$\frac{d^2y}{dx^2} + y = 0,$$

rewriting in operator form

$$(m^2 + 1)y = 0.$$

Solving,

$$m = \frac{0 \pm \sqrt{0 - 4}}{2} = \pm\sqrt{-1}$$

In most engineering work $\sqrt{-1}$ is given the symbol j. (In mathematical texts it is denoted by i.). Now,

$$m = \pm j$$

and the solution is written

$$y_t = c_1 e^{jx} + c_2 e^{-jx} \tag{3.43}$$

This is a perfectly good solution from a mathematical standpoint, but Euler's identity can be used to put the solution in a more useable form.

$$e^{jx} = \cos x + j \sin x \tag{3.44}$$

This equation can be restated in many ways geometrically and analytically, and can be verified by adding the series expansion of cos x to the series expansion of j sin x. Now Equation 3.43 may be expressed

$$y_t = c_1 (\cos x + j \sin x) + c_2 \; [\cos \; (-x) + j \sin (-x)]$$

$$y_t = (c_1 + c_2) \cos x + j \; (c_1 - c_2) \sin x \tag{3.45}$$

or without loss of generality

$$y_t = c_3 \cos x + c_4 \sin x \tag{3.46}$$

An equivalent expression to Equation 3.46 is

$$y_t = \sqrt{c_3^2 + c_4^2} \left[\frac{c_3}{\sqrt{c_3^2 + c_4^2}} \cos x + \frac{c_4}{\sqrt{c_3^2 + c_4^2}} \sin x \right] \tag{3.47}$$

If the arbitrary constants c_3 and c_4 are related as shown in Figure 3.2,

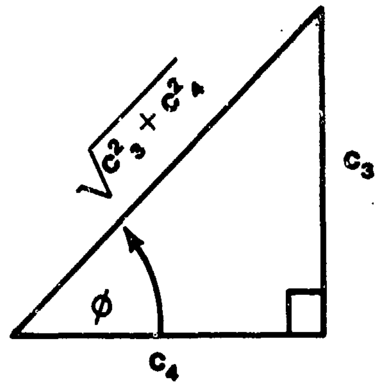


FIGURE 3.2. DEFINITION OF c_3 AND c_4

then

$$\frac{c_3}{\sqrt{c_3^{\,2} + c_4^{\,2}}} = \sin\phi$$

$$\frac{4}{\sqrt{c_3^{\,2} + c_4^{\,2}}} = \cos\phi$$

and

$$\sqrt{c_3^{\,2} + c_4^{\,2}} = A$$

where A and ϕ are also arbitrary constants, Equation 3.47 becomes

$$y_t = A\,(\sin\phi\cos x + \cos\phi\sin x)$$

or using a common trigonometric identity

$$y_t = A\sin(x + \phi) \tag{3.48}$$

Note also that Equation 3.48 could be written in the equivalent form

$$y_t = A\cos(x - \theta)$$

where

$$\theta = 90^{\circ} - \phi$$

To summarize, if the roots of the operator polynomial are purely imaginary, they will be numerically equal but opposite in sign, and the solution will have the form of Equation 3.46, 3.48, or 3.49.

EXAMPLE

Given the homogeneous equation

$$\frac{d^2y}{dx^2} + 4y = 0$$

rewriting in operator form

$$(m^2 + 4)y = 0$$

which gives the roots

$$m^2 = \pm 2j$$

Alternate solutions can immediately be written as

$$y_t = c_3 \cos 2x + c_4 \sin 2x$$

or

$$y_t = A \sin (2x + \phi)$$

where c_3, c_4, A, and ϕ are arbitrary constants.

3.4.1.4 Case 4: Roots Complex Conjugates.

EXAMPLE

Given the homogeneous equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$$

rewriting in operator form

$$(m^2 + 2m + 2)y = 0$$

Solving gives a complex pair of roots

$$m = \frac{-2 + \sqrt{4 - 8}}{2} = -1 \pm \sqrt{-1}$$

or

$$m = -1 + j, -1 - j$$

The solution can be written

$$y_t = c_1 e^{(-1 + j)x} + c_2 e^{(-1 - j)x}$$

Factoring out the exponential term gives

$$y_t = e^{-x}\left[c_1 e^{jx} + c_2 e^{-jx}\right]$$

or, using the results from Equations 3.46 and 3.48, alternate solutions can immediately be written as

$$y_t = e^{-x}\left[c_3 \cos x + c_4 \sin x\right] \tag{3.50}$$

or

$$y_t = e^{-x} A \sin (x + \phi) \tag{3.51}$$

3.4.2 Particular Solution

The particular solution to a linear differential equation can be obtained by the method of undetermined coefficients. This method consists of assuming a solution of the same general form as the input (forcing function), but with undetermined constant coefficients. Substitution of this assumed solution into the differential equation enables the coefficients to be evaluated. The method of undetermined coefficients applies when the forcing function or input is a polynomial, or of the form

$$\sin ax, \cos ax, e^{ax}$$

or combinations of sums and products of these terms. The general solution to the differential equation with constant coefficients is then given by Equation 3.34,

$$y = y_t + y_p \tag{3.34}$$

which is the summation of the solution to the complementary equation (transient solution), plus the particular solution.

Consider the equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x) \tag{3.52}$$

The particular solution which results from a given input, f(x), can be solved for using the method of undetermined coefficients. The method is best illustrated by considering examples.

3.4.2.1 Constant Forcing Functions.

EXAMPLE

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 6 \tag{3.53}$$

The input is a constant (trivial polynomial), so a solution of form $y_p = K$ is assumed.

Then

$$\frac{dy_p}{dx} = \frac{dK}{dx} = 0$$

and

$$\frac{d^2y_p}{dx^2} = \frac{d^2K}{dx^2} = 0$$

Substituting into Equation 3.53,

$$0 + 4(0) + 3K = 6$$

$$y_p = K = 2$$

Therefore, $y_p = 2$ is a particular solution. The homogeneous equation can be solved using operator form

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 0 \tag{3.54}$$

$$(m^2 + 4m + 3)y = 0$$

or

$$m = -1, -3$$

and the transient solution can be written as

$$y_t = c_1e^{-x} + c_2e^{-3x} \tag{3.55}$$

The general solution of Equation 3.53 is

$$y = \underbrace{c_1e^{-x} + c_2e^{-3x}}_{\text{transient solution}} + \underbrace{2}_{\text{particular (or steady state) solution}} \tag{3.56}$$

3.4.2.2 Polynomial Forcing Function.

EXAMPLE

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = x^2 + 2x \tag{3.57}$$

The form of f(x) for Equation 3.57 is a polynomial of second degree, so a particular solution for y_p of second degree is assumed:

$$y_p = Ax^2 + Bx + C$$

Then

$$\frac{dy_p}{dx} = 2Ax + B$$

and

$$\frac{d^2y_p}{dx} = 2A$$

Substituting into Equation 3.57,

$$(2A) + 4\ (2Ax + B) + 3\ (Ax^2 + Bx + C) = x^2 + 2x$$

or

$$(3A)\ x^2 + (8A + 3B)\ x + (2A + 4B + 3C) = x^2 + 2x$$

Equating like powers of x,

$$x^2{:}\quad 3A = 1$$

$$A = 1/3$$

$$x{:}\quad 8A + 3B = 2$$

$$3B = 2 - 8/3$$

$$B = -\,2/9$$

$$x^0{:}\quad 2A + 4B + 3C = 0$$

$$3C = 8/9 - 2/3$$

$$C = 2/27$$

Therefore,

$$y_p = 1/3\ x^2 - 2/9\ x + 2/27$$

The total general solution of Equation 3.57 is given by

$$y = c_1e^{-x} + c_2e^{-3x} + 1/3\ x^2 - 2/9\ x + 2/27 \tag{3.58}$$

since the transient solution is Equation 3.55. As a general rule, if the forcing function is a polynomial of degree n, assume a polynomial solution of degree n.

3.4.2.3 EXPONENTIAL FORCING FUNCTION.

EXAMPLE

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = e^{2x} \tag{3.59}$$

The forcing function is e^{2x} so assume a solution of the form

$$y_p = Ae^{2x}$$

$$\frac{d}{dx}(Ae^{2x}) = 2Ae^{2x}$$

$$\frac{d^2}{dx^2}(Ae^{2x}) = 4Ae^{2x}$$

Substituting into Equation 3.59,

$$4Ae^{2x} + 4(2Ae^{2x}) + 3(Ae^{2x}) = e^{2x}$$

$$e^{2x}(4A + 8A + 3A) = e^{2x}$$

The coefficients on both sides of the equation must be the same. Therefore, $4A + 8A + 3A = 1$, or $15A = 1$, and $A = 1/15$. The particular solution of Equation 3.59 then is $y_p = 1/15\ e^{2x}$. The transient solution is still Equation 3.55. A final example will illustrate a pitfall sometimes encountered using this method.

3.4.2.4 Exponential Forcing Function (special case).

EXAMPLE

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = e^{-x} \tag{3.60}$$

The forcing function is e^{-x}, so assume a solution of the form $y_p = Ae^{-x}$.

Then

$$\frac{d}{dx}(Ae^{-x}) = -Ae^{-x}$$

and

$$\frac{d^2}{dx^2}(Ae^{-x}) = Ae^{-x}$$

Substituting into Equation 3.60,

$$Ae^{-x} + 4(-Ae^{-x}) + 3(Ae^{-x}) = e^{-x}$$

$$(A - 4A + 3A)e^{-x} = e^{-x}$$

$$(0)e^{-x} = e^{-x}$$

Obviously, this is an incorrect statement. To locate the difficulty, the procedure to solve differential equations will be reviewed.

To solve an equation of the form

$$(m + a)(m + b)y = e^{-x}$$

solve the homogeneous equation to get

$$(m + a)(m + b)y = 0$$

$$m = -a, -b$$

$$y_t = c_1e^{-ax} + c_2e^{-bx}$$

If $y_p = Ae^{-ax}$ is assumed for a particular solution, then

$$y = y_t + y_p = c_1e^{-ax} + c_2e^{-bx} + Ae^{-ax} = (c_1 + A)e^{-ax} + c_2e^{-bx}$$

$$= c_3e^{-ax} + c_2e^{-bx}$$

$$= y_t$$

However, y_t is the solution only when the right side of the equation is zero, and will not solve the equation when there is a forcing function of the form given. Assuming a particular solution of the form

$$y_p = Axe^{-ax}$$

will lead to a solution, then

$$y = y_p + y_t = c_1e^{-ax} + c_2e^{-bx} + Axe^{-ax} = (c_1 + Ax)e^{-ax} + c_2e^{-bx} = y_t$$

Similarly, the equation

$$(m + aj)\ (m - aj)\ y = \sin ax$$

has the transient solution

$$y_t = c_1 \sin ax + c_2 \cos ax$$

If $y_p = A \sin ax + B \cos ax$ is assumed for a particular solution, then

$$y = y_t + y_p = (c_1 + A) \sin ax + (c_2 + b) \cos ax$$

$$y = c_3 \sin ax + c_4 \cos ax = y_t$$

which, as in the previous example, does not provide a solution when there is a forcing function of the form given. But, assuming a solution of the form

$$y_p = Ax \sin ax + Bx \cos ax$$

does lead to a solution

$$y = (c_1 + Ax) \sin ax + (c_2 + Bx) \cos ax = y_t$$

Continuing with the solution of Equation 3.60, a valid solution can be found by assuming $y_p = Axe^{-x}$, then

$$\frac{d}{dx}(Axe^{-x}) = A(-xe^{-x} + e^{-x})$$

and

$$\frac{d^2}{dx}(Axe^{-x}) = A(xe^{-x} - 2e^{-x})$$

Substituting into Equation 3.60,

$$A(xe^{-x} - 2e^{-x}) + 4A(-xe^{-x} + e^{-x}) + 3(Axe^{-x}) = e^{-x}$$

$$(A - 4A + 3A)xe^{-x} + (-2A + 4A)e^{-x} = e^{-x}$$

$$(0)xe^{-x} + 2Ae^{-x} = e^{-x}$$

and

$$A = 1/2$$

Thus,

$$y_p = (1/2)xe^{-x}$$

is a particular solution of Equation 3.60, and the general solution is given by

$$y = c_1e^{-x} + c_2e^{-3x} + 1/2\ xe^{-x}$$

The key to successful application of the method of undetermined coefficients is to assume the proper form for a trial or candidate particular solution. Table 3.1 summarizes the results of this discussion. When f(x) in Table 3.1 consists of a sum of several terms, the appropriate choice for y_p is the sum of y_p expressions corresponding to these terms individually. Whenever a term in any of the y_p's listed in Table 3.1 duplicates a term already in the complementary function, <u>all</u> terms in that y_p must be multiplied by the lowest positive integral power of x sufficient to eliminate the duplication.

TABLE 3.1
CANDIDATE PARTICULAR SOLUTIONS

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = f(x)$$

Forcing Function $f(x)$	Assumed Solution y_p
Constant: K_1	A
Polynomial: K_1x^n	$A_0x^n + A_1x^{n-1} + \ldots + A_{n-1}x + A_n$
Sine: $K_1 \sin K_2x$	$A \cos K_2x + B \sin K_2x$
Cosine: $K_1 \cos K_2x$	$A \cos K_2x + B \sin K_2x$
Exponential: $K_1 e^{K_2x}$	$A e^{K_2x}$

3.4.3 Solving For Constants of Integration

As discussed previously, the number of arbitrary constants in the solution of a linear differential equation is equal to the order of the equation. The constants of integration can be determined by initial or boundary conditions. That is, to solve for the constants the physical state (position, velocity, etc.) of the system must be known at some time. The number of initial or boundary conditions given must equal the number of constants to be solved for. Many times these conditions are given at time

equal to zero, in which case they are called initial conditions. A system which has zero initial condition, i.e., initial position, velocity, and acceleration all equal to zero, is frequently called a quiescent system.

The arbitrary constants of the solution must be evaluated from the total general solution, that is, the transient plus the steady state solution. The method of evaluating the constants of integration will be illustrated with an example.

EXAMPLE

$$\ddot{x} + 4\dot{x} + 13x = 3 \tag{3.61}$$

where the dot notation indicates derivatives with respect to time, that is, $\dot{x} = dx/dt$, $\ddot{x} = d^2x/dt^2$. The initial conditions given are $x(0) = 5$, and $\dot{x}(0) = 8$. The transient solution is given by

$$m^2 + 4m + 13 = 0$$

$$m = -2 \pm \sqrt{4 - 13} = -2 \pm j3$$

$$x_t = e^{-2t}(A \cos 3t + B \sin 3t)$$

Assume the particular solution of the form

$$x_p = D$$

$$\dot{x} = \frac{dx_p}{dt} = 0$$

$$\ddot{x}_p = 0$$

Substituting into Equation 3.61, $D = 3/13$ for the total general solution

$$x(t) = e^{-2t}(A \cos 3t + B \sin 3t) + 3/13$$

To solve for A and B, the initial conditions specified above are used.

$$x(0) = 5 = A + 3/13$$

or

$$A = 62/13$$

Differentiating the total general solution,

$$\dot{x}(t) = e^{-2t} [(3B \cos 3t - 3A \sin 3t) - 2e^{-2t} (A \cos 3t + B \sin 3t)]$$

Substituting the second initial condition

$$\dot{x}(0) = 8 = 3B - 2A$$

$$B = \frac{76}{13}$$

Therefore, the complete solution to Equation 3.61 with the given initial conditions is

$$x(t) = e^{-2t} [(62/13) \cos 3t + (76/13) \sin 3t] + 3/13$$

First and second order differential equations have been discussed in some detail. It is of great importance to note that many higher order systems quite naturally decompose into first and second order systems. For example, the study of a third order equation (or system) may be conducted by examining a first and a second order system, a fourth order system analyzed by examining two second order systems, etc. All these cases are handled by solving the characteristic equation to get a transient solution and then obtaining the particular solution by any convenient method.

A few remarks are appropriate regarding the second order linear differential equation with constant coefficients. Although the equation is interesting in its own right, it is of particular value because it is a mathematical model for several problems of physical interest.

$$a \frac{dy^2}{dx^2} + b \frac{dy}{dx} + cy = f(x) \quad \text{(mathematical model)} \tag{3.62}$$

$$M \frac{d^2x}{dt^2} + D \frac{dx}{dt} + Kx = f(t) \quad \text{(describes a mass spring damper system)} \tag{3.63}$$

$$L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} = E(t) \quad \text{(describes a series LRC electrical circuit)} \tag{3.64}$$

Equations 3.62, 3.63, and 3.64 are all the same mathematically, but are expressed in different notation. Different notations or symbols are employed to emphasize the physical parameters involved, or to force the solution to appear in a form that is easy to interpret. In fact, the similarity of these last two equations may suggest how one might design an electrical circuit to simulate the operation of a mechanical system.

3.5 APPLICATIONS AND STANDARD FORMS

Up to this point, differential equations in general and linear differential equations with constant coefficients have been considered. Methods for solving first and second order equations of the following type have been developed:

$$a \frac{dx}{dt} + bx = f(t) \tag{3.65}$$

$$a \frac{d^2x}{dt^2} + b \frac{dx}{dt} + cx = f(t) \tag{3.66}$$

These two equations are mathematical models or forms. These same forms may be used to describe diverse physical systems. This section will concentrate on the transient response of the systems under investigation.

3.5.1 First Order Equation

EXAMPLE

$$4\dot{x} + x = 3 \tag{3.67}$$

Physically, x can represent distance or displacement, where t is used to represent time. The transient solution can be found from the homogeneous equation.

$$4\dot{x} + x = 0$$

$$(4m + 1)x = 0$$

$$4m + 1 = 0$$

$$m = -1/4$$

Thus

$$x_t = ce^{-t/4}$$

The particular solution is found by assuming

$$x_p = A$$

$$\frac{dx_p}{dt} = 0$$

Substitute

$$A = 3$$

or

$$x_p = 3$$

The total general solution is then

$$x = ce^{-t/4} + 3 \tag{3.68}$$

The first term on the right of Equation 3.68 represents the transient response of the physical system described by Equation 3.67, and the second term represents the steady state response if the transient decays. A term useful in describing the physical effect of a negative exponential term is time constant which is denoted by τ. The time constant is defined by

$$\tau = -\frac{1}{m}$$

Thus, Equation 3.68 could be rewritten as

$$x = ce^{-t/\tau} + 3 \tag{3.69}$$

where $\tau = 4$.

Note the following points:

1. The time constant is discussed only if m is negative. If m is positive, the exponent of e is positive, and the transient solution will not decay.

2. If m is negative, τ is positive.

3. τ is the negative reciprocal of m, so that small numerical values of m give large numerical values of τ (and vice versa).

4. The value of τ is the time, in seconds, required for the displacement to decay to 1/e of its original displacement from equilibrium or steady value. To get a better understanding of this statement, examine Equation 3.69

$$x = ce^{-t/\tau} + 3 \tag{3.69}$$

and let $t = \tau$. Then

$$x = ce^{-1} + 3 = c\frac{1}{e} + 3 \tag{3.70}$$

Thus, when $t = \tau$, the exponential portion of the solution has decayed to 1/e of its original displacement as shown in Figure 3.3.

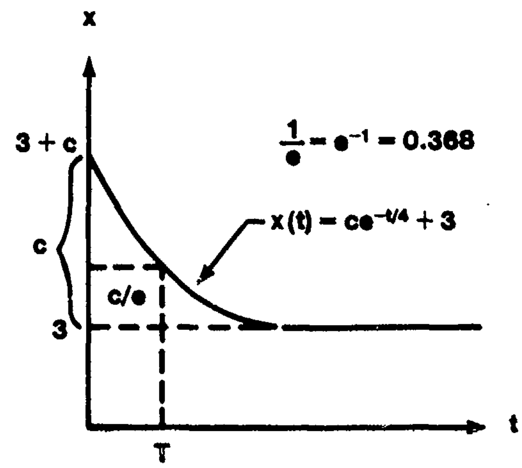


FIGURE 3.3. EXAMPLE OF FIRST ORDER EXPONENTIAL DECAY WITH AN ARBITRARY CONSTANT

Other measures of time are sometimes used to describe the decay of the exponential of a solution. If T_1 is used to denote the time it takes for the transient to decay to one-half its original amplitude, then

$$T_1 = 0.693\ \tau \tag{3.71}$$

This relationship can be easily shown by investigating

$$x = c_1 e^{-at} + c_2 \tag{3.72}$$

By definition, $\tau = 1/a$. T_1 is the value of t at which $x_t = 1/2\ x_t(0)$. Solving

$$x_t = c_1 e^{-at}$$

$$1/2\ x_t(0) = 1/2\ c_1 = c_1 e^{-aT_1}$$

$$e^{-aT_1} = 1/2$$

$$-\ln 1/2 = aT_1$$

$$T_1 = \frac{-\ln 1/2}{a} = \frac{0.693}{a} = 0.693\tau$$

The solution of Equation 3.67 can be completed by specifying a boundary condition and evaluating the arbitrary constant. Let $x = 0$ at $t = 0$.

$$x = ce^{-t/4} + 3$$

$$x(0) = 0 = c + 3$$

$$c = -3$$

The complete solution for this boundary condition is

$$x = -3e^{-t/4} + 3$$

as shown in Figure 3.4.

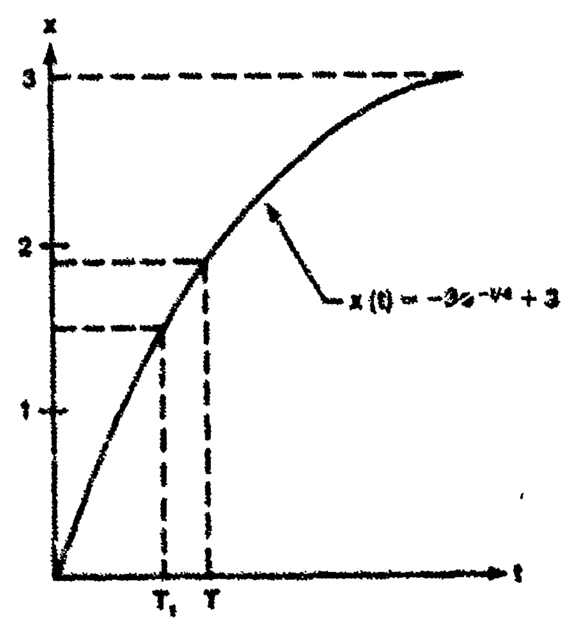


FIGURE 3.4. EXAMPLE OF FIRST ORDER EXPONENTIAL DECAY

3.5.2 Second Order Equations

Consider an equation of the form of Equation 3.66

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t) \tag{3.66}$$

As discussed earlier, the characteristic equation can be written in operator notation as

$$m^2 + bm + c = 0 \tag{3.39}$$

where roots can be represented by

$$m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3.40}$$

These quadratic roots determine the form of the transient solution. The physical implications of solutions for various values of m will now be discussed.

3.5.2.1 Case 1: Roots Real and Unequal. When the roots are real and unequal, the transient solution has the form

$$x_t = c_1e^{m_1t} + c_2e^{m_2t} \tag{3.73}$$

When m_1 and m_2 are both negative, the system decays and there will be a time constant associated with each exponential as shown in Figure 3.5.

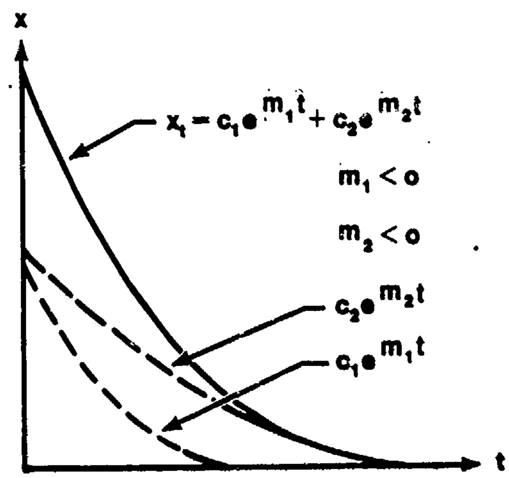


FIGURE 3.5. SECOND ORDER TRANSIENT RESPONSE WITH REAL, UNEQUAL, NEGATIVE ROOTS

When m_1 or m_2 (or both) is positive, the system will generally diverge as shown in Figures 3.6 and 3.7.

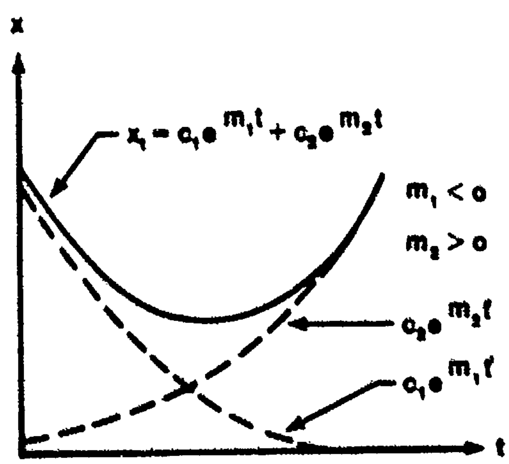


FIGURE 3.6. SECOND ORDER TRANSIENT RESPONSE WITH ONE POSITIVE AND ONE NEGATIVE REAL, UNEQUAL ROOTS

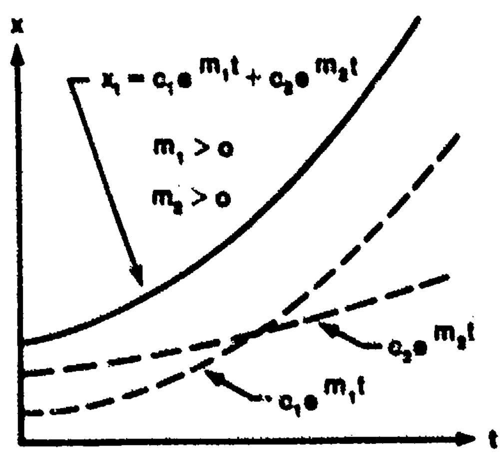


FIGURE 3.7. SECOND ORDER TRANSIENT RESPONSE WITH REAL, UNEQUAL, POSITIVE ROOTS

3.5.2.2 Case 2: Roots Real and Equal. When $m_1 = m_2$, the transient solution has the form

$$x_t = c_1 e^{mt} + c_2 t e^{mt} \tag{3.74}$$

When m is negative, the system will usually decay as shown in Figure 3.8. If m is very small, the system may initially exhibit divergence.

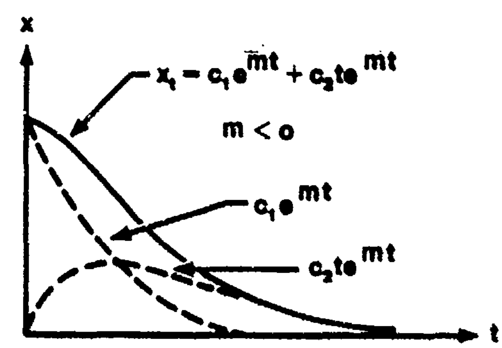


FIGURE 3.8. SECOND ORDER TRANSIENT RESPONSE WITH REAL, EQUAL, NEGATIVE ROOTS

When m is positive, the system will diverge much the same way as shown in Figure 3.7.

3.5.2.3 Case 3: Roots Purely Imaginary. When $m = \pm jk$, the transient solution has the form

$$x_t = c_1 \sin kt + c_2 \cos kt \tag{3.75}$$

or

$$x_t = A \sin (kt + \phi) \tag{3.76}$$

or

$$x_t = A \cos (kt + \theta) \tag{3.77}$$

The system executes oscillations of constant amplitude with a frequency k as shown in Figure 3.9.

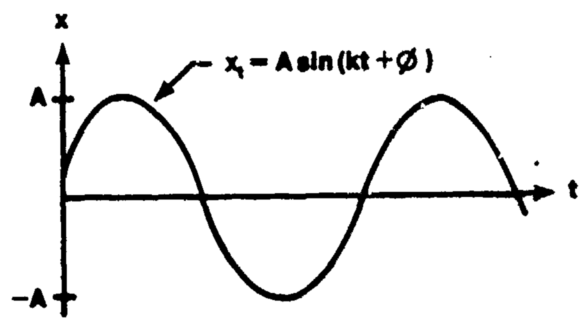


FIGURE 3.9. SECOND ORDER TRANSIENT RESPONSE WITH IMAGINARY ROOTS

3.5.2.4 <u>Case 4: Roots Complex Conjugates</u>. When the roots are given by $m = k_1 \pm jk_2$, the form of the transient solution is

$$x_t = e^{k_1 t} (c_1 \cos k_2 t + c_2 \sin k_2 t) \tag{3.78}$$

or

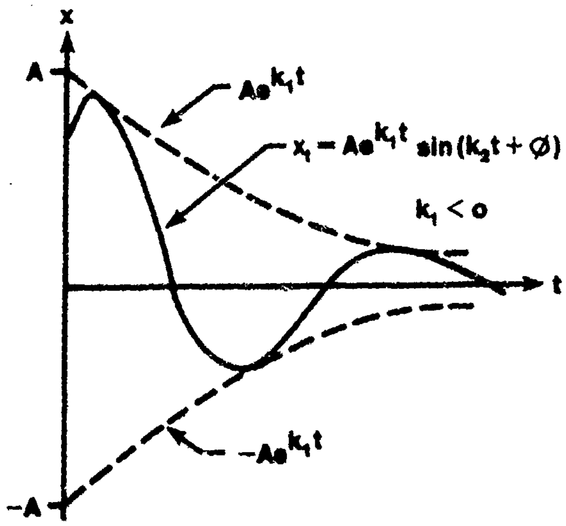
$$x_t = A e^{k_1 t} \sin (k_2 t + \phi) \tag{3.79}$$

or

$$x_t = A e^{k_1 t} \cos (k_2 t + \theta) \tag{3.80}$$

The system executes periodic oscillations contained in an envelope given by $x = \pm e^{k_1 t}$.

When k_1 is negative, the system decays or converges as shown in Figure 3.10. When k_1 is positive, the system diverges as shown in Figure 3.11.



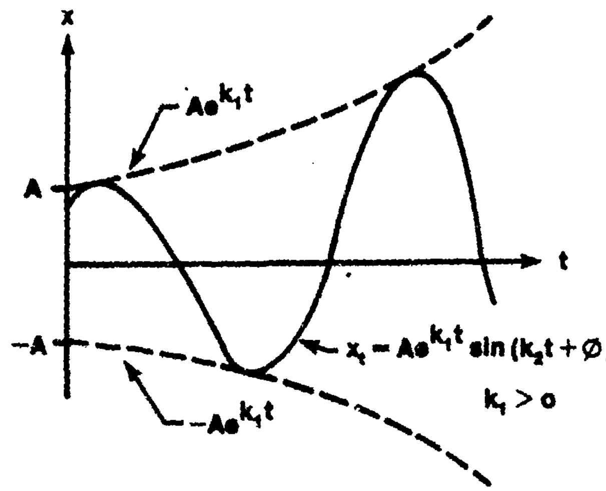


FIGURE 3.10. SECOND ORDER CONVERGENT TRANSIENT RESPONSE WITH COMPLEX CONJUGATE ROOTS

FIGURE 3.11. SECOND ORDER DIVERGENT TRANSIENT RESPONSE WITH COMPLEX CONJUGATE ROOTS

The discussion of transient solutions above reveals only part of the picture presented by Equation 3.66. The input or forcing function is still left to consider, that is, f(t). In practice, a linear system that possesses a divergence (without input) may be changed to a damped system by carefully selecting or controlling the input. Conversely, a nondivergent linear system with weak damping may be made divergent by certain types of inputs. Chapter 13, Linear Control Theory, will examine these problems in detail.

3.5.3 Second Order Linear Systems

Consider the physical model shown in Figure 3.12. The system consists of an object suspended by a spring, with a spring constant of K. The mass represented by M may move vertically and is subject to gravity, input, and damping, with the total viscous damping constant equal to D.

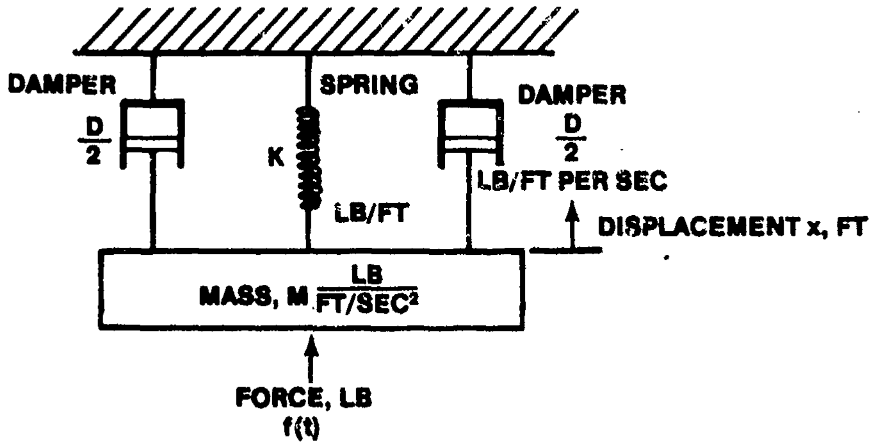


FIGURE 3.12. SECOND ORDER MASS, SPRING, DAMPER SYSTEM

The equation for this system is given by

$$M\ddot{x} + D\dot{x} + Kx = f(t) \tag{3.81}$$

The characteristic equation in operator notation is given by

$$Mm^2 + Dm + K = 0 \tag{3.82}$$

The roots of this equation can be written

$$m_{1,2} = \frac{-D}{2M} \pm \sqrt{\left(\frac{D}{2M}\right)^2 - \frac{K}{M}} \tag{3.83}$$

$$m_{1,2} = \frac{-D}{2M} \pm \sqrt{\frac{K}{M}} \sqrt{\frac{D^2}{4KM} - 1} \tag{3.84}$$

For simplicity, and for reasons that will be obvious later three constants are defined

$$\zeta \equiv \frac{D}{2\sqrt{MK}} \tag{3.85}$$

the term ζ is called the damping ratio, and is a value which indicates the damping strength in the system.

$$\omega_n \equiv \sqrt{\frac{K}{M}} \tag{3.86}$$

ω_n is the undamped natural frequency of the system. This is the frequency at which the system would oscillate if there were no damping present.

$$\omega_d \equiv \omega_n \sqrt{1 - \zeta^2} \tag{3.87}$$

ω_d is the damped frequency of the system. It is the frequency at which the system oscillates when a damping ratio of ζ is present.

Substituting the definitions of ζ and ω_n into Equation 3.84 gives

$$m_{1,2} = -\zeta\,\omega_n \pm j\omega_n \sqrt{1 - \zeta^2} \tag{3.88}$$

With these roots, the transient solution becomes

$$x_t = c_1 e^{m_1 t} + c_2 e^{m_2 t} \tag{3.89}$$

which can be written as

$$x_t = e^{-\zeta\omega_n t}\left[c_3 \cos \omega_n \sqrt{1 - \zeta^2}\, t + c_4 \sin \omega_n \sqrt{1 - \zeta^2}\, t\right] \tag{3.90}$$

or

$$x_t = A\, e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1 - \zeta^2}\, t + \phi\right) \tag{3.91}$$

The solution will lie within an exponentially decreasing envelope which has a time constant of $1/(\zeta\omega_n)$. This damped oscillation is shown in Figure 3.13. From Equation 3.91 and Figure 3.14, note that the numerical value of damping ratio has a powerful effect on system response.

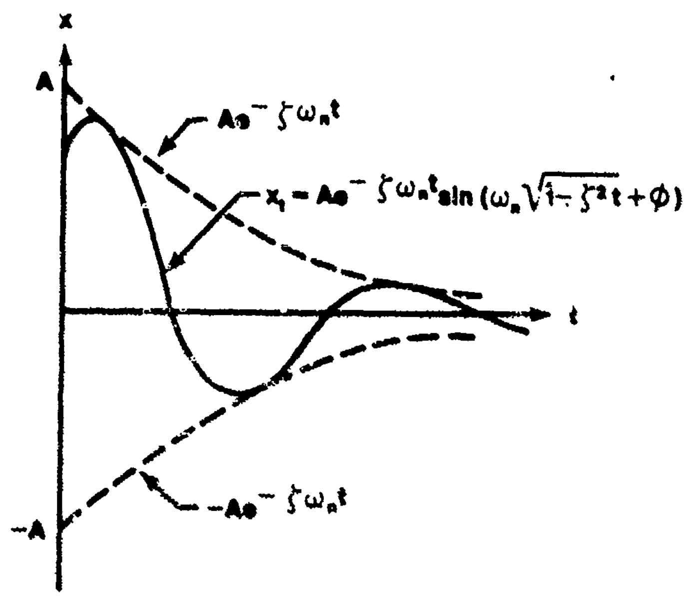


FIGURE 3.13. SECOND ORDER DAMPED OSCILLATIONS

If Equation 3.81 is divided by M

$$\ddot{x} + \frac{D}{M}\dot{x} + \frac{K}{M}x = \frac{f(t)}{M} \tag{3.92}$$

or, rewriting using ω_n and ζ defined by Equations 3.85 and 3.86

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{f(t)}{M} \tag{3.93}$$

Equation 3.93 is a form of Equation 3.81 that is useful in analyzing the behavior of any second order linear system. In general, the magnitude and sign of damping ratio determine the response properties of the system. There are five distinct cases which are given names descriptive of the response associated with each case. These are:

1. $\zeta = 0$, undamped

2. $0 < \zeta > 1$, underdamped

3. $\zeta = 1$, critically damped

4. $\zeta > 1$, overdamped

5. $\zeta < 0$, unstable.

Each case will be examined in turn, making use of Equation 3.88, repeated below

$$m_{1,2} = -\zeta\omega_n \pm j\omega_n \sqrt{1 - \zeta^2} \qquad (3.88)$$

3.5.3.1 Case 1: $\zeta = 0$, Undamped. For this condition, the roots of the characteristic equation are

$$m_{1,2} = \pm j\omega_n$$

giving a transient solution of the form

$$x_t = c_1 \cos \omega_n t + c_2 \sin \omega_n t \qquad (3.94)$$

or

$$x_t = A \sin (\omega_n t + \phi) \qquad (3.95)$$

showing the system to have the transient response of an undamped sinusoidal oscillation with frequency ω_n. Hence, the designation of ω_n as the "undamped natural frequency." Figure 3.9 shows an undamped system.

3.5.3.2 Case 2: $0 < \zeta < 1.0$, Underdamped. For this case, m is given by Equation 3.88.

$$m_{1,2} = -\zeta\omega_n \pm j\omega_n \sqrt{1 - \zeta^2} \qquad (3.88)$$

The transient solution has the form

$$x_t = A\,e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1 - \zeta^2}\, t + \phi\right) \qquad (3.96)$$

This solution shows that the system oscillates at the damped frequency, ω_d, and is bounded by an exponentially decreasing envelope with time constant $1/(\zeta\,\omega_n)$. Figure 3.14 shows the effect of increasing the damping ratio from 0.1 to 1.0.

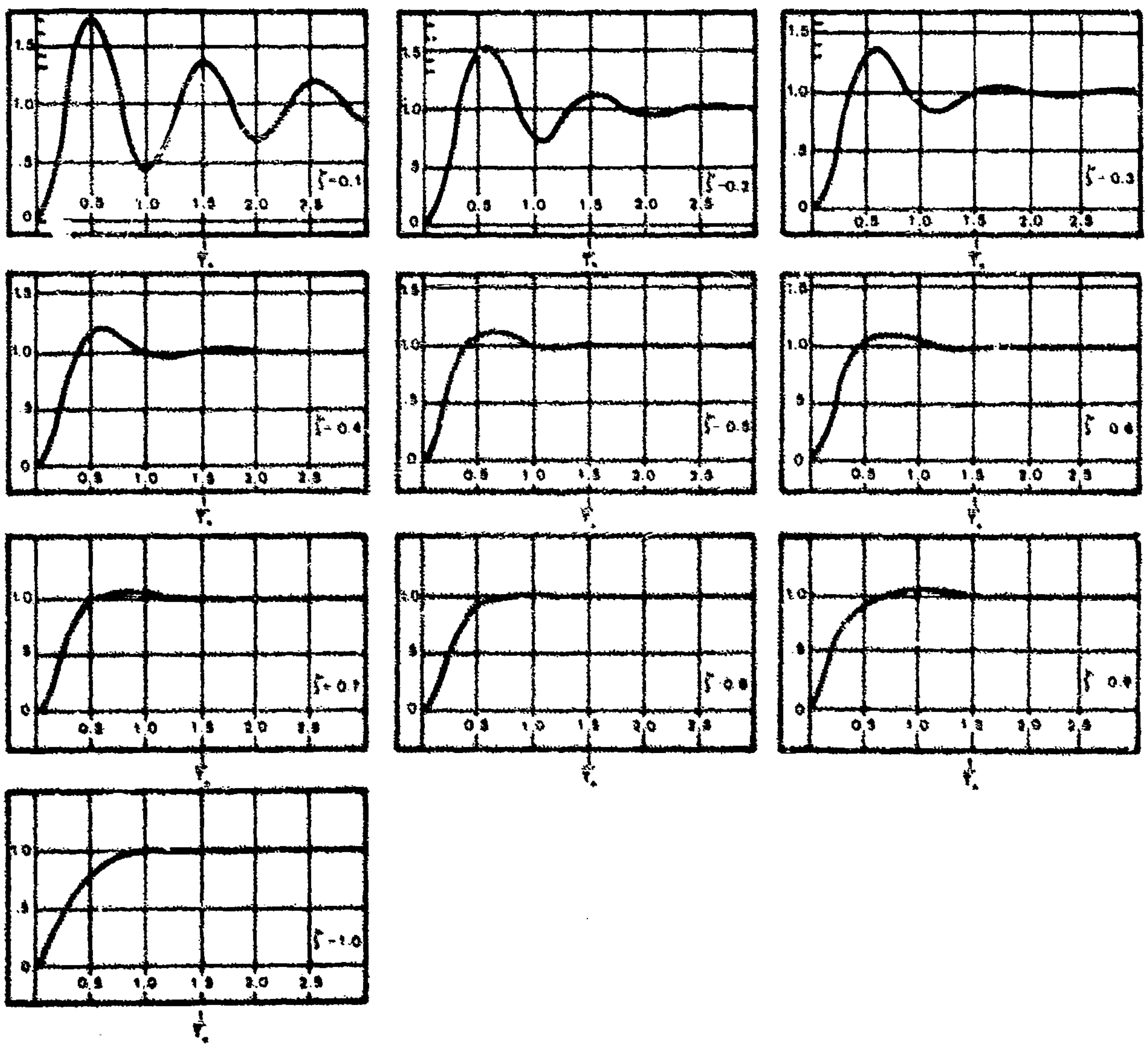


FIGURE 3.14. SECOND ORDER SYSTEM RESPONSE FOR DAMPING RATIOS BETWEEN ZERO AND ONE

3.5.3.3 Case 3: ζ 1.0, Critically Damped. For this condition, the roots of the characteristic equation are

$$m_{1,2} = -\omega_n \tag{3.97}$$

which gives a transient solution of the form

$$x_t = c_1 e^{-\omega_n t} + c_2 t e^{-\omega_n t} \tag{3.98}$$

This is called the critically damped case and generally will not overshoot. It should be noted, however, that large initial values of x can cause one overshoot. Figure 3.14 shows a response when $\zeta = 1.0$.

3.5.3.4 Case 4: $\zeta > 1.0$, Overdamped. In this case, the characteristic roots are

$$m_{1,2} = -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1} \tag{3.99}$$

which shows that both roots are real and negative. The system will have a transient which has an exponential decay without sinusoidal motion. The transient response is given by

$$x_t = c_1 e^{-\omega_n \left[\zeta - \sqrt{(\zeta^2 - 1)}\right] t} + c_2 e^{-\omega_n \left[\zeta + \sqrt{(\zeta^2 - 1)}\right] t} \tag{3.100}$$

This response can also be written as

$$x_t = c_1 e^{-t/\tau_1} + c_2 e^{-t/\tau_2} \tag{3.101}$$

where τ_1 and τ_2 are time constants for each exponential term.

This solution is the sum of two decreasing exponentials, one with time constant τ_1 and the other with time constant τ_2. The smaller the value of τ,

the quicker the transient decays. Usually the larger the value of ζ, the larger τ_1 is compared to τ_2. Figure 3.5 shows an overdamped system.

3.5.3.5 Case 5: $-1.0 < \zeta < 0$, Unstable. For the first Case 5 example, the roots of the characteristic equation are

$$m_{1,2} = -\zeta\omega_n \pm j\omega_n \sqrt{1 - \zeta^2} \quad . \tag{3.102}$$

These roots are the same as for the underdamped case, except that the exponential term in the transient solution shows an exponential increase with time.

$$x_t = e^{-\zeta\omega_n t} \left[c_1 \cos \omega_n \sqrt{1 - \zeta^2}\, t + c_2 \sin \omega_n \sqrt{1 - \zeta^2}\, t \right] \tag{3.103}$$

Whenever a term appearing in the transient solution grows with time (and especially an exponential growth), the system is generally unstable. This means that whenever the system is disturbed from equilibrium the disturbance will increase with time. Figure 3.11 shows an unstable system

Case 5: $\zeta = -1.0$, Unstable. For this second Case 5 example, the roots of the characteristic equation are

$$m_{1,2} = +\omega_n \tag{3.104}$$

and

$$x_t = e^{\omega_n t} (c_1 + c_2 t) \tag{3.105}$$

This case diverges much the same way as shown in Figure 3.7.

$$m_{1,2} = -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1} \tag{3.99}$$

The response can be written as the sum of two exponential terms

$$x_t = c_1 e^{m_1 t} + c_2 e^{m_2 t}$$

where the values of m can be determined from Equation 3.99.

Five examples will illustrate some of these system response cases.

Case 5: $\zeta < -1.0$, Unstable. This third Case 5 example is similar to Case 4, except that the system diverges as shown in Figure 3.7.

$$m_{1,2} = -\zeta\,\omega_n \pm \omega_n \sqrt{\zeta^2 - 1} \qquad (3.99)$$

The response can be written as the sum of two exponential terms

$$x_t = c_1 e^{m_1 t} + c_2 e^{m_2 t}$$

where the values of m can be determined from Equation 3.99.

Five examples will illustrate some of these system response cases.

EXAMPLE

Given the homogeneous equation,

$$\ddot{x} + 4x = 0$$

from Equation 3.93,

$$\zeta = 0$$

and

$$\omega_n = 2.0$$

The system is undamped with a solution

$$x_t + A \sin (2t + \phi)$$

where A and ϕ are constants of integration which could be determined by substituting boundary conditions into the total general solution.

EXAMPLE

Given the homogeneous equation

$$\ddot{x} + \dot{x} + x = 0$$

from Equation 3.93,

$$\omega_n = 1.0$$

and

$$\zeta = 0.5$$

Also from Equation 3.87, the definition of damped frequency

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 0.87$$

The system is underdamped with a solution

$$x_t = Ae^{-0.5t} \sin (0.87\, t + \phi)$$

EXAMPLE

Given the homogeneous equation

$$\frac{\ddot{x}}{4} + \dot{x} + x = 0$$

Multiply by four to get the equation in the form of Equation 3.93.

Then

$$\ddot{x} + 4\dot{x} + 4x = 0$$

and

$$\omega_n = 2.0$$

$$\zeta = 1.0$$

The system is critically damped and has a solution given by

$$x_t = c_1 e^{-2t} + c_2 t e^{-2t} \quad .$$

EXAMPLE

Given the homogeneous equation

$$\ddot{x} + 8\dot{x} + 4x = 0$$

from Equation 3.93,

$$\omega_n = 2.0$$

and

$$\zeta = 2.0$$

The system is overdamped and has a solution

$$x_t = c_1 e^{-7.46t} + c_2 e^{-0.54t}$$

EXAMPLE

Given the homogeneous equation

$$\ddot{x} - 2\dot{x} + 4x = 0$$

from Equation 3.93,

$$\omega_n = 2.0$$

and

$$\zeta = -0.5$$

From Equation 3.87, the definition of damped frequency

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 1.7$$

The solution is unstable (negative damping) and has the form

$$x_t = Ae^{t} \sin (1.7\ t + \phi)$$

In summary, the best damping ratio for a system is determined by the intended use of the system. If a fast response is desired and the size and number of overshoots is inconsequential, then a small value of damping ratio would be desired. If it is essential that the system not overshoot and response time is not too critical, a critically damped (or even an overdamped) system could be used. The value of damping ratio of 0.7 is often referred to as an optimum damping ratio since it gives a small overshoot and a relatively quick response. The optimum damping ratio will change as the requirements of the physical system change.

3.6 ANALOGOUS SECOND ORDER LINEAR SYSTEMS

3.6.1 Mechanical System

The second order equation which has been examined in detail represents the mass-spring-damper system of Figure 3.12 and has a differential equation which was given by

$$M\ddot{x} + D\dot{x} + Kx = f(t) \tag{3.81}$$

Using the definitions

$$\zeta \equiv \frac{D}{2\sqrt{MK}} \quad (3.85) \quad \text{and} \quad \omega_n \equiv \sqrt{\frac{K}{M}} \tag{3.86}$$

Equation 3.81 was rewritten as

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{f(t)}{M} \tag{3.93}$$

3.6.2 Electrical System

The second order equation can also be applied to the series LRC circuit shown in Figure 3.15.

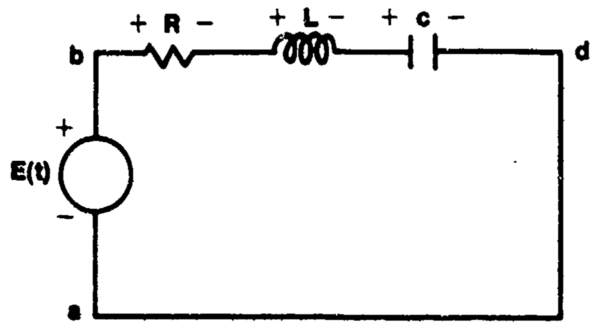


FIGURE 3.15. SERIES ELECTRICAL CIRCUIT

where

L = inductance

R = resistance

C = capacitance

q = charge

i = current

Assume $q(0) = \dot{q}(0) = 0$, then Kirchhoff's voltage law gives

$$\Sigma V_{abd} = 0$$

or

$$E(t) - V_R - V_L - V_C = 0$$

$$E(t) - iR - L\frac{di}{dt} - \frac{1}{C}\int_0^t idt = 0$$

Since

$$i = \frac{dq}{dt}$$

$$E(t) = L\ddot{q} + R\dot{q} + \frac{q}{c} \quad (3.106)$$

The following parameters can now be defined.

$$\omega_n \equiv \sqrt{\frac{1}{LC}} \quad (3.107)$$

$$\zeta \equiv \frac{R}{2\sqrt{L/C}} \quad (3.108)$$

and

$$2\zeta\omega_n = \frac{R}{L} \quad (3.109)$$

Using these parameters, Equation 3.106 can be written

$$\ddot{q} + 2\zeta\omega_n\dot{q} + \omega_n^{2} q = \frac{E(t)}{L} \quad (3.110)$$

3.6.3 Servomechanisms

For linear control systems work in Chapter 13, the applicable second order equation is

$$I\ddot{\theta}_0 + f\,\dot{\theta}_0 + \mu\,\theta_0 = \mu\,\theta_i \quad (3.111)$$

where

I = inertia

f = friction

μ = gain

θ_i = input

θ_0 = output

Rearranging Equation 3.111

$$\ddot{\theta}_0 + \frac{f}{I}\dot{\theta}_0 + \frac{\mu}{I}\theta_0 = \frac{\mu}{I}\theta_i \tag{3.112}$$

or

$$\ddot{\theta}_0 + 2\,\zeta\omega_n\,\dot{\theta}_0 + \omega_n^{\,2}\,\theta_0 = \omega_n^{\,2}\,\theta_i \tag{3.113}$$

where the following parameters are defined

$$\omega_n \equiv \sqrt{\frac{\mu}{I}} \tag{3.114}$$

$$\zeta \equiv \frac{f}{2\sqrt{\mu\, I}} \tag{3.115}$$

Thus, in general, any second order differential equation can be written in the form

$$\ddot{x} + 2\,\zeta\omega_n\,\dot{x} + \omega_n^{\,2}x = f_1(t) \tag{3.116}$$

where each term has the same qualitative significance, but different physical significance.

3.7 LAPLACE TRANSFORMS

A technique has been presented for solving linear differential equations with constant coefficients, with and without inputs or forcing functions. The method has limitations. It is suited for differential equations with inputs of only certain forms. Further, solution procedures require looking for special cases which require careful handling. However, these procedures have the remarkable property of changing or "transforming" a problem of integration into a problem in algebra, that is, solving a quadratic equation in the case of linear second order differential equations. This is accomplished by making an assumption involving the number e.

Given the second order homogeneous equation

$$a\ddot{x} + b\dot{x} + cx = 0 \tag{3.117}$$

The following solution is assumed

$$x_t = e^{mt} \tag{3.118}$$

Substituting into Equation 3.117 gives

$$am^2e^{mt} + bme^{mt} + ce^{mt} = 0 \tag{3.119}$$

and, factoring the exponential term

$$e^{mt}\,(am^2 + bm + c) = 0 \tag{3.120}$$

leading to the assertion that Equation 3.118 will produce a solution to Equation 3.117 if m is a root of the characteristic equation

$$am^2 + bm + c = 0 \tag{3.121}$$

Introducing operator notation, $p = d/dt$, the characteristic equation can be written by inspection.

$$ap^2 + bp + c = 0 \tag{3.122}$$

Equation 3.122 can then be solved for p to give a solution of the form

$$x_t = c_1e^{p_1t} + c_2e^{p_2t} \tag{3.123}$$

Of course, the great shortcoming of this method is that it does not provide a solution to an equation of the form

$$a\ddot{x} + b\dot{x} + cx = f(t) \tag{3.124}$$

It works only for the homogeneous equation. Still, a solution to the equation can be found by obtaining a particular solution and adding it to the transient solution of the homogeneous equation. The technique used to obtain the particular solution, the method of undetermined coefficients, also provides a solution by algebraic manipulation.

However, there is a technique which exchanges (transforms) the whole differential equation, including the input and initial conditions into an algebra problem. Fortunately, the method applies to linear first and second order equations with constant coefficients.

In Equation 3.124, x is a function of t. For emphasis, Equation 3.124 can be rewritten

$$a\ddot{x}(t) + b\dot{x}(t) + cx(t) = f(t) \qquad (3.125)$$

Multiplying each term of Equation 3.125 by the integrating factor e^{mt} gives

$$a\ddot{x}(t)e^{mt} + b\dot{x}(t)e^{mt} + cx(t)e^{mt} = f(t)e^{mt} \qquad (3.126)$$

It is now possible that Equation 3.126 can be integrated term by term on both sides of the equation to produce an algebraic expression in m. The algebraic expression can then be manipulated to eventually obtain the solution of Equation 3.125.

The new integrating factor e^{mt} should be distinguished from the previous integrating factor used in developing the operator techniques for solving the homogeneous equation. In order to accomplish this, m will be replaced by -s. The reason for the minus sign will be apparent later. In order to integrate the terms in Equation 3.126, limits of integration are required. In most physical problems, events of interest take place subsequent to a given starting time which is called t = 0. To be sure to include the duration of all significant events, the composite of effects from time t = 0 to time t = ∞ will be included. Equation 3.126 now becomes

$$\int_0^\infty a\ddot{x}(t)\, e^{-st}\, dt + \int_0^\infty b\,\dot{x}(t)\, e^{-st}\, dt + \int_0^\infty c\, x(t)\, e^{-st}\, dt$$

$$= \int_0^\infty f(t)\, e^{-st}\, dt \tag{3.127}$$

Equation 3.127 is called the Laplace transform of Equation 3.125. The problem now is to integrate the terms in the equation.

3.7.1 Finding the Laplace Transform of a Differential Equation

The integrals of the terms of Equation 3.127 must now be found. The Laplace transform is defined as

$$\int_0^\infty x(t)\, e^{-st}\, dt \equiv L\ [x(t)] \equiv X(s) \tag{3.128}$$

where the letter L is used to signify a Laplace transform. X(s) must, for the present, remain an unknown. (m was carried along as an unknown until the characteristic equation evolved, at which time m was solved for explicitly.) Since Equation 3.128 transforms x(t) into a function of the variable, s,

then

$$\int_0^\infty c\, x(t)\, e^{-st}\, dt = c\int_0^\infty x(t)\, e^{-st}\, dt = cX(s) \tag{3.129}$$

and X(s) will be carried along until such time that it can be solved for.

The transform for the second term, b x(t) is given by

$$\int_0^\infty b\, \dot{x}(t)\, e^{-st}\, dt = b\int_0^\infty \dot{x}(t)\, e^{-st}\, dt \tag{3.130}$$

To solve Equation 3.130, a useful formula known as integration by parts is used

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du \tag{3.131}$$

Applying this formula to Equation 3.130, let

$$u = e^{-st}$$

and

$$dv = \dot{x}(t)\,dt$$

then

$$du = -se^{-st}\,dt$$

and

$$v = x(t)$$

Substituting these values into Equation 3.131 and integrating from $t = 0$ to $t = \infty$

$$\int_0^\infty \dot{x}(t)e^{-st}\,dt = x(t)e^{-st}\Big|_0^\infty - \int_0^\infty x(t)\left[-se^{-st}\right]dt$$

$$= x(t)e^{-st}\Big|_0^\infty + s\int_0^\infty x(t)e^{-st}\,dt$$

$$= x(t)e^{-st}\Big|_0^{\infty} + sX(s) \tag{3.132}$$

Now

$$x(t)e^{-st}\Big|_0^{\infty} = \lim_{t \to \infty} x(t)e^{-st} - x(0) \tag{3.133}$$

and assume that the term e^{-st} "dominates" the term $x(t)$ as $t \to \infty$. The reason for using the minus sign in the exponent should now be apparent. Thus, $\lim_{t \to \infty} x(t)e^{-st} = 0$, and Equation 3.131 becomes

$$\int_0^{\infty} \dot{x}(t)e^{-st}\,dt = 0 - x(0) + sX(s) = sX(s) - x(0) \tag{3.134}$$

Equations 3.129 and 3.134 can now be abbreviated to signify Laplace transformations.

$$L\ \{x(t)\} = X(s) \tag{3.135}$$

$$L\ \{cx(t)\} = cX(s) \tag{3.136}$$

$$L\ \{\dot{x}(t)\} = sX(s) - x(0) \tag{3.137}$$

$$L\ \{b\dot{x}(t)\} = b\Big[sX(s) - x(0)\Big] \tag{3.138}$$

Equation 3.138 can be extended to higher order derivatives. Such an extension gives

$$L\ \{a\ddot{x}(t)\} = a\Big[s^2X(s) - sx(0) - \dot{x}(0)\Big] \tag{3.139}$$

Returning to Equation 3.127, note that the Laplace transforms of all the terms except the forcing function have been found. To solve this transform, the forcing function must be specified. A few typical forcing functions will be considered to illustrate the technique for finding Laplace transforms.

EXAMPLE

$$f(t) = A = \text{constant}$$

Then

$$L\,[A] = \int_0^{\infty} Ae^{-st}\,dt = -\frac{A}{s}\int_0^{\infty} e^{-st}(-s dt) = -\frac{A}{s}\,e^{-st}\Big|_0^{\infty}$$

or

$$L\,[A] = \frac{A}{s} \tag{3.140}$$

EXAMPLE

$$f(t) = t$$

Then

$$L\,[t] = \int_0^{\infty} te^{-st}\,dt$$

To integrate by parts, let

$$u = t$$

$$dv = e^{-st}\,dt$$

Then

$$du = dt$$

$$v = -\frac{1}{s}\,e^{-st}$$

Substituting into Equation 3.131

$$\int_0^\infty te^{-st}\,dt = \frac{-te^{-st}}{s}\Bigg|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt$$

$$= 0 - \frac{1}{s^2}e^{-st}\Bigg|_0^\infty = 0 + \frac{1}{s^2}$$

or

$$L(t) = \frac{1}{s^2} \tag{3.141}$$

EXAMPLE

$$f(t) = e^{2t}$$

Then

$$L(e^{2t}) = \int_0^\infty e^{2t}e^{-st}\,dt = \int_0^\infty e^{(2-s)t}dt = \frac{1}{s-2}$$

or

$$L(e^{2t}) = \frac{1}{s-2} \tag{3.142}$$

EXAMPLE

$$f(t) = \sin at$$

Then

$$L[\sin at] = \int_0^\infty \sin at\, e^{-st}\,dt$$

Integrate by parts, letting

$$u = \sin at$$

$$dv = e^{-st}\, dt$$

Then

$$du = a\,(\cos at)\, dt$$

$$v = -\frac{1}{s} e^{-st}$$

Substituting into Equation 3.131

$$\int_0^\infty (\sin t)\, e^{-st} = \left.\frac{-(\sin at)(e^{-st})}{s}\right|_0^\infty + \frac{a}{s}\int_0^\infty (\cos at)\, e^{-st}\, dt$$

or

$$\int_0^\infty (\sin at)\, e^{-st}\, dt = 0 + \frac{a}{s}\int_0^\infty (\cos at)\, e^{-st}\, dt \qquad (3.143)$$

The expression $(\cos at)\, e^{-st}$ can also be integrated by parts, letting

$$u = \cos at$$

$$dv = e^{-st}\, dt$$

and

$$du = -a\,(\sin at)\, dt$$

$$v = \frac{1}{s} e^{-st}$$

Giving

$$\int_0^\infty (\cos at)\, e^{-st}\, dt = \left.\frac{-(\cos at)(e^{-st})}{s}\right|_0^\infty - \frac{a}{s}\int_0^\infty (\sin at)\, e^{-st}\, dt$$

or

$$\int_0^\infty (\cos at)\, e^{-st}\, dt = \frac{1}{s} - \frac{a}{s} L\{\sin at\} \qquad (3.144)$$

Substituting Equation 3.144 into Equation 3.143

$$L\{\sin at\} = 0 + \frac{a}{s}\left[\frac{1}{s} - \frac{a}{s} L\{\sin at\}\right] = \frac{a}{s^2} - \frac{a^2}{s^2} L\{\sin at\}$$

which "obviously" yields

$$L\{\sin at\} = \frac{a}{s^2 + a^2} \qquad (3.145)$$

Also note that Equation 3.143 may be written as

$$L\{\sin at\} = \frac{a}{s} L\{\cos at\}$$

which yields

$$L\{\cos at\} = \frac{s}{s^2 + a^2} \qquad (3.146)$$

The Laplace transforms of more complicated functions may be quite tedious to derive, but the procedure is similar to that above. Fortunately, it is not necessary to derive Laplace transforms each time they are needed. Extensive tables of transforms exist in most advanced mathematics and control system textbooks. All of the transforms needed for this course are listed in Table 3.2 Page 3.73.

The technique of using Laplace transforms to assist in the solution of a

differential equation is best described by an example.

EXAMPLE

Given the differential equation

$$\ddot{x} + 4\dot{x} + 4x = 4e^{2t} \tag{3.147}$$

with initial conditions $x(0) = 1$, $\dot{x}(0) = -4$. Taking the Laplace transform of the equation gives

$$s^2X(s) - sx(0) - \dot{x}(0) + 4\,[sX(s) - x(0)] + 4X(s) = \frac{4}{s-2}$$

or

$$[s^2 + 4s + 4]\;X(s) + [-s + 4 - 4] = \frac{4}{s-2}$$

Solving for X(s)

$$X(s) = \frac{s^2 - 2s + 4}{(s-2)(s+2)^2} \tag{3.148}$$

In order to continue with the solution, it is necessary to discuss partial fraction expansions.

3.7.2 Partial Fractions

The method of partial fractions enables the separation of a complicated rational proper fraction into a sum of simpler fractions. If the fraction is not proper (the degree of the numerator less than the degree of the denominator), it can be made proper by dividing the fraction and considering the remaining expression. Given a fraction of two polynomials in the variable s as shown in Equation 3.148 there occur several cases:

3.7.2.1 Case 1: Distinct Linear Factors. To each linear factor such as $(as + b)$, occurring once in the denominator, there corresponds a single partial fraction of the form, $A/(as + b)$.

EXAMPLE

$$\frac{7s - 4}{s(s - 1)(s + 2)} = \frac{A}{s} + \frac{B}{s - 1} + \frac{C}{s + 2} \tag{3.149}$$

where A, B, and C are constants to be determined.

3.7.2.2 Case 2: Repeated Linear Factors. To each linear factor, $(as + b)$, occurring n times in the denominator there corresponds a set of n partial fractions.

EXAMPLE

$$\frac{s^2 - 9s + 17}{(s - 2)^2(s + 1)} = \frac{A}{(s + 1)} + \frac{B}{(s - 2)} + \frac{C}{(s - 2)^2} \tag{3.150}$$

where A, B, and C are constants to be determined.

3.7.2.3 Case 3: Distinct Quadratic Factors. To each irreducible quadratic factor, $as^2 + bs + c$, occurring once in the denominator, there corresponds a single partial fraction of the form, $(As + B)/(as^2 + bs + c)$.

EXAMPLE

$$\frac{3s^2 + 5s + 8}{(s + 2)(s^2 + 1)} = \frac{A}{s + 2} + \frac{Bs + C}{s^2 + 1} \tag{3.151}$$

where A, B, and C are constants to be determined.

3.7.2.4 Case 4: Repeated Quadratic Factors. To each irreducible quadratic factor, $as^2 + bs + c$, occurring n times in the denominator, there corresponds a set of n partial fractions.

EXAMPLE

$$\frac{10\,s^2 + s + 36}{(s - 4)(s^2 + 4)^2} = \frac{A}{s - 4} + \frac{Bs + C}{s^2 + 4} + \frac{Ds + E}{(s^2 + 4)^2} \tag{3.152}$$

where A, B, C, D, and E are constants to be determined.

The "brute-force" technique for finding the constants will be illustrated by solving Equation 3.152. Start by finding the common denominator on the right side of Equation 3.152

$$\frac{10\,s^2 + s + 36}{(s-4)(s^2+4)^2} = \frac{A(s^2+4)^2 + (Bs+C)(s-4)(s^2+4) + (Ds+E)(s-4)}{(s-4)(s^2+4^2)} \tag{3.153}$$

Then the numerators are set equal to each other

$$10s^2 + s + 36 = A(s^2+4)^2 + (Bs+C)(s^2+4)(s-4) + (Ds+E)(s-4) \tag{3.154}$$

Since Equation 3.154 must hold for all values of s, enough values of s are substituted into Equation 3.154 to find the five constants.

1. Let $s = 4$, then Equation 3.154 becomes

$$(10)(16) + 4 + 36 = 400A$$

and

$$A = 1/2$$

2. Let $s = 2j$, then Equation 3.154 becomes

$$-40 + 2j + 36 = -4D + 2je - 8jd = 4E$$

$$-4 + 2j = -4(D+E) + 2j\,(E - 4D)$$

The real and imaginary parts must be equal to their counterparts on the opposite side of the equal sign, thus

$$(D+E) = 1$$

and

$$E - 4D = 1$$

or

$$D = 0$$

and

$$E = 1$$

3. Now let $s = 0$, then Equation 3.154 becomes

$$36 = 16A - 16\ (C) - 4E$$

and from steps 1 and 2

$$A = 1/2,\ E = 1$$

hence

$$36 = 8 - 16C - 4$$

and

$$C = -2$$

4. Let $s = 1$, then Equation 3.154 becomes

$$47 = 25\ (1/2) + (B - 2)\ (-15)\ -3$$

$$94 = 25 - 30B + 60 - 6$$

or

$$B = -\ 1/2$$

Now Equation 3.155 may be written by substituting the values of A, B, C, D, and E into Equation 3.152

$$\frac{10\ s^2 + s + 36}{(s-4)\ (s^2+4)^2} = 1/2\left(\frac{1}{s-4}\right) - 1/2\left(\frac{s+4}{s^2+4}\right)\ \frac{1}{(s^2+4)^2} \qquad (3.155)$$

Returning now to the example Laplace solution of the differential equation

$$\ddot{x} + 4\dot{x} + 4x = 4e^{2t} \tag{3.147}$$

The Laplace transformed equation was

$$X(s) = \frac{s^2 - 2s + 4}{(s - 2)(s + 2)^2} \tag{3.148}$$

which can now be expanded by partial fractions

$$\frac{s^2 - 2s + 4}{(s - 2)(s + 2)^2} = \frac{A}{s - 2} + \frac{B}{s + 2} + \frac{C}{(s + 2)^2} \tag{3.156}$$

Taking the common denominator, and setting numerators equal

$$s^2 - 2s + 4 = A(s + 2)^2 + B(s + 2)(s - 2) + C(s - 2) \tag{3.157}$$

The "brute-force" technique could again be used to solve for the constants A, B, and C by substituting different values of s into Equation 3.157. An alternate method exists for solving for the constants. Multiplying the right side of Equation 3.157 gives

$$s^2 - 2s + 4 = As^2 + 4As + 4A + Bs^2 - 4B + Cs - 2C$$

$$s^2 - 2s + 4 = (A + B)s^2 + (4A + C)s + (4A - 4B - 2C)$$

Now the coefficients of like powers of s on both sides of the equation must be equal (that is, the coefficient of s^2 on the left side equals the coefficient of s^2 on the right side, etc.). Equating gives

$$s^2 : 1 = A + B$$

$$s^1 : -2 = 4A + C$$

$$s^0 : 4 = 4A - 4B - 2C$$

Solving for the constants gives

$$A = 1/4$$

$$B = 3/4$$

$$C = -3$$

Substituting the constants into Equation 3.156 results in the expanded right side

$$X(s) = 1/4\left(\frac{1}{s-2}\right) + 3/4\left(\frac{1}{s+2}\right) - 3\left(\frac{1}{s+2}\right)^2 \qquad (3.158)$$

Another expansion method called the Heaviside Expansion Theorem can be used to solve for the constants in the numerator of distinct linear factors. This method of expansion is used extensively in Chapter 13, Linear Control Theory. If the denominator of an expansion term has a distinct linear factor, $(s - a)$, the constant for that factor can be found by multiplying $X(s)$ by $(s - a)$ and evaluating the remainder of $X(s)$ at $s = a$.

Stated mathematically the Heaviside Expansion Theorem is

$$X(s) = \frac{A}{s-a} + \ldots$$

$$A = (s-a)\,X(s)\Big|_{s = a}$$

EXAMPLE

$$X(s) = \frac{7s-4}{s(s-1)(s+2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+2}$$

$$A = sX(s)\Big|_{s=0} = \frac{7s-4}{(s-1)(s+2)}\Big|_{s=0} = \frac{-4}{(-1)(2)} = 2$$

$$B = (s-1)X(s)\Big|_{s=1} = \frac{7s-4}{s(s+2)}\Big|_{s=1} = \frac{7-4}{(1)(3)} = 1$$

$$C = (s+2)X(s)\Big|_{s=-2} = \frac{7s-4}{s(s-1)}\Big|_{s=-2} = \frac{-14-4}{(-2)(-3)} = -3$$

As another example, the constant A in the first term on the right side of Equation 3.156 can be evaluated using the Heaviside Expansion Theorem.

$$\frac{s^2-2s+4}{(s-2)(s+2)^2} = \frac{A}{s-2} + \frac{B}{s+2} + \frac{C}{(s+2)^2} \tag{3.156}$$

$$A = (s-2)X(s)\Big|_{s=2} = \frac{s^2-2s+4}{(s+2)^2}\Big|_{s=2} = \frac{4}{16} = \frac{1}{4}$$

which is the same result obtained earlier by equating like powers of s.

3.7.3 Finding the Inverse Laplace Transform

Now that methods to expand the right side of X(s) have been discussed in detail, all that remains is to transform the expanded terms back to the time domain. This is easily accomplished using any suitable transform table.

Returning to the Laplace transformed and expanded equation in the example

$$X(s) = 1/4\left(\frac{1}{s-2}\right) + 3/4\left(\frac{1}{s+2}\right) - 3\left(\frac{1}{s+2}\right)^2 \tag{3.158}$$

Using Table 3.2, it can be easily verified that Equation 3.158 can be transformed to

$$x(t) = 1/4\,e^{2t} + 3/4\,e^{-2t} - 3t\,e^{-2t} \tag{3.159}$$

In summary, the strength of the Laplace transform is that it converts linear differential equations with constant coefficients into algebraic equations in the s-domain. All that remains to do is to take the inverse transform of the explicit solutions to return to the time domain. Although the applications here at the School will consider time as the independent variable, a linear differential equation with <u>any</u> independent variable may be solved by Laplace transforms.

3.7.4 Laplace Transform Properties

There are several important properties of the Laplace transform which should be included in this discussion.

In the general case

$$L\left\{\frac{d^n x(t)}{dt^n}\right\} = s^n X(s) - \left(s^{n-1} x(0) + s^{n-2} \frac{dx(0)}{dt} + \ldots + \frac{d^{n-1}x(0)}{dt^{n-1}}\right) \qquad (3.160)$$

For quiescent systems

$$L\left\{\frac{d^n x(t)}{dt^n}\right\} = s^n X(s) \qquad (3.161)$$

This result enables transfer functions to be written by inspection.

EXAMPLE

Given the differential equation

$$\ddot{x} + 4\dot{x} + 4x = 4e^{2t} \qquad (3.162)$$

with quiescent initial conditions, the Laplace transform can immediately be written by inspection as

$$X(s)(s^2 + 4s + 4) = \frac{4}{s - 2} \qquad (3.163)$$

In most cases, reference to Table 3.2 will probably be needed to transform the right side forcing function (input).

Another significant transform is that of an indefinite integral. In the general case

$$L\left\{\iiint\ldots x(t)\,dt^n\right\} = \frac{X(s)}{s^n} + \frac{\int x(t)\,dt\Big|_{t=0}}{s^n} + \frac{\iint x(t)\,dt\Big|_{t=0}}{s^{n-1}} + \ldots \quad (3.164)$$

Equation 3.164 allows the transformation of integro-differential equations such as those arising in electrical engineering.

For the case where all integrals of f(t) evaluated at zero are zero (quiescent system) the transform becomes

$$L\left\{\iiint\ldots x(t)\,dt^n\right\} = \frac{X(s)}{s^n} \quad (3.165)$$

EXAMPLE

Given the differential equation

$$\ddot{x} + 4\dot{x} + 4x + \int x\,dt = 4e^{2t} \quad (3.166)$$

TABLE 3.2.
LAPLACE TRANSFORMS

	X(s)	x(t)
1.	$aX(s)$	$ax(t)$
2.	$a[sX(s) - x(0)]$	$a\dot{x}(t)$
3.	$a[s^2X(s) - sx(0) - \dot{x}(0)]$ (which can be extended to any necessary order)	$a\ddot{x}(t)$
4.	$\frac{1}{s}$	1
5.	$\frac{1}{s^2}$	t
6.	$\frac{n!}{s^{n+1}}$ ([illegible], 2, ...)	t^n
7.	$\frac{1}{s + a}$	e^{-at}
8.	$\frac{1}{(s + a)^2}$	te^{-at}
9.	$\frac{n!}{(s + a)^{n+1}}$ $(n = 1, 2, \ldots)$	$t^n e^{-at}$
10.	$\frac{1}{(s + a)(s + b)}$, $a \neq b$	$\frac{1}{b - a}(e^{-at} - e^{-bt})$
11.	$\frac{s}{(s + a)(s + b)}$ $a \neq b$	$\frac{1}{a - b}(ae^{-at} - be^{-bt})$
12.	$\frac{1}{(s + a)(s + b)(s + c)}$	$\frac{(b-c)e^{-at} - (a-c)e^{-bt} + (a-b)e^{-ct}}{(a-b)(b-c)(a-c)}$

TABLE 3.2
LAPLACE TRANSFORMS (continued)

	X(s)	x(t)
13.	$\frac{a}{s^2 + a^2}$	sin at
14.	$\frac{s}{s^2 + a^2}$	cos at
15.	$\frac{a^2}{s(s^2 + a^2)}$	1 - cos at
16.	$\frac{a^3}{s^2\,(s^2 + a^2)}$	at - sin at
17.	$\frac{2a^3}{(s^2 + a^2)^2}$	sin at - at cos at
18.	$\frac{2as}{(s^2 + a^2)^2}$	t sin at
19.	$\frac{2as^2}{(s^2 + a^2)^2}$	sin at + at cos at
20.	$\frac{s^2 - a^2}{(s^2 + a^2)^2}$	t cos at
21.	$\frac{(b^2 - a^2)s}{(s^2 + a^2)(s^2 + b^2)}$ $(a^2 \neq b^2)$	cos at - cos bt
22.	$\frac{b}{(s + a)^2 + b^2}$	e^{-at} sin bt
23.	$\frac{s + a}{(s + a)^2 + b^2}$	e^{-at} cos bt

with quiescent initial conditions, the Laplace transform can immediately be written by inspection as

$$X(s)\ (s^2 + 4s + 4) + \frac{X(s)}{s} = \frac{4}{s - 2}$$

The right side transform is the same as Equation 3.163. Factoring results in

$$X(s)\ (s^2 + 4s + 4 + \frac{1}{s}) = \frac{4}{s - 2} \qquad (3.167)$$

Multiplying Equation 3.167 by s gives

$$X(s)\ (s^3 + 4s^2 + 4s + 1) = \frac{4s}{s - 2}$$

which raises the order of the left side and acts to differentiate the right side (input).

The usefulness of the Laplace transform technique will be demonstrated by solving several example problems.

EXAMPLE

Solve the given equation for x(t),

$$\dot{x} + 2x = 1 \qquad (3.168)$$

when $x(0) = 1$.

By Laplace transform of Equation 3.168

$$L[\dot{x}] = sX(s) - x(0)$$

$$L[2x] = 2X(s)$$

$$L[1] = \frac{1}{s}$$

Thus

$$(s + 2)\ X(s) = \frac{1}{s} + 1$$

$$X(s) = \frac{s + 1}{s(s + 2)} = \frac{A}{s} + \frac{B}{s + 2}$$

Solving,

$$A = 1/2$$

and

$$B = 1 - 1/2 = 1/2$$

$$X(s) = \frac{1/2}{s} + \frac{1/2}{s + 2}$$

Inverse Laplace transforming gives

$$x(t) = 1/2 - 1/2\ e^{-2t} \tag{3.169}$$

EXAMPLE

Given the differential equation

$$\dot{x} + 2x = \sin t,\ x(0) = 5 \tag{3.170}$$

solve for x(t).

Taking the Laplace transform of Equation 3.170

$$sX(s) - x(0) + 2Xs = \frac{1}{s^2 + 1}$$

and

$$X(s) = \frac{1}{(s^2 + 1)(s + 2)} + \frac{5}{s + 2} \tag{3.171}$$

Expanding the first term on the right side of the equation gives

$$\frac{1}{(s^2 + 1)(s + 2)} = \frac{As + B}{s^2 + 1} + \frac{C}{s + 2} \tag{3.172}$$

Taking the common denominator and equating numerators gives

$$1 = (As + B)(s + 2) + C(s^2 + 1)$$

Substituting values of s leads to

$$A = -1/5$$

$$B = 2/5$$

$$C = 1/5$$

and substituting back into Equation 3.171 gives

$$X(s) = \frac{-1/5\ s}{s^2 + 1} + \frac{2/5}{s^2 + 1} + \frac{1/5}{s + 2} + \frac{5}{s + 2}$$

Inverse Laplace transforming gives the solution

$$X(t) = -1/5 \cos t + 2/5 \sin t + 5\ 1/5\ e^{-2t} \tag{3.173}$$

EXAMPLE

Given the differential equation

$$\ddot{x} + 5\dot{x} + 6x = 3e^{-3t},\ x(0) = \dot{x}(0) = 1 \tag{3.174}$$

solve for x(t).
Taking the Laplace transform of Equation 3.174

$$s^2X(s) - sx(0) - \dot{x}(0) + 5sX(s) - 5x(0) + 6X(s) = \frac{3}{s + 3} \tag{3.175}$$

or

$$X(s) = \frac{s^2 + 9s + 21}{(s + 3)\quad (s^2 + 5s + 6)} \tag{3.176}$$

Factoring the denominator,

$$X(s) = \frac{s^2 + 9s + 21}{(s + 3)(s + 2)(s + 3)} \tag{3.177}$$

$$X(s) = \frac{s \quad + 9s + 21}{(s + 3)^2(s + 2)} \tag{3.178}$$

$$X(s) = \frac{A}{s + 3} + \frac{B}{(s + 3)^2} + \frac{C}{s + 2} \tag{3.179}$$

Finding the common denominator of Equation 3.179, and setting the resultant numerator equal to the numerator of Equation 3.178.

$$s^2 + 9s + 21 = A(s + 3)(s + 2) + B(s + 2) + C(s + 3)^2$$

which can be solved easily for

$$A = -6$$

$$B = -3$$

$$C = 7$$

Now X(s) is given by

$$X(s) = \frac{-6}{s + 3} - \frac{3}{(s + 3)^2} + \frac{7}{s + 2}$$

which can be inverse Laplace transformed to

$$X(t) = -6e^{-3t} - 3te^{-3t} + 7e^{-2t} \tag{3.180}$$

EXAMPLE

Given the differential equation

$$\ddot{x} + 2\dot{x} + 10x = 3t + 6/10 \tag{3.181}$$

$$x(0) = 3$$

$$\dot{x}(0) = -27/10$$

solve for x(t).

Laplace transforming Equation 3.181 and solving for X(s) gives

$$X(s) = \frac{3s^3 + 3.3s^2 + 0.6s + 3}{s^2\ (s^2 + 2s + 10)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 + 2s + 10} \tag{3.182}$$

where

$$A = 0$$

$$B = 0.3$$

$$C = 3$$

$$D = 3$$

Thus,

$$X(s) = \frac{0.3}{s^2} + \frac{3s + 3}{s^2 + 2s + 10} \tag{3.183}$$

To make the inverse Laplace transform easier, Equation 3.183 is rewritten as

$$X(s) = \frac{0.3}{s^2} + 3\left[\frac{(s + 1)}{(s + 1)^2 + 3^2}\right] \tag{3.184}$$

which is readily inverse transformable to

$$x(t) = 0.3t + 3e^{-t}\cos 3t \tag{3.185}$$

3.8 TRANSFER FUNCTIONS

A transfer function is defined in Chapter 13, Linear Control Theory as, "The ratio of the output to the input expressed in operator or Laplace notation with zero initial conditions." The term "transfer function" can be thought of as what is done to the input to produce the output. A transfer function is essentially a mathematical model of a system and embodies all the physical characteristics of the system. A linear system can be completely described by its transfer function. Consider the following quiescent system.

$$a\ddot{x} + b\dot{x} + cx = f(t) \tag{3.186}$$

$$x(0) = \dot{x}(0) = 0$$

Taking the Laplace transform of Equation 3.186 results in

$$as^2X(s) + bsX(s) + cX(s) = F(s) \tag{3.187}$$

factoring gives

$$X(s)(as^2 + bs + c) = F(s) \tag{3.188}$$

or

$$\frac{X(s)}{F(s)} = \frac{1}{as^2 + bs + c} \tag{3.189}$$

Since Equation 3.186 represents a system whose input is f(t) and whose output is x(t) the following transforms can be defined

$$X(s) \equiv \text{output transform}$$

$$F(s) \equiv \text{input transform}$$

The transfer function can then be given the symbol TF and defined as

$$TF \equiv \frac{X(s)}{F(s)} \tag{3.190}$$

In the example represented by Equation 3.189

$$TF = \frac{1}{as^2 + bs + c} \tag{3.191}$$

Note that the denominator of the transfer function is algebraically the same as the characteristic equation appearing in the Equation 3.186. The characteristic equation completely defines the transient solution, and the total solution is only altered by the effect of the particular solution due to the input (or forcing function). Thus, from a physical standpoint, the transfer function completely characterizes a linear system.

The transfer function has several properties that will be used in control system analysis. Suppose that two systems are characterized by the differential equations

$$a\ddot{x} + b\dot{x} + cx = f(t) \tag{3.192}$$

and

$$d\ddot{y} + e\dot{y} + gy = x(t) \tag{3.193}$$

From the equations it can be seen that the first system has an input f(t), and an output x(t). The second system has an input x(t) and an output y(t). Note that the input to the second system is the output of the first system. Taking the Laplace transform of these two equations gives

$$(as^2 + bs + c)\ X(s) = F(s) \tag{3.194}$$

and

$$(ds^2 + es + g)\ Y(s) = X(s) \tag{3.195}$$

Finding the transfer functions,

$$TF_1 = \frac{X(s)}{F(s)} = \frac{1}{as^2 + bs + c} \tag{3.196}$$

$$TF_2 = \frac{Y(s)}{X(s)} = \frac{1}{ds^2 + es + g} \tag{3.197}$$

Now, both of these systems can be represented schematically by the block diagrams shown in Figure 3.16.

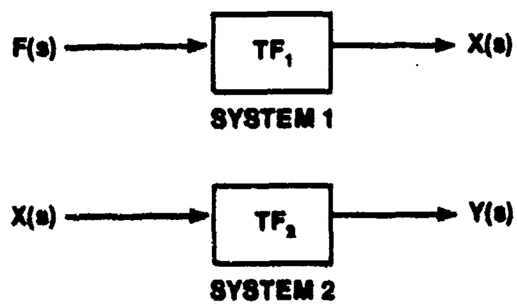


FIGURE 3.16. EXAMPLE BLOCK DIAGRAM NOTATION

If it is desired to find the output y(t) of System 2 due to the input f(t) of System 1, it is not necessary to find x(t) since the two systems can be linked using transfer functions as shown in Figure 3.17.

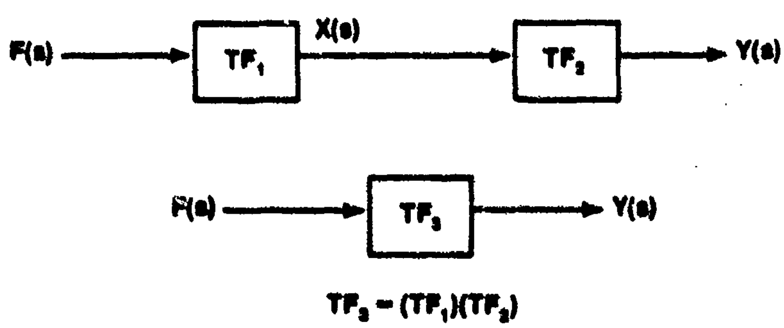


FIGURE 3.17. COMBINING TRANSFER FUNCTIONS

The solution y(t) is then given by the inverse transform of Y(s), or

$$Y(s) = \left[TF_3\right] F(s) \tag{3.198}$$

or

$$Y(s) = \left[TF_1\right] \left[TF_2\right] F(s) \tag{3.199}$$

This method of solution can be logically extended to include any desired number of systems.

3.9 SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS

In many physical problems the mathematical description of the system can most conveniently be written as simultaneous differential equations with constant coefficients. The basic procedure for solving a system of n ordinary differential equations in n dependent variables consists in obtaining a set of equations from which all but one of the dependent variables, say x, can be eliminated. The equation resulting from the elimination is then solved for the variable x. Each of the other dependent variables is then obtained in a similar manner.

A very effective means of handling simultaneous linear differential equations is to take the Laplace transform of the set of equations and reduce the problem to a set of algebraic equations that can be solved explicitly for the dependent variable in s. This method is demonstrated below.

Given the set of equations

$$3\frac{d^2x}{dt^2} + x + \frac{d^2y}{dt^2} + 3y = f(t) \tag{3.200}$$

$$2\frac{d^2x}{dt^2} + x + \frac{d^2y}{dt^2} + 2y = g(t) \tag{3.201}$$

where $x(0) = \dot{x}(0) = y(0) = \dot{y}(0) = 0$, find x(t) and y(t). Taking the

Laplace transform of this system yields

$$(3s^2 + 1)\ X(s) + (s^2 + 3)\ Y(s) = F(s) \tag{3.202}$$

$$(2s^2 + 1)\ X(s) + (s^2 + 2)\ Y(s) = G(s) \tag{3.203}$$

Cramer's rule will now be used to solve this set of equations. Cramer's rule can be stated in its simplest form as, given the equations

$$P_1(s)\ X(s) + P_2(s)\ Y(s) = F_1(s) \tag{3.204}$$

$$Q_1(s)\ X(s) + Q_2(s)\ Y(s) = F_2(s) \tag{3.205}$$

then,

$$X(s) = \frac{\begin{vmatrix} F_1 & P_2 \\ F_2 & Q_2 \end{vmatrix}}{\begin{vmatrix} P_1 & P_2 \\ Q_1 & Q_2 \end{vmatrix}} \tag{3.206}$$

for unknown X(s), and

$$Y(s) = \frac{\begin{vmatrix} P_1 & F_1 \\ Q_1 & F_2 \end{vmatrix}}{\begin{vmatrix} P_1 & P_2 \\ Q_1 & Q_2 \end{vmatrix}} \tag{3.207}$$

for the unknown Y(s).

The X(s) unknown in Equations 3.202 and 3.203 can be solved for in this fashion by applying Cramer's rule

$$X(s) = \frac{\begin{vmatrix} F(s) & (s^2+3) \\ G(s) & (s^2+2) \end{vmatrix}}{\begin{vmatrix} (3s^2+1) & (s^2+3) \\ (2s^2+1) & (s^2+2) \end{vmatrix}} \tag{3.208}$$

In a similar manner,

$$Y(s) = \frac{\begin{vmatrix} (3s^2+1) & F(s) \\ (2s^2+1) & G(s) \end{vmatrix}}{\begin{vmatrix} (3s^2+1) & (s^2+3) \\ (2s^2+1) & (s^2+2) \end{vmatrix}} \tag{3.209}$$

For the particular inputs $f(t) = t$ and $g(t) = 1$,

$$X(s) = \frac{\begin{vmatrix} \frac{1}{s^2} & (s^2+3) \\ \frac{1}{s} & (s^2+2) \end{vmatrix}}{(s^4-1)} = \frac{-s^3+s^2-3s+2}{s^2\,(s^4-1)} \tag{3.210}$$

Expanded as a partial fraction

$$X(s) = \frac{A}{s^2} + \frac{B}{s} + \frac{Cs+D}{(s^2+1)} + \frac{E}{s-1} + \frac{F}{(s+1)} = \frac{-s^3+s^2-3s+2}{s^2\,(s^4-1)} \tag{3.211}$$

Solving for A, B, etc.,

$$X(s) = \frac{-2}{s^2} + \frac{3}{s} + \frac{1/2-s}{s^2+1} - \frac{7/4}{s+1} - \frac{1/4}{s-1} \tag{3.212}$$

which yields a solution

$$x(t) = -2t + 3 - 7/4e^{-t} - 1/4e^{t} + 1/2 \sin t - \cos t \qquad (3.213)$$

A similar approach will obtain the solution for y(t).

In the case of three simultaneous differential equations, the application of Laplace transforms and use of Cramer's rule will yield the solution.

$$P_1(s)X(s) + P_2(s)Y(s) + P_3(s)Z(s) = F_1(s) \qquad (3.214)$$

$$Q_1(s)X(s) + Q_2(s)Y(s) + Q_3(s)Z(s) = F_2(s) \qquad (3.215)$$

$$R_1(s)X(s) + R_2(s)Y(s) + R_3(s)Z(s) = F_3(s) \qquad (3.216)$$

where

$$X(s) = \frac{\begin{vmatrix} F_1 & P_2 & P_3 \\ F_2 & Q_2 & Q_3 \\ F_3 & R_2 & R_3 \end{vmatrix}}{\begin{vmatrix} P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \\ R_1 & R_2 & R_3 \end{vmatrix}} \qquad (3.217)$$

Y(s) and Z(s) will have similar forms.

3.10 ROOT PLOTS

Some insight into the response of a second order system can be gained by examining the roots of the differential equation describing the system on a root plot. A root plot is a plot of the roots of the characteristic equation in the complex plane. Root plots are used in Chapter 8, Dynamics, to describe aircraft longitudinal and lateral directional modes of motion. These plots are also used extensively in Chapter 13, Linear Control Theory, for linear control system analysis.

It was shown earlier that a second order linear system can be put into the following form:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{f(t)}{M} \tag{3.93}$$

whose roots can be written as

$$p_{1,2} = -\zeta\omega_n \pm j\,\omega_n\sqrt{1-\zeta^2} \tag{3.88}$$

or

$$p_{1,2} = -\zeta\omega_n \pm j\,\omega_d \tag{3.218}$$

Figure 3.18 is a plot of the two roots of Equations 3.88 and 3.218 in the complex plane.

$$p_{1,2} = \underbrace{-\zeta\omega_n \pm j\,\omega_n}_{\text{real}}\underbrace{\sqrt{1-\zeta^2}}_{\text{imaginary}} \tag{3.88}$$

where the first term is plotted on the real axis and the second term plotted on the imaginary (j) axis.

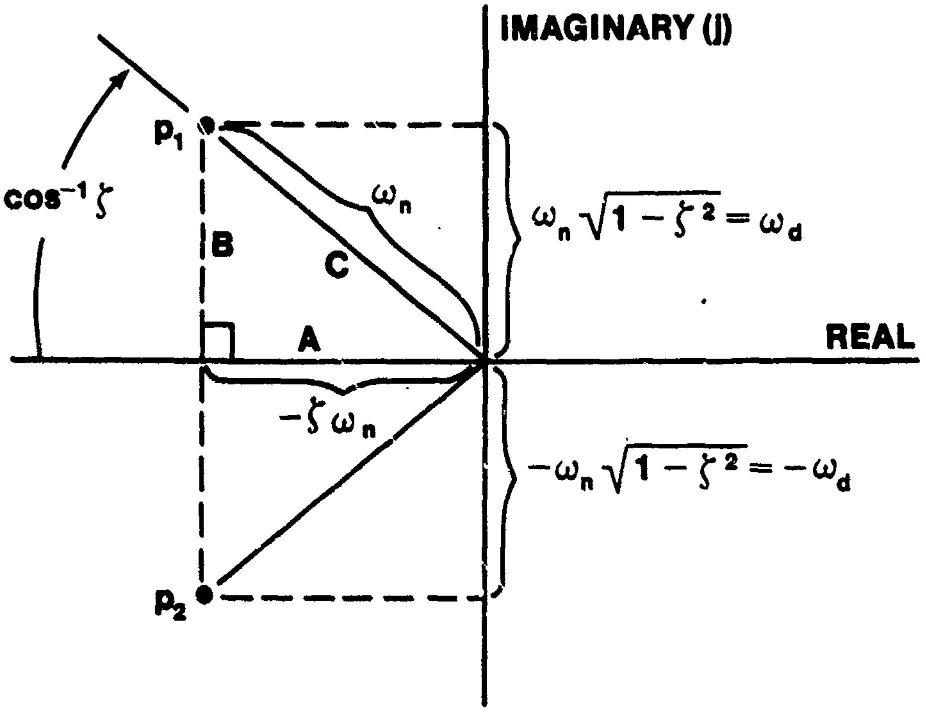


FIGURE 3.18. GENERAL ROOT PLOT IN THE COMPLEX PLANE

From the right triangle relationship shown in Figure 3.18, it can be easily shown that the length of the line from the origin to either point p_1 or p_2 is equal to ω_n.

$$A^2 + B^2 = C^2$$

$$(\zeta\,\omega_n)^2 + (\omega_n\sqrt{1-\zeta^2})^2 = C^2$$

$$\zeta^2\,\omega_n{}^2 + \omega_n{}^2\,(1-\zeta^2) = C^2$$

$$\zeta^2\,\omega_n{}^2 + \omega_n{}^2 - \zeta^2\,\omega_n^2 = C^2$$

$$\omega_n{}^2 = C^2$$

$$C = \omega_n$$

The five distinct damping cases previously discussed can be examined on root plots through the use of Equation 3.88.

1. $\zeta = 0$, undamped (Figure 3.19)

$$p_{1,2} = -\zeta\,\omega_n \pm j\,\omega_n \sqrt{1-\zeta^2} \qquad (3.88)$$

$$p_{1,2} = 0 \pm j\,\omega_n$$

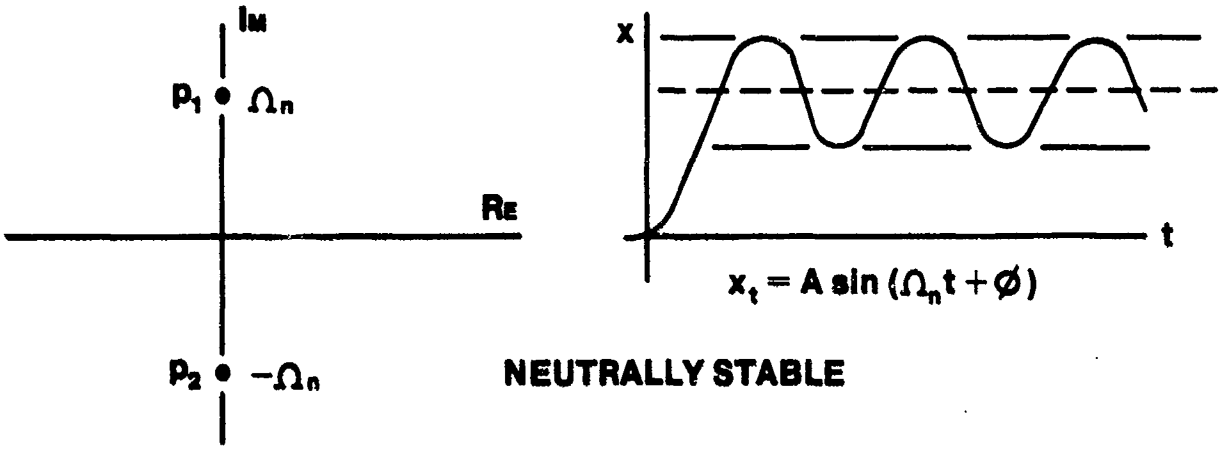


FIGURE 3.19. NEUTRALLY STABLE UNDAMPED RESPONSE

2. $0 < \zeta < 1.0$, Underdamped (Figure 3.20)

$$p_{1,2} = -\zeta\omega_n \pm j\,\omega_n \sqrt{1-\zeta^2} \qquad (3.88)$$

$$p_{1,2} = (-) \pm j\,(+)$$

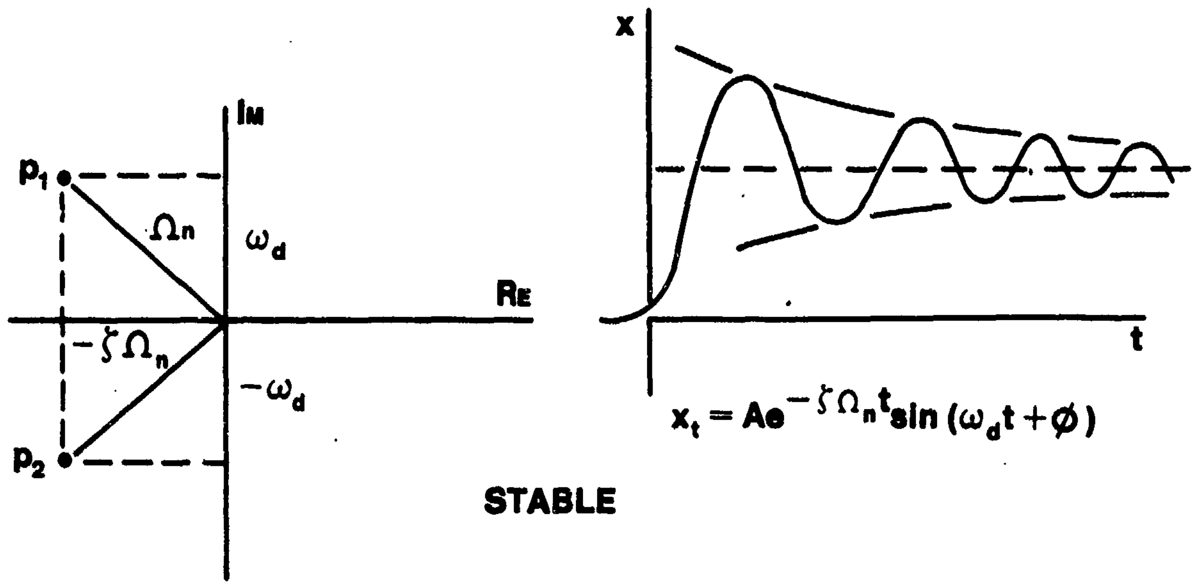


FIGURE 3.20. STABLE UNDERDAMPED RESPONSE

3. ζ = 1.0 Critically damped (Figure 3.21)

$$p_{1,2} = -\zeta\,\omega_n \pm j\,\omega_n\sqrt{1-\zeta^2} \tag{3.88}$$

$$p_{1,2} = -\omega_n$$

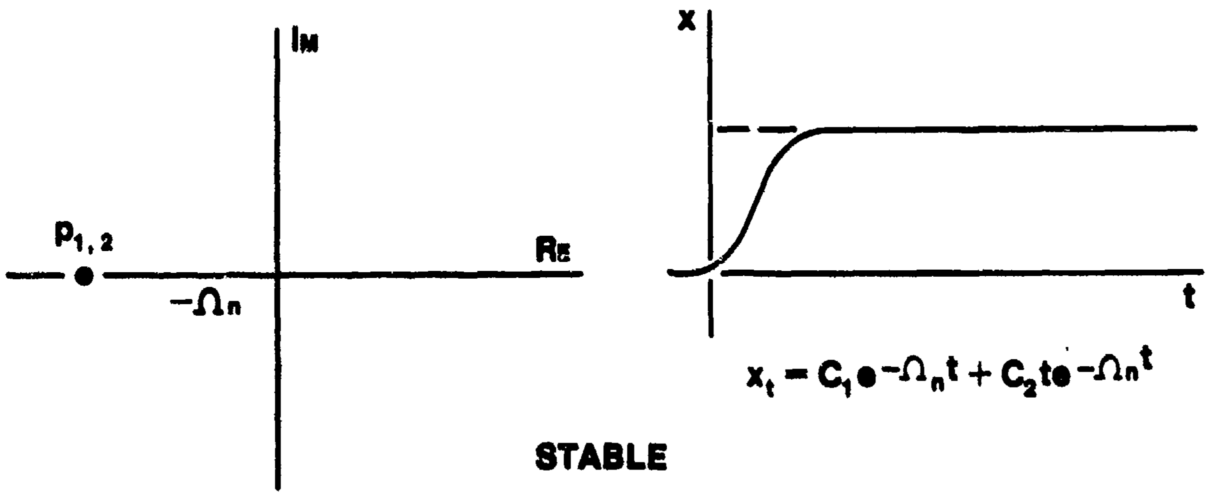


FIGURE 3.21. STABLE CRITICALLY DAMPED RESPONSE

4. $\zeta > 1.0$ Overdamped (Figure 3.22)

$$p_{1,2} = -\zeta\,\omega_n \pm j\,\omega_n\,\sqrt{1-\zeta^2} \tag{3.88}$$

$$p_{1,2} = -\zeta\,\omega_n \pm \omega_n\,\underbrace{\sqrt{\zeta^2-1}}_{\text{real}}$$

$$p_{1,2} = (-),\ (-)$$

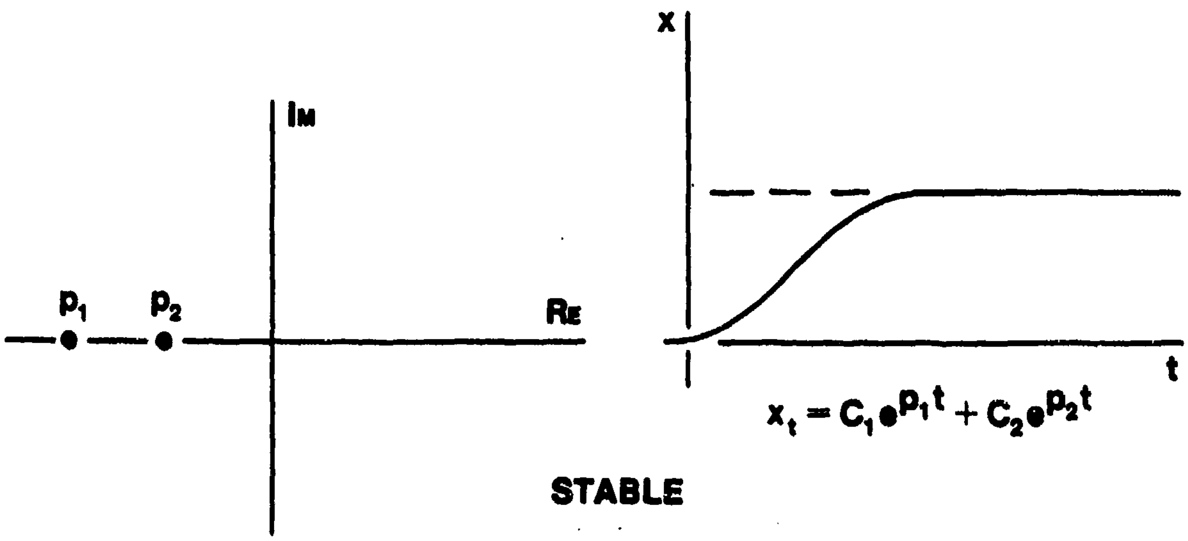


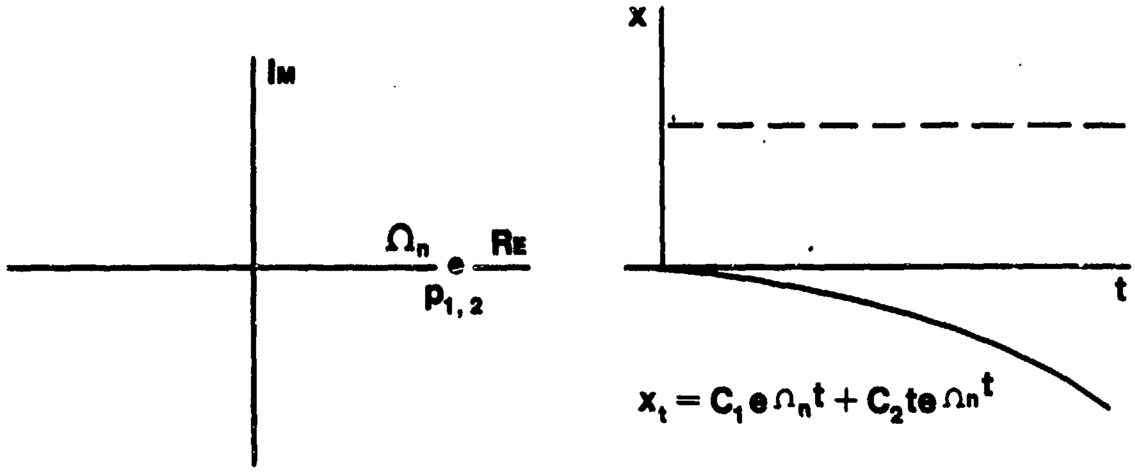
FIGURE 3.22. STABLE OVERDAMPED RESPONSE

5. $\zeta < 0$ Unstable

$\zeta = -1.0$ (Figure 3.23)

$$p_{1,2} = -\zeta\,\omega_n \pm j\,\omega_n\,\sqrt{1-\zeta^2} \tag{3.88}$$

$$p_{1,2} = \omega_n$$



UNSTABLE

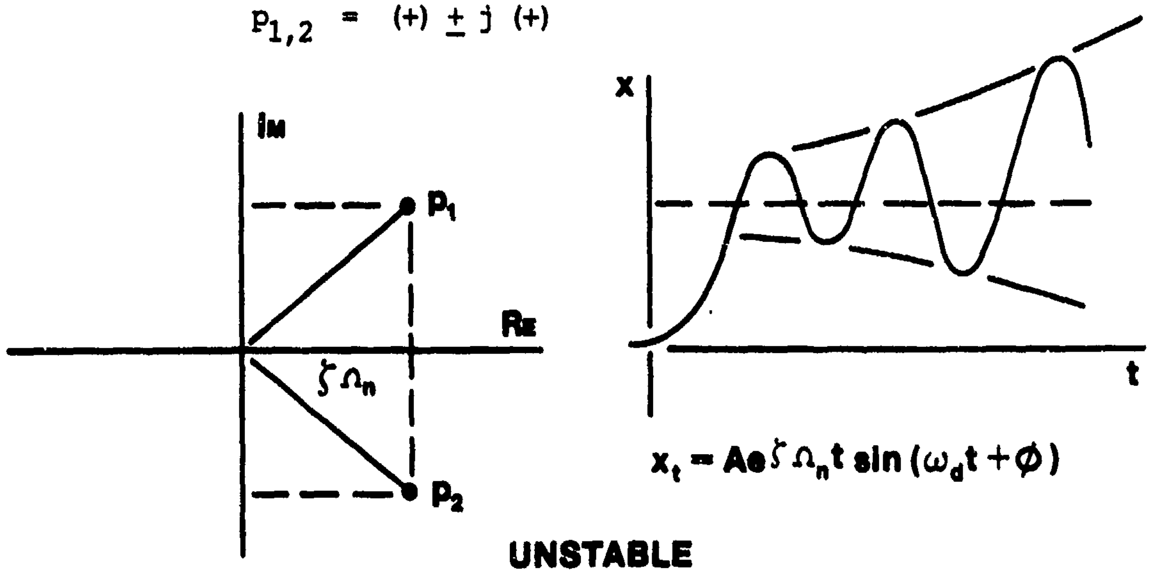
FIGURE 3.23. UNSTABLE RESPONSE

$-1.0 < \zeta < 0$ (Figure 3.24)

$$p_{1,2} = -\zeta\,\omega_n \pm j\,\omega_n\sqrt{1-\zeta^2} \qquad (3.88)$$

$$p_{1,2} = (+) \pm j\,(+)$$

$$p_{1,2} = (+) \pm j\,(+)$$



UNSTABLE

FIGURE 3.24 UNSTABLE RESPONSE

$\zeta < -1.0$ Both roots positive (Figure 3.25)

$$p_{1,2} = -\zeta\,\omega_n \pm j\,\omega_n \sqrt{1-\zeta^2} \tag{3.88}$$

$$p_{1,2} = (+),\ (+)$$

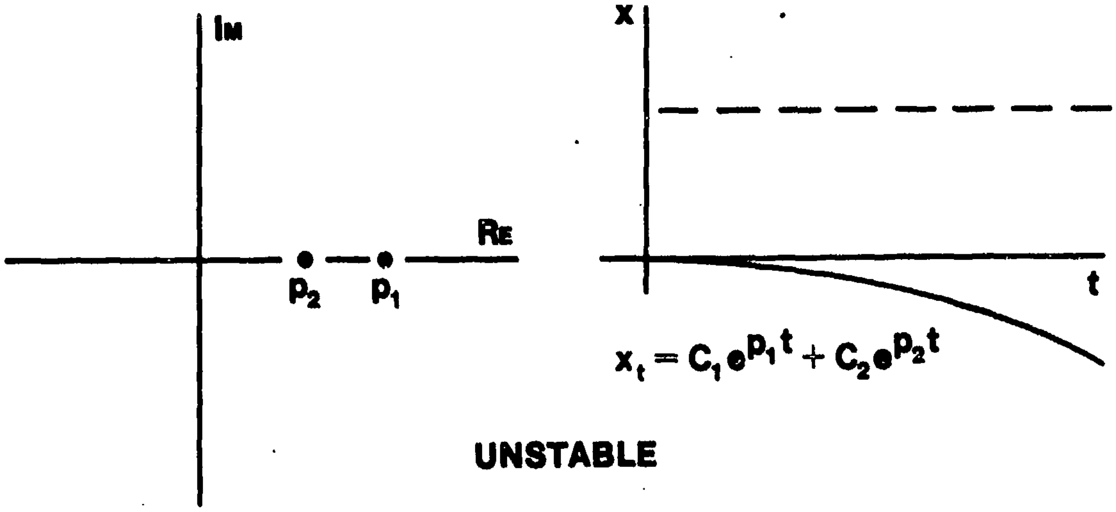


FIGURE 3.25. UNSTABLE RESPONSE

In summary, a second order system with both roots located to the left of the imaginary axis is stable. If both roots are on the imaginary axis the system is neutrally stable, and if one or more roots are located to the right of the imaginary axis the system is unstable. These conditions are shown in Figure 3.26.

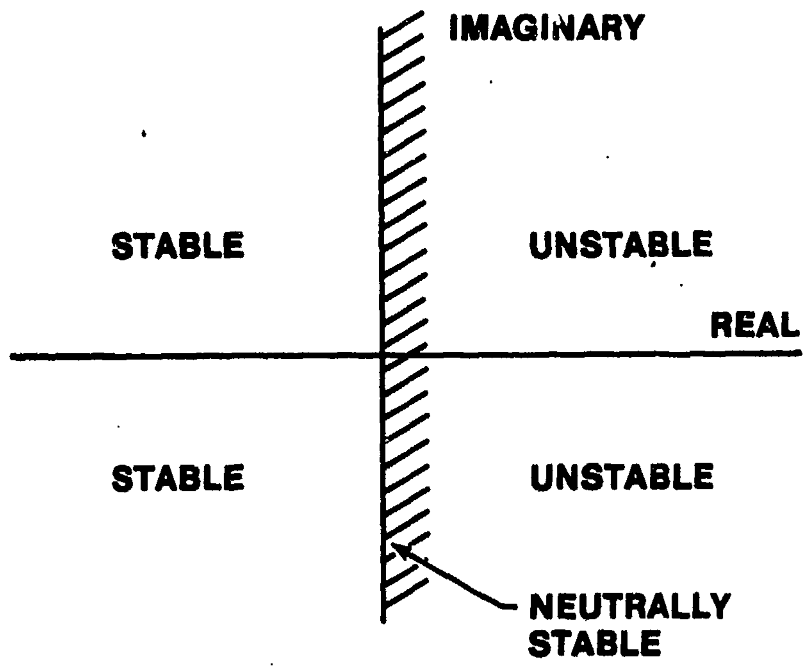


FIGURE 3.26. ROOT LOCUS STABILITY

Root plots can be used for analysis of the aircraft modes of motion. For example, the longitudinal static statiblity of an aircraft is greatly influenced by center of gravity (cg) position. Figure 3.27 shows how the roots of the characteristic equation describing one of the longitudinal motion modes change position as the cg is moved aft. This plot is called a root locus plot.

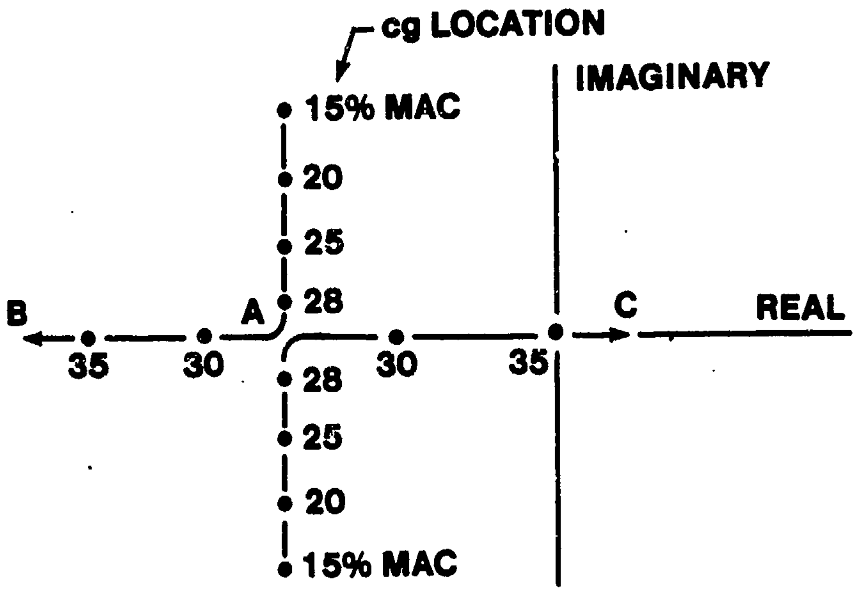


FIGURE 3.27 EFFECT OF CG SHIFT ON LONGITUDINAL STATIC STABILITY OF A TYPICAL AIRCRAFT

Note that as the cg is moved aft of its initial location at 15% MAC, damping of this mode of motion (short period) increases while the frequency decreases. Zero frequency is reached between a cg location of 28% and 30% MAC. The root locus then splits into a pair of real roots, branches AB and AC of the locus. These branches represent damped aperiodic (nonoscillatory) motion. The short period mode of motion goes unstable at a cg location of 35% MAC. The location of the cg where this instability occurs (35% MAC in this example) is known as the maneuver point and it is discussed in detail in Chapter 6, Maneuvering Flight.

PROBLEMS

Solve for y.

3.1. $\frac{dy}{dx} = x^4 + 4x + \sin 6x$

3.2. $\frac{d^2y}{dx^2} = e^{-x} + \sin \omega x$

3.3. $\frac{d^3y}{dx^3} = x^5$

3.4. $y \frac{dy}{dx} + 3x^2 = 0$

3.5. $(x - 1)^2 y\,dx + x^2 (y - 1)\,dy = 0$
Just find a solution. Solving for y is tough.

Test for exactness and solve if exact.

3.6. $(y^2 - x)\,dx + (x^2 - y)\,dy = 0$

3.7. $(2x^3 + 3y)\,dx + (3x + y - 1)\,dy = 0$

3.8. $(2xy^4 e^y + 2xy^3 + y)\,dx + (x^2y^4e^y - x^2y^2 - 3x)\,dy = 0$

3.9. Multiply Problem 3.8 by $1/y^4$ and solve for y. Note this assumes that $y = 0$.

Solve for y_t

3.10. $5y' + 6y = 0$

3.11. $y''' - 5y'' - 24y' = 0$

3.12. $y'' + 12y' + 36y = 0$

3.13. $y'' + 4y' + 13y = 0$

Solve for y_t and y_p in Problems 3.14 - 3.17, then solve for the general solution.

3.14. $\ddot{y} + 5\dot{y} + 6y = 3e^{-3t}$ $\qquad y(0) = 1, \ \dot{y}(0) = 6$

3.15. $\ddot{y} + 4\dot{y} + 4y = \cos t$ $\qquad y(0) = \frac{28}{25}, \ \dot{y}(0) = -\frac{104}{25}$

3.16. $2\ddot{x} + 4\dot{x} + 20x = 6t + \frac{6}{5}$, $\qquad x(0) = 3, \ x(0) = -\frac{27}{10}$

3.17. $3\dot{x} + 2x = -4e^{-2t}$, $\qquad x(3) = -0.14$

3.18. Find ω_n, ω_d, ζ, and τ and describe system damping (i.e., underdamped, overdamped, etc.) where applicable.

$$\ddot{y} + 5\dot{y} + 6y = 3e^{-3t}$$

3.19. $\ddot{y} + 4\dot{y} + 4y = \cos t$

3.20. $2\ddot{x} + 4\dot{x} + 20x = 6t + \frac{6}{5}$

3.21. $3\dot{x} + 2x = -4e^{-2t}$

In problems 3.22 - 3.24 find x(s), do not find the inverse transform.

3.22. $3\ddot{x} + \dot{x} + 6x = \sin 6t, \qquad x(0) = \dot{x}(0) = 0$

3.23. $\ddot{x} - 2\dot{x} + 5x = e^{-t} \sin 3t, \qquad x(0) = -1,\ \dot{x}(0) = 9$

3.24. $4\ddot{x} + 3\dot{x} - x = t^3 - t \sin 2t, \qquad x(0) = 3,\ \dot{x}(0) = -2$

In Problems 3.25 - 3.27, expand X(s) by partial fractions and find the inverse transforms.

3.25. $X(s) = \dfrac{5s^2 + 29s + 36}{(s + 2)\ (s^2 + 4s + 3)}$

3.26. $X(s) = \dfrac{2s^2 + 6s + 5}{(s^2 + 3s + 2)\ (s + 1)}$

3.27. $X(s) = \dfrac{2s^4 + 7s^3 + 27s^2 + 51s + 27}{(s^3 + 9s)\ (s^2 + 3s + 3)}$

Solve the following problems by Laplace transform techniques.

3.28. $\dot{x} + 2x = \sin t, \qquad x(0) = 5$

3.29. $\ddot{x} + 5\dot{x} + 6x = 3e^{-3t}, \qquad x(0) = \dot{x}(0) = 1$

Solve using Laplace Transforms

3.30. $\dot{x} + 3x - y = 1 \qquad x(0) = y(0) = 0$

$\dot{x} + 8x + \dot{y} = 2$

3.31. Read the question and circle the correct answer, true (T) or false (F):

T F The particular solution to a second order differential equation contains two arbitrary constants that are solved for using initial conditions and the transient solution too.

T F Solutions to linear differential equations are generally nonlinear functions.

T F Differential equation solutions are free of derivatives.

T F Direct integration will give solutions to some differential equations without the necessity of arbitrary constants.

T F In general, the number of arbitrary constants in the solution of a differential equation is equal to the order of the differential equation.

T F There is no known way to determine if a differential equation is exact.

T F The solution to a first order linear differential equation with constant coefficients is always of exponential form.

T F The Laplace variable s can be real, imaginary, or complex.

T F Inverse Laplace transforms are used to return from the s to the time domain.

T F Quiescent systems have zero initial conditions.

T F First order equation roots cannot be plotted on root plots.

T F A transfer function can be defined as input transform divided by output transform.

T F The characteristic equation completely describes the transient solution.

T F The method of undetermined coefficients is used to solve for the particular solution.

T F $\ddot{x} + 4\dot{x} + 13x = 3$, is a second degree equation.

T F $\ddot{x} + 4\dot{x} + 13x = 3$, is a second order equation.

T F $4\dot{x} + 13x = 3$, is a first order equation.

T F It is impossible to have a linear, second degree equation.

T F $13x = 3$, is a linear equation.

T F 13x = 3, is a differential equation.

T F Damping ratio and natural frequency have no physical significance.

T F The time constant and time to half amplitude for a first order system are equal.

T F The Laplace transform converts a differential equation from the time domain to the s domain.

T F The transient response is dependent on the input.

T F Laplace transforms are easy to derive.

T F In general, it is easier to check a candidate solution to see if it is a solution than to determine the candidate solution.

T F Superposition can be used for adding linear differential equation solutions.

T F The method of partial fractions is used to solve for the particular solution of a differential equation.

T F The number "e" is a variable.

T F The Laplace transform of the characteristic equation appears in the denominator of the transfer function.

T F There is a general technique which can be used to solve any linear differential equation.

T F Cramer's rule is in centimeters.

T F Cramer's rule is an outdated method of solving simultaneous equations.

T F The transfer function completely characterizes a linear system.

T F The Heaviside Expansion theorem is often cited by weight watchers.

T F A root plot is a short hand method of presenting transient time response.

T F The settling time is a measure of damping ratio of a system without regard for the damped frequency.

T F If y = f(x), then y is the dependent and x the independent variable.

3.32. The following terms are important. Define and provide symbols for those you are not sure of.

Differential Equation

Dependent Variable

Independent Variable

Ordinary Differential Equation

Partial Differential Equation

Exact Differential Equation

Linear Differential Equation

Degree of a Differential Equation

Order of a Differential Equation

General solution

Transient solution

Particular solution

Steady-state solution

Forcing function

Input to a system (related to the differential equation)

Output of a system (related to the differential equation)

Time Constant

Damping ratio

Damped natural frequency

Natural Frequency

Undamped response

Underdamped response

Overdamped response

Unstable system response

Critical Damping

Linear system

Laplace Transform

Inverse Laplace Transform

Unit Step

Unit Impulse

Ramp function

Transfer function

Pole

Zero

Root Plot

Root Locus

Rise time

Settling Time

Peak Overshoot

Time to peak overshoot

Steady-state error

3.33. Solve the following problems. Sketch root locus plots, and find ζ, ω_n, ω_d, and τ where appropriate.

A. $\frac{d^2y}{dx^2} = x^2 + 4x$

B. $dy = \frac{-2xydx}{(x^2 + \cos y)}$

C. $\frac{dx}{dt} + t^3x = 0$

D. $\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + 6x = 0$

E. $\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = 0$

F. $\frac{d^2x}{dt^2} + 4x = 0$

G. $\frac{d^2x}{dt^2} + 7\frac{dx}{dt} + 22x = 0$

H. Given: $y_t = 2 \sin 3x + 2 \cos 3x$

Find A and ϕ in the expression

$$y_t = A \sin (3x + \phi)$$

The following problems are the same as D thru G with forcing functions.

I. $\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + 6x = 9$

J. $\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = e^{2t}$

K. $\frac{d^2x}{dt^2} + 4x = \sin 3t$

L. $\frac{d^2x}{dt^2} + 7\frac{dx}{dt} + 22x = t$

The following problems are the same as D thru G with forcing functions and initial conditions:

M. $\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + 6x = 9$ $\quad x(0) = 3/2$

$\dot{x}(0) = 2$

N. $\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = e^{2t}$ $\quad x(0) = 2$

$\dot{x}(0) = 4$

O. $\frac{d^2x}{dt^2} + 4x = \sin 3t$ $\quad x(0) = 0$

$\dot{x}(0) = -3/10$

P. $\frac{d^2x}{dt^2} + 7\frac{dx}{dt} + 22x = t$ $\quad x(0) = 0$

$\dot{x}(0) = 1/22$

3.34. Solve the following problems using Laplace techniques:

A. $\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + 6x = 9$ $\quad x(0) = 3/2$

$\dot{x}(0) = 2$

B. $\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = e^{2t}$ $\quad x(0) = 2$

$\dot{x}(0) = 4$

3.35. Given the set of equations

$$3\frac{dx}{dt} + \frac{dy}{dt} = t$$

$$2\frac{dx}{dt} + \frac{dy}{dt} = 1$$

where $x(0) = y(0) = 0$, find $y(t)$ using Laplace transform methods.

ANSWERS

3.1. $y = \frac{x^5}{5} + 2x^2 - \frac{\cos 6x}{6} + C$

3.2. $y = e^{-x} - \frac{\sin \omega x}{\omega^2} + C_{1x} + C_2$

3.3. $y = \frac{x^8}{336} + \frac{C_1 x^2}{2} + C_2 x + C_3$

3.4. $y^2 = -2x^3 + C$

3.5. $y^{et} = Cx^2 e^{\frac{1-x^2}{x}}$

3.6. Not exact.

3.7. $f = \frac{x^4}{2} + 3xy + \frac{y^2}{2} - y + C$

3.8. Not exact.

3.9. $f = x^2 e^y + \frac{x^2}{y} + \frac{x}{y^3} = C$

3.10. $y_t = Ce^{-6/5t}$

3.11. $y_t = C_1 + C_2 e^{-3t} + C_3 e^{8t}$

3.12. $y_t = C_1 e^{-6t} + C_2\, t\, e^{-6t}$

3.13. $y_t = C\,e^{-2t}\cos(3t + \phi)$

3.14. $y = -11e^{-3t} + 12e^{-2t} - 3te^{-3t}$

3.15. $y = e^{-2t} - \frac{58}{25}\,t\,e^{-2t} + \frac{3}{25}\cos t + \frac{4}{25}\sin t$

3.16. $y = 3e^{-t}\cos 3t + 3/10\ t$

3.17. $y = -e^{-2/3t} + e^{-2t}$

3.18. $\omega_n = \sqrt{6}$

$\zeta = 1.02$

3.19. $\omega_n = 2$

$\zeta = 1$

$\omega_d = 0$

3.20. $\omega_n = \sqrt{10}$

$\zeta = 0.316$

$\omega_d \cong 3.0$

3.21. $\tau = 3/2$

3.22. $$x(s) = \frac{\frac{6}{s^2 + 36}}{3s^2 + s + 6}$$

3.23. $$x(s) = \frac{\frac{3}{(s + 1)^2 + 9} - s + 11}{s^2 - 2s + 5}$$

3.24. $x(s) = \dfrac{\dfrac{6}{s^4} - \dfrac{4s}{(s^2+4)^2} + 12s + 1}{4s^2 + 3s - 1}$

3.25. $y(t) = 2e^{-2t} - 3e^{-3t} + 6e^{-t}$

3.26. $y(t) = e^{-2t} + e^{-t} + te^{-t}$

3.27. $y(t) = 1 + 2/3 \sin 3t + e^{-3/2t} \cos\sqrt{3/4}\, t + 1/2\, e^{-3/2t} \sin\sqrt{3/4}\, t$

3.28. $x(t) = 5\ 1/5\ e^{-2t} = 1/5 \cos t + 2/5 \sin t$

3.29. $x(t) = -6e^{-3t} - 3te^{-3t} + 7e^{-2t}$

3.30. $x(t) = 1/4\ (1 - e^{-2t}\ (\cos 2t - \sin 2t))$

$y(t) = 1/4\ (-1\ te^{-2t}\ (\cos 2t + 3 \sin 2t))$

3.33. A. $y = \dfrac{x^4}{12} + 2/3\, x^3 + Cx + C_1$

B. $x^2y + \sin y = C$

C. $x = Ce^{-t^4/4}$

D. $x(t) = C_1 e^{2t} + C_2 e^{3t}$

E. $\omega_n = 2$

$\omega_d = 0$

$\zeta = -1$

F. $\zeta = 0$

$\omega_n = 2$

$\omega_d = 2$

G. $\omega_n = 4.69$

$\omega_d = 3.12$

$\zeta = 0.746$

H. $A = \sqrt{8}$

$\phi = \pi/4$

I. $x = C_1 e^{2t} + C_2 e^{3t} + 3/2$

J. $x = C_1e^{2t} + C_2te^{2t} + 1/2\ t^2e^{2t}$

K. $x = C_1 \cos 2t + C_2 \sin 2t - 1/5 \sin 3t$

L. $x = e^{-7/2t}\left(C_1 \cos \frac{\sqrt{39}}{2} t + C_2 \sin \frac{\sqrt{39}}{2} t\right) + 1/22\ t - \frac{7}{484}$

M. $x \doteq -2e^{2t} + 2e^{3t} + 3/2$

N. $x = 2e^{2t} + 1/2\ t^2\ e^{2t}$

O. $x = 3/20 \sin 2t - 1/5 \sin 3t$

P. $x = e^{-7/2t}\left(\frac{7}{484} \cos \frac{\sqrt{39}}{2} t + .016 \sin \frac{\sqrt{39}}{2} t\right) + \frac{1}{22} t - \frac{7}{484}$

3.34. A. $X(t) = \frac{3}{2} - 2e^{2t} + 2e^{3t}$

B. $X(t) = 2e^{-2t} + 1/2\ t^2\ e^{2t}$

3.35. $y(t) = 3t - t^2$

$x(t) = 1/2\, t^2 - t$

CHAPTER 4

EQUATIONS OF MOTION

4.1 INTRODUCTION

This chapter presents aircraft equations of motion used in the Flying Qualities phase of the USAF Test Pilot School curriculum.

The theory incorporates certain simplifying assumptions to make the main elements of the subject clearer. The equations developed are by no means suitable for design of modern aircraft, but the basic method of attacking the problem is valid. With the aid of high speed computers the aircraft designers' more rigorous theoretical calculations, modified by data obtained from the wind tunnel, often give results which closely predict the flying qualities of new airplanes. However, neither the theoretical nor the wind tunnel results are infallible. Therefore, there is still a valid requirement for the test pilot in the development cycle of new aircraft.

4.2 TERMS AND SYMBOLS

There will be many terms and symbols used during the Flying Qualities Phase. Some of these will be familiar, but many will be new. It will be a great asset to be able to recall at a glance the definitions represented by these symbols. Below is a condensed list of the terms and symbols used in this course.

Terms:

Stability Derivatives - Nondimensional quantities expressing the variation of the force or moment coefficient with a disturbance from steady flight.

$$C_{m_\alpha} = \frac{\partial C_m}{\partial \alpha}$$

$$C_{m_q} = \left(\frac{2U_0}{c}\right) \frac{\partial C_m}{\partial q}$$

Stability Parameters - A quantity that expresses the variation of force or moment on aircraft caused by a disturbance from steady flight.

$$\mathcal{M}_u = \frac{\rho U_0 Sc}{I_y}\left[C_{m_0} + \frac{U_0}{2}\frac{\partial C_m}{\partial u}\right]$$ (Change in pitching moment caused by a change in velocity)

$$L_q = \frac{\rho Sc}{4m} C_{L_q}$$ (Change in lift caused by a change in pitch rate)

Static Stability - The initial tendency of an airplane to return to steady state flight after a disturbance.

Dynamic Stability - The time history of an airplane's response to a disturbance in which the aircraft ultimately returns to a steady state flight.

Neutral Stability -

a) Static - The airplane would have no tendency to move from its disturbed condition.

b) Dynamic - The airplane would sustain a steady oscillation caused by a disturbance.

Static Instability - A characteristic of an aircraft such that when disturbed from steady flight, its tendency is to depart further or diverge from the original condition of steady flight.

Dynamic Instability - Time history of an aircraft response to a disturbance in which the aircraft ultimately diverges.

Flight Control Sign Convention - Any control movement or deflection that causes a positive movement or moment (right yaw, pitch up, right roll) on the airplane shall be considered a positive control movement. This sign convention does not conform to the convention used by NASA and some reference text books. This convention is the easiest to remember and is used at the Flight Test Center, therefore, it will be used in the School (Figure 4.1).

Degrees of Freedom - The number of paths that a physical system is free to follow.

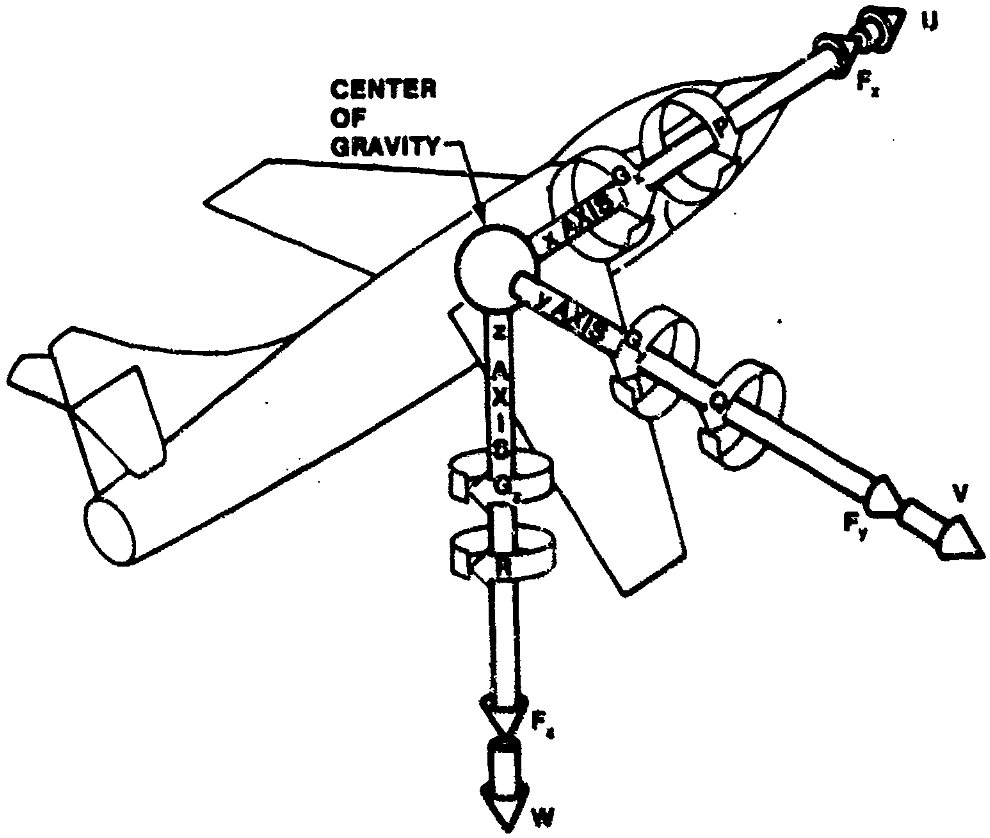


FIGURE 4.1. VEHICLE FIXED AXIS SYSTEM
AND NOTATION

Symbols:

Symbol	Definition
a.c.	Aerodynamic Center: A point located on the wing chord (approximately one quarter of the chord length back of the leading edge for subsonic flight) about which the moment coefficient is practically constant for all angles of attack.
b	Wingspan
C	Chordwise Force: The component of the resultant aerodynamic force that is parallel to the aircraft reference axis. (i.e., fuselage reference line).

c	Mean Aerodynamic Chord: The theoretical chord for a wing which has the same force vector as the actual wing (also MAC).
c.p.	Center of Pressure: Theoretical point on the chord through which the resultant force acts.
D	Drag: The component of the resultant aerodynamic force parallel to the relative wind. It too must be specified whether this applies to a complete aircraft or to parts thereof.
$\overline{F}$	Applied force vector.
F_a, F_e, F_r	Control forces on the aileron, elevator, and rudder, respectively
F_x, F_y, F_z	Components of applied forces on respective body axes.
$\overline{G}$	Applied moment vector.
G_x, G_y, G_z	Components of the applied moments on the respective body axes.
$\overline{H}$	Angular momentum vector.
H_x, H_y, H_z	Components of the angular momentum vector on the body axes.
I	Moments of Inertia: With respect to any given axis, the moment of inertia is the sum of the products of the mass of each elementary particle by the square of its distance from the axis. It is a measure of the resistance of a body to angular acceleration. $I = \sum mr^2$
$\overline{i}, \overline{j}, \overline{k}$	Unit vectors in the body axis system.
I_x, I_y, I_z	Moments of inertia about respective body axes. $I_x = \sum m (y^2 + z^2)$
I_{xy}, I_{yz}, I_{xz}	Products of inertia, a measure of symmetry.
L	Lift: The component of the resultant aerodynamic force perpendicular to the relative wind. It must be specified whether this applies to a complete aircraft or to parts thereof.
$\mathcal{L}, \mathcal{M}, \mathcal{N}$	Aerodynamic moments about x, y, and z vehicle axis.
N	Normal Force: The component of the resultant aerodynamic force that is perpendicular to the aircraft reference axis.
P,Q,R	Angular rates about the x, y, and z vehicle axes, respectively.
p,q,r	Perturbed values of P,Q,R, respectively.

Symbol	Definition
$\overline{R}$	Resultant Aerodynamic Force: The vector sum of the lift and drag forces on an airfoil or airplane.
S	Wing area.
U_0	Component of velocity along the x vehicle axis at zero time (i.e., equilibrium condition).
U,V,W	Components of velocity along the x, y, and z vehicle axes.
u,v,w	Perturbed valued of U,V,W, respectively.
X,Y,Z	Aerodynamic force components on respective body axes (Caution: Also used as aces in "Moving Earth Axis System" in derivation of Euler angle equations.)
x,y,z	Axes in the body axis system.
α	Angle of attack.
β	Sideslip angle.
$\delta_a,\delta_e,\delta_r$	Deflection angle of the ailerons, elevator, and rudder, respectively.
ε	Thrust angle.
θ,ϕ,ω	Euler angles: pitch, roll, and yaw, respectively.
$\overline{\omega}$	Total angular velocity vector of an aircraft.
∇	Dimensionless derivative with respect to time.

4.3 OVERVIEW

The purpose of this section is to derive a set of equations that describes the motion of an airplane. An airplane has six degrees of freedom (i.e., it can move forward, sideways, and down, and it can rotate about its axes with yaw, pitch, and roll). In order to solve for these six unknowns, six simultaneous equations will be required. To derive these, the following relations will be used:

START WITH NEWTON'S SECOND LAW

$$\underbrace{\bar{F}}_{\text{externally applied force}} = \frac{d}{dt}\underbrace{(m\,\bar{V})}_{\text{linear momentum}} \quad \text{(3 linear degrees of freedom).}$$

$$\underbrace{\bar{G}}_{\text{externally applied moment}} = \frac{d}{dt}\underbrace{(\bar{H})}_{\text{angular momentum}} \quad \text{(3 rotational degrees of freedom).}$$

Six equations for the six degrees of freedom of a rigid body.

Equations are valid with respect to inertial space only.

OBTAIN THE 6 AIRCRAFT EQUATIONS OF MOTION

Longitudinal

$$F_x = m\,(\dot{U} + QW - RV) \tag{4.1}$$

$$F_z = m\,(\dot{W} + PV - QU) \tag{4.2}$$

$$G_y = \dot{Q}\,I_y - PR\,(I_z - I_x) + (P^2 - R^2)\,I_{xz} \tag{4.3}$$

Lateral-Directional

$$F_y = m\,(\dot{V} + RU - PW) \tag{4.4}$$

$$G_x = \dot{P}\,I_x + QR\,(I_z - I_y) - (\dot{R} + PQ)\,I_{xz} \tag{4.5}$$

$$G_z = \dot{R}\,I_z + PQ\,(I_y - I_x) + (QR - \dot{P})\,I_{xz} \tag{4.6}$$

The Left-Hand Side (LHS) of the equation represents the applied forces and moments on the airplane while the Right-Hand Side (RHS) stands for the airplane's response to these forces and moments. Before launching into the development of these equations, it will first be necessary to cover some basics.

4.4 COORDINATE SYSTEMS

There are many coordinate systems that are useful in the analysis of vehicle motion. According to generally accepted notation, all coordinate systems will be right-hand orthogonal.

4.4.1 Inertial Coordinate System

An axis system fixed in space that has no relative motion and in which Newton's laws apply (Figure 4.2).

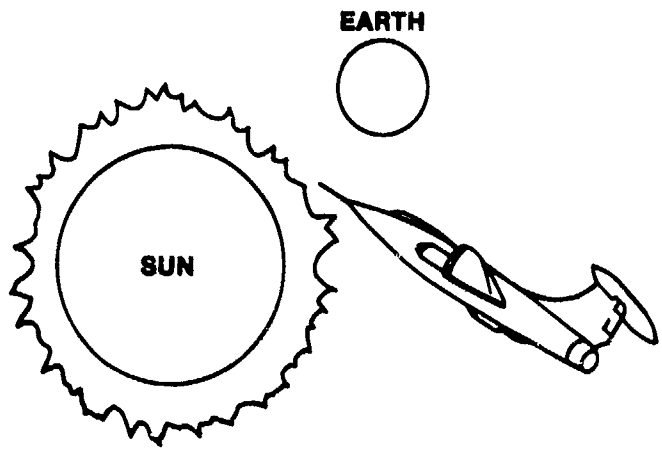


FIGURE 4.2. TRUE INERTIAL COORDINATE SYSTEM

Experience with physical observations determines whether a particular reference system can properly be assumed to be an inertial frame for the application of Newton's laws to a particular problem.

> Location of origin: unknown.
> Approximation for space dynamics: the center of the sun.
>
> Approximation for aircraft: the center of the earth. (earth axis system)

4.4.2 Earth Axis System

There are two earth axis systems, the fixed and the moving. An example of a moving earth axis system is an inertial navigation platform. An example of a fixed earth axis is a radar site (Figure 4.3).

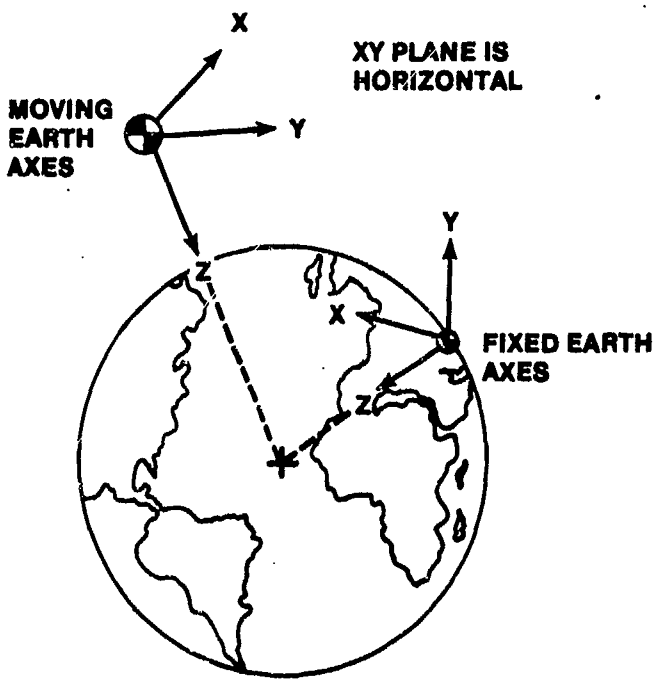


FIGURE 4.3. THE EARTH AXIS SYSTEMS

Location of origin:

Fixed System: arbitrary location
Moving System: at the vehicle cg

The Z-axis points toward the center of the earth along the local gravitational vector, $\overline{g}$

The XY Plane
parallel to local horizontal.

The Orientation of the X-axis is arbitrary; may be North or on the initial vehicle heading.

4.4.3 Vehicle Axis Systems

These coordinate systems have origins fixed to the vehicle. There are many different types, e.g.,

Body Axis System.
Stability Axis System.
Principal Axis System.
Wind Axis System.

The body and the stability axis systems are the only two that will be used during this course.

4.4.3.1 Body Axis System. The body axis system (Figure 4.4) is the most general kind of axis system in which the origin and axes are fixed to a rigid body. The use of axes fixed to the vehicle ensures that the measured rotary inertial terms in the equations of motion are constant, to the extent that mass can also be considered constant, and that aerodynamic forces and moments depend only upon the relative velocity orientation angles α and β.

The body fixed axis system has another virtue; it is the natural frame of reference for most vehicle-borne observations and measurements of the vehicle's motion.

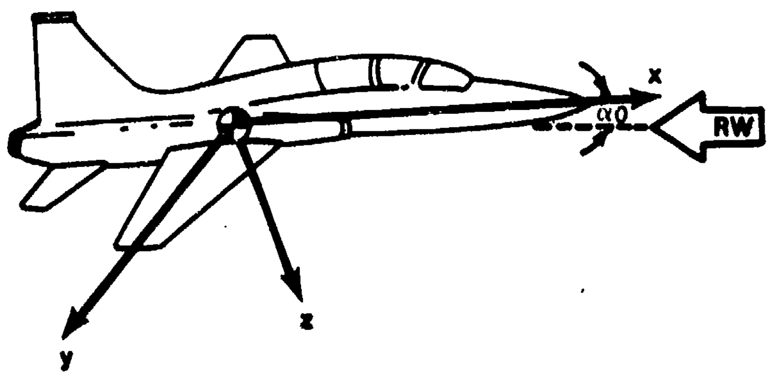


FIGURE 4.4. BODY AXIS SYSTEM

In the body axis system;

The Unit Vectors are $\bar{i}$, $\bar{j}$, $\bar{k}$

The Origin is at the cg

The x-z plane is in the vehicle plane of symmetry

The positive x-axis points forward along a vehicle horizontal reference line

The positive y-axis points out the right wing

The positive z-axis points downward out the bottom of the vehicle

4.4.3.2 Stability Axis System. Stability axes are specialized body axes (see Figure 4.5) in which the orientation of the vehicle axes system is determined by the equilibrium flight condition. The x_s-axis is selected to be coincident with the relative wind at the start of the motion. This initial alignment does not alter the body-fixed nature of the axis system; however, the alignment of the frame with respect to the body changes as a function of the equilibrium condition. If the reference flight condition is not symmetric, i.e. with sideslip, then the x_s-axis is chosen to lie on the projection of V_T in the plane of symmetry, with z_s also in the plane of symmetry. The moment-of-inertia and product-of-inertia terms vary for each equilibrium flight condition. However, they are constant in the equations of motion.

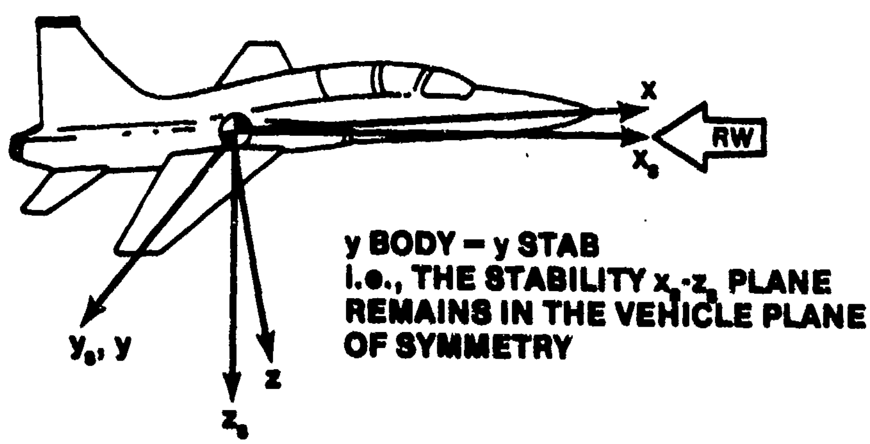


FIGURE 4.5. STABILITY AXIS SYSTEM

In the stability axis system;

The unit vectors are $\bar{i}_s$, $\bar{j}_s$, $\bar{k}_s$

The origin is at the cg

The positive x_s-axis points forward coincident with the equilibrium position of the relative wind.

The x_s-z_s plane must remain in the vehicle plane of symmetry.

The positive z_s-axis points downward out the bottom of the vehicle, normal to the x_s-axis

4.4.3.3 Principal Axes. These are a special set of body axes aligned with the principal axes of the vehicle and are used for certain applications. The convenience of principal axes results from the fact that all of the products-of-inertia are reduced to zero. The equations of motion are thus greatly simplified.

4.4.3.4 Wind Axes. The wind axes use the vehicle translational velocity as the reference for the axis system. Wind axes are thus oriented with respect to the flight path of the vehicle, i.e., with respect to the relative wind, V_T. If the reference flight condition is symmetric, i.e., V_T lies in the vehicle plane of symmetry, then the wind axes coincide with the stability axes, but depart from it, moving with the relative wind during the disturbance.

The relation between general wind axes and vehicle body axes of a rigid body defines the angle of attack, α, and the sideslip angle, β. These angles are convenient independent variables for use in the expression of aerodynamic force and moment coefficients.

Wind axes are not generally used in the analysis of the motion of a rigid body, because, as in the case of the earth axes, the moment-of-inertia and product-of-inertia terms in the three rotational equations of motion vary with time, angle of attack, and sideslip angle.

4.5 VECTOR DEFINITIONS

The Equations of Motion describe the vehicle motion in terms of four vectors ($\bar{F}$, $\bar{G}$, $\bar{V}_T$, $\bar{\omega}$). The components of these vectors resolved along the body axis system are shown below.

1. $$\bar{F} = F_x\bar{i} + F_y\bar{j} + F_z\bar{k}$$

2. $$\bar{G} = G_x\bar{i} + G_y\bar{j} + G_z\bar{k} = \text{Total Moment (applied)}$$

$$\bar{G} = \bar{G}_{aerodynamic} + \bar{G}_{other\ sources}$$

$$\bar{G}_{aerodynamic} = L_{aero}\bar{i} + M_{aero}\bar{j} + N_{aero}\bar{k}$$

$$\bar{G}_{aerodynamic} = \mathcal{L}\bar{i} + \mathcal{M}\bar{j} + \mathcal{N}\bar{k}$$

NOTE: Control deflections that tend to produce positive $\mathcal{L}$, $\mathcal{M}$, or $\mathcal{N}$, are defined at the USAF TPS to be positive (i.e., Right δ_r is positive).

3. $$\bar{V}_T = U\bar{i} + V\bar{j} + W\bar{k} = \text{True Velocity}$$

where

U = forward velocity
V = side velocity
W = vertical velocity

Angle of sideslip, β, and angle of attack, α, can be expressed in terms of the velocity components (Figure 4.6).

$$\sin\beta = \frac{V}{V_T}$$

For a small β

$$\sin\beta \cong \beta$$

or

$$\beta \cong \frac{V}{V_T}$$

Also, for small α and β

$$\cos\beta \cong 1$$

or

$$V_T \cos\beta \cong V_T$$

Hence,

$$\sin\alpha = \frac{W}{V_T \cos\beta} \cong \frac{W}{V_T}$$

or

$$\alpha \cong \frac{W}{V_T}$$

Some texts also define

$$\alpha = \tan^{-1} \frac{W}{U}$$

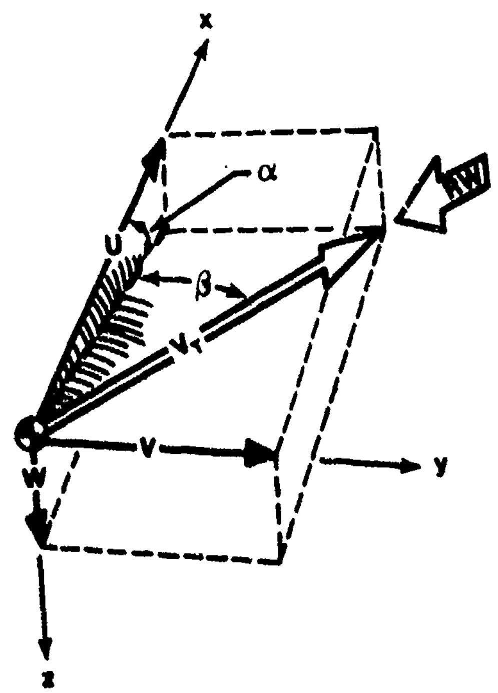


FIGURE 4.6. VELOCITY COMPONENTS AND THE AERODYNAMIC ORIENTATION ANGLES, α AND β

4. $\bar{\omega}$ - Angular Velocity

$$\bar{\omega} = P\bar{i} + Q\bar{j} + R\bar{k}$$

where

P = roll rate
Q = pitch rate
R = yaw rate.

4.6 EULER ANGLES

The orientation of any reference frame relative to another can be given by three angles, which are consecutive rotations about the axes z, y, and x, in that order, that carry one frame into coincidence with the other. In flight dynamics, the Euler angles used are those which rotate the vehicle carried moving earth axis system into coincidence with the relevant vehicle axis system (Figure 4.7).

The importance of the sequence of the Euler angle rotations cannot be overemphasized. Finite angular displacements do not behave as vectors. Therefore, if the sequence is performed in a different order than ψ, θ, ϕ, the final result will be different. This fact is clearly illustrated by the final aircraft attitudes in Figure 4.8 in which two rotations of equal magnitude have been performed about the x and y axes, but in opposite order. Addition of a rotation about a third axis does nothing to improve the outcome.

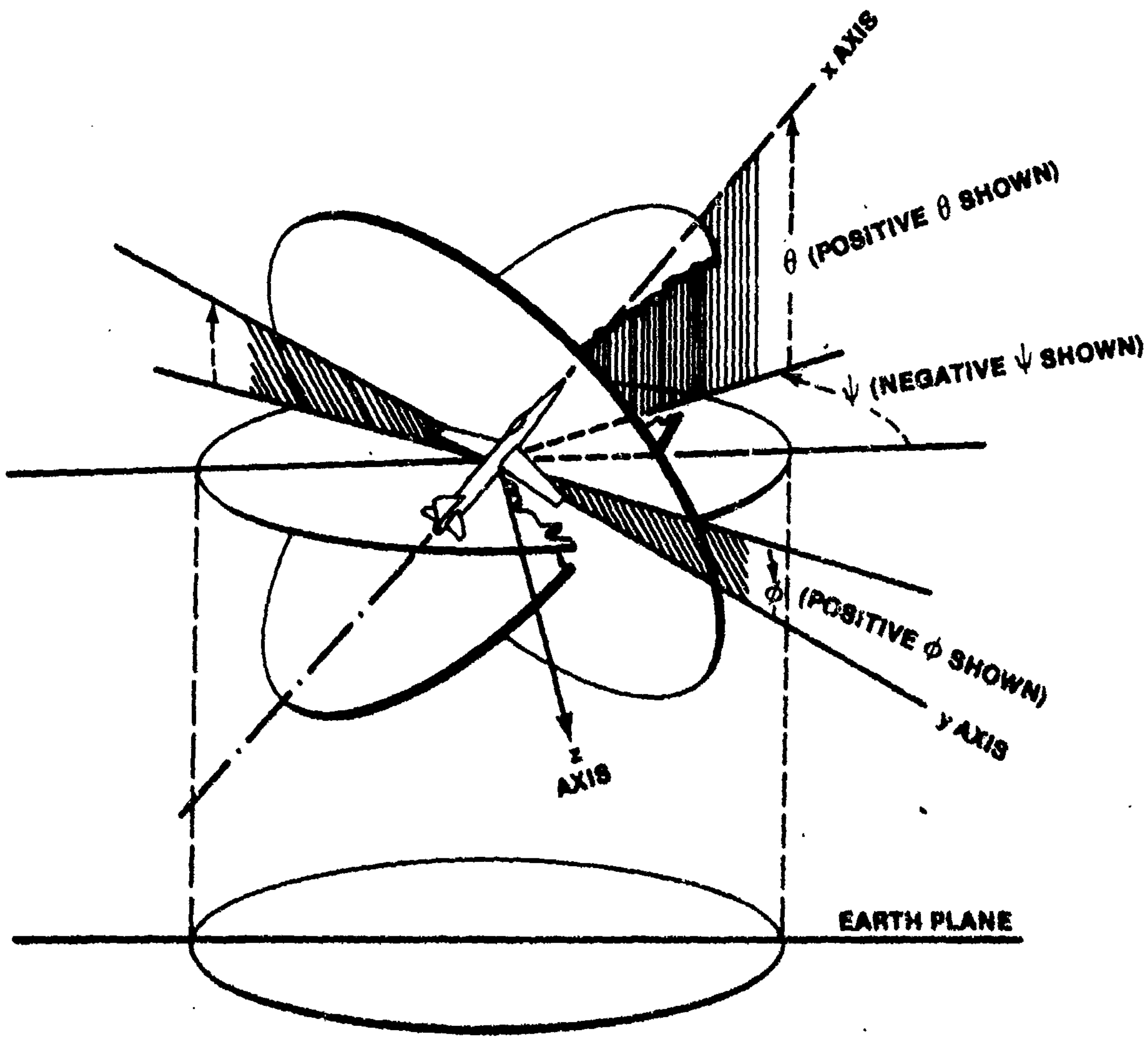


FIGURE 4.7. THE EULER ANGLE ROTATIONS

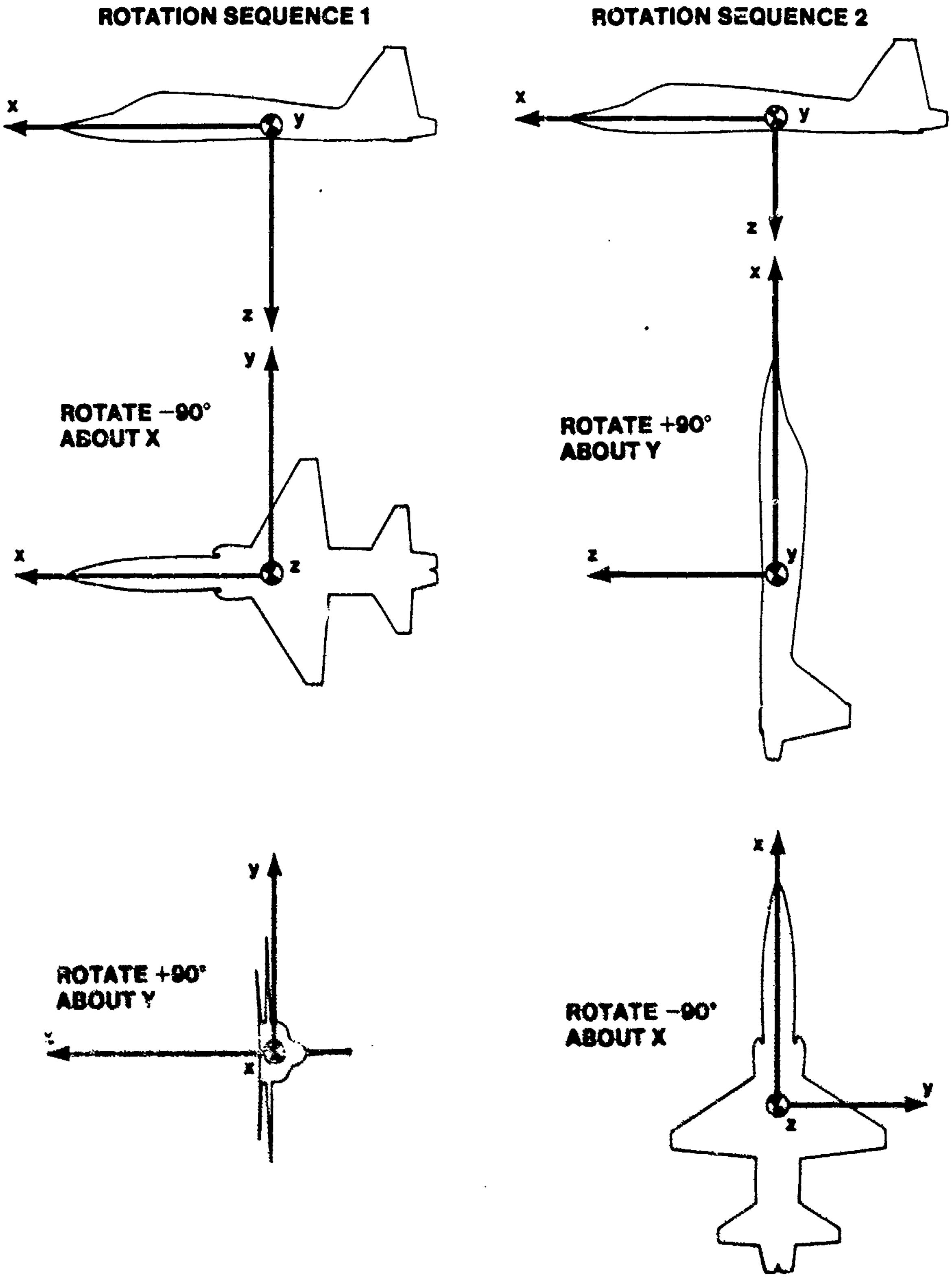


FIGURE 4.8. DEMONSTRATION THAT FINITE ANGULAR DISPLACEMENTS DO NOT BEHAVE AS VECTORS

Euler angles are very useful in describing the orientation of flight vehicles with respect to inertial space. Consequently, angular rates in an inertial system ($\dot{\phi}$, $\dot{\theta}$, $\dot{\psi}$) can be transformed to angular rates in the vehicle axes (P, Q, R) using Euler angle transformations. For example, if an Inertial Navigation System (INS) is available, data can be taken directly in Euler angles. P, Q, and R can then be determined using transformations.

Euler angles are expressed as YAW (ψ), PITCH (θ), and ROLL (ϕ). The sequence (YAW, PITCH, ROLL) must be maintained to arrive at the proper set of Euler angles.

ψ - Yaw Angle - The angle between the projection of x vehicle axes onto the horizontal plane and the initial reference position of the X earth axis. (Yaw angle is the vehicle heading only if the initial reference is North).

θ - Pitch Angle - The angle measured in a vertical plane between the x vehicle axis and the horizontal plane.

ϕ - Roll Angle - The angle, measured in the yz plane of the vehicle system, between the y axis and the horizontal plane. For a given ψ and θ, bank angle is a measure of the rotation about the x axis to put the aircraft in the desired position from a wing's horizontal condition.

The accepted limits on the Euler angles are:

$$-180^\circ \leq \psi \leq +180^\circ$$

$$-90^\circ \leq \theta \leq +90^\circ$$

$$-180^\circ \leq \phi \leq +180^\circ$$

4.7 ANGULAR VELOCITY TRANSFORMATION

The following relationships, derived by vector resolution, will be useful later in the study of dynamics. (See Appendix A.)

$$\overline{P} = (\dot{\phi} - \dot{\psi} \sin \theta)\overline{i} \tag{4.7}$$

$$\overline{Q} = (\dot{\theta} \cos \phi + \dot{\psi} \sin \phi \cos \theta)\overline{j} \tag{4.8}$$

$$\overline{R} = (\dot{\psi} \cos \phi \cos \theta - \dot{\theta} \sin \phi)\overline{k} \tag{4.9}$$

The above equations transform the angular rates from the moving earth axis system ($\dot{\psi}$, $\dot{\theta}$, $\dot{\phi}$) into angular rates about the vehicle axis system (P, Q, R) for any aircraft attitude.

For example, it is easy to see that when an aircraft is pitched up and banked, the vector $\bar{\dot{\psi}}$ will have components along the x, y, and z body axes (Figure 4.9). Remember, $\bar{\dot{\psi}}$ is the angular velocity about the Z axis of the <u>Moving Earth Axis System</u> (it can be thought of as the rate of change of aircraft heading). Although it is not shown in Figure 4.9, the aircraft may have a value of $\bar{\dot{\theta}}$ and $\bar{\dot{\phi}}$. In order to derive the transformation equations, it is easier to analyze one vector at a time. First resolve the components of $\bar{\dot{\psi}}$ on the body axes. Then do the same with $\bar{\dot{\theta}}$ and $\bar{\dot{\phi}}$. The components can then be added and the total transformation will result.

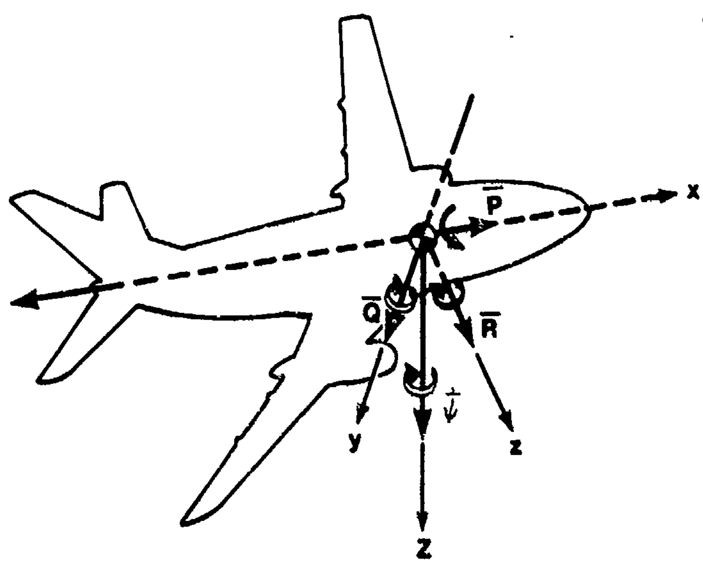


FIGURE 4.9 COMPONENTS OF $\bar{\dot{\psi}}$ ALONG x, y, AND z BODY AXES. (NOTE: THE X AND Y AXES OF THE MOVING EARTH AXIS SYSTEM ARE NOT SHOWN.)

<u>Step 1</u> - Resolve the components of $\bar{\dot{\psi}}$ along the body axes for any aircraft attitude.

It is easy to see how $\bar{\dot{\psi}}$ reflects to the body axis by starting with an aircraft in straight and level flight and changing the aircraft attitude one

angle at a time. In keeping with convention, the sequence of change will be yaw, pitch and bank.

First, it can be seen from Figure 4.10 that the Z axis of the Moving Earth Axis System remains aligned with the z axis of the Body Axis System regardless of the angle ψ if θ and ϕ are zero; therefore, the effect of $\dot{\psi}$ on P,Q, and R does not change with the yaw angle ψ.

$$\therefore R = \dot{\psi} \text{ (when } \theta = \phi = 0)$$

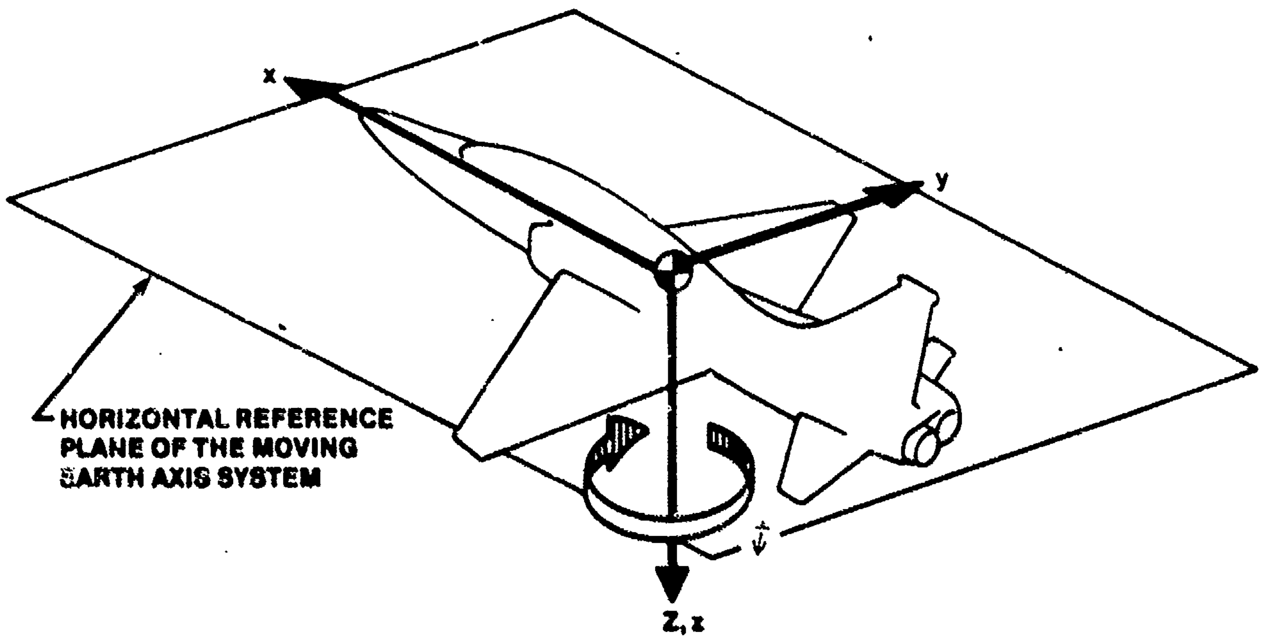


FIGURE 4.10. DEVELOPMENT OF AIRCRAFT ANGULAR VELOCITIES BY THE EULER ANGLE YAW RATE (ψ ROTATION)

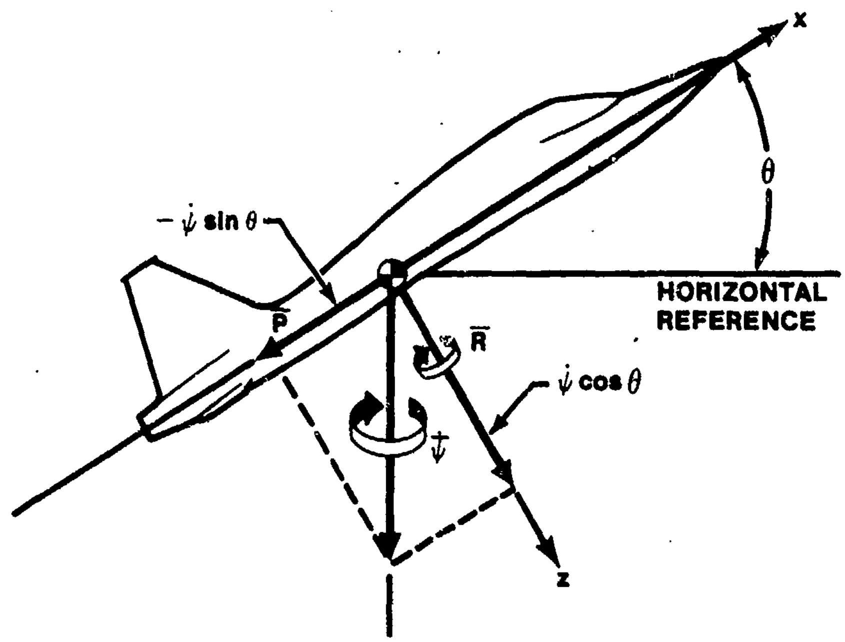


FIGURE 4.11. DEVELOPMENT OF AIRCRAFT ANGULAR VELOCITIES BY THE EULER ANGLE YAW RATE (θ ROTATION)

Next, consider pitch up. In this attitude, $\dot{\psi}$ has components on the x and z-body axes as shown in Figure 4.11. As a result, $\dot{\psi}$ will contribute to the angular rates about these axis.

$$P = -\dot{\psi} \sin \theta$$

$$R = \dot{\psi} \cos \theta$$

With just pitch, the Z axis remains perpendicular to the y-body axis, so Q is not affected by $\dot{\psi}$ in this attitude.

Next, bank the aircraft. leaving the pitch as it is (Figure 4.12).

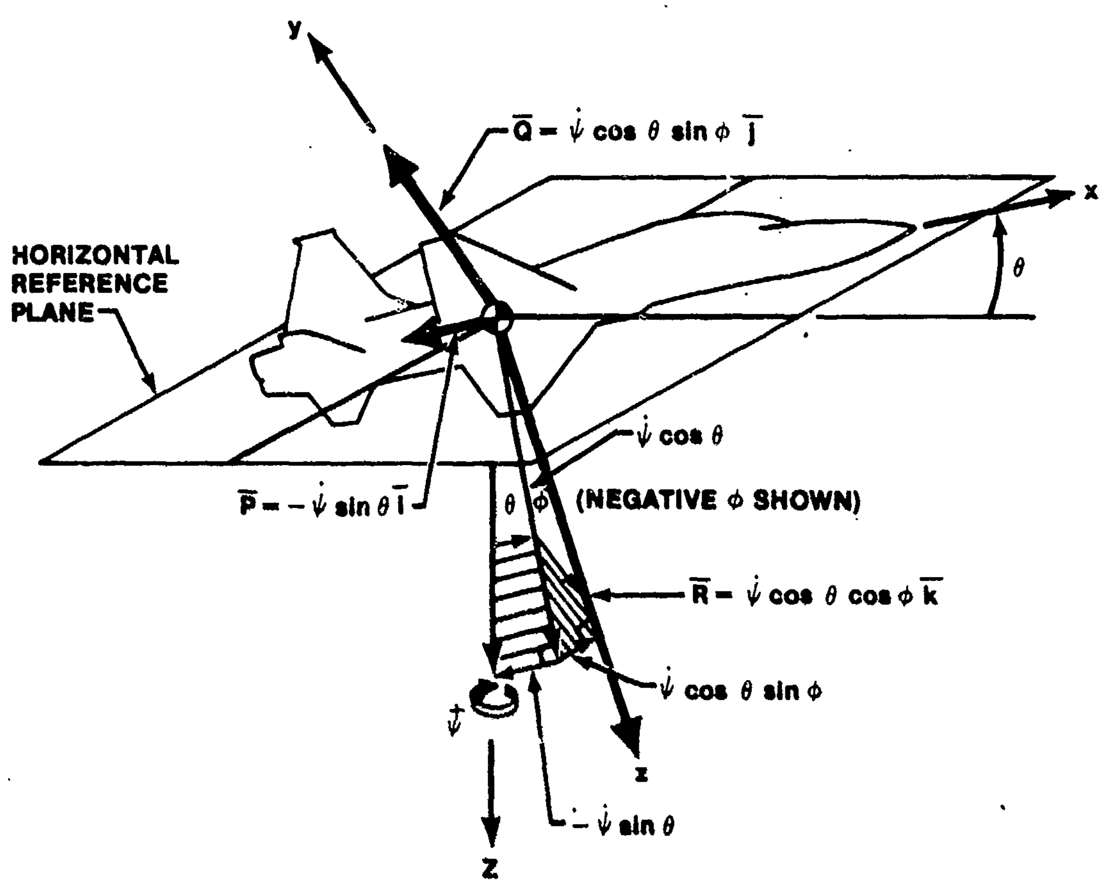


FIGURE 4.12. DEVELOPMENT OF AIRCRAFT ANGULAR VELOCITIES BY THE EULER ANGLE YAW RATE (ϕ ROTATION)

All of the components are now illustrated. Notice that roll did not change the effect of $\dot{\psi}$ on P. The components, therefore, of $\dot{\psi}$ in the body axes for any aircraft attitude are

$$\text{Effect of } \dot{\psi} \text{ only} \quad \begin{aligned} P &= -\dot{\psi} \sin\theta \\ Q &= \dot{\psi} \cos\theta \sin\phi \\ R &= \dot{\psi} \cos\theta \cos\phi \end{aligned}$$

Step 2 - Resolve the components of $\dot{\theta}$ along the body axes for any aircraft attitude.

Remember, θ is the angle between the x-body axis and the local horizontal (Figure 4.13). Once again, change the aircraft attitude by steps in the sequence of yaw, pitch, and bank and analyze the effects of $\dot{\theta}$.

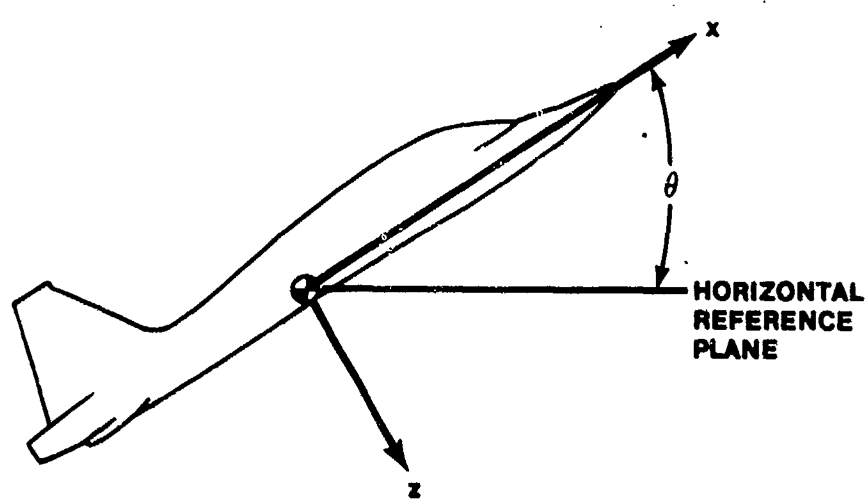


FIGURE 4.13. CONTRIBUTION OF THE EULER PITCH ANGLE RATE TO AIRCRAFT ANGULAR VELOCITIES (θ ROTATION)

It can be seen immediately that the yaw angle has no effect. Likewise when pitched up, the y-body axis remains in the horizontal plane. Therefore, $\dot{\theta}$ is the same as $\bar{Q}$ in this attitude and the component is equal to

$$Q = \dot{\theta}$$

Now bank the aircraft.

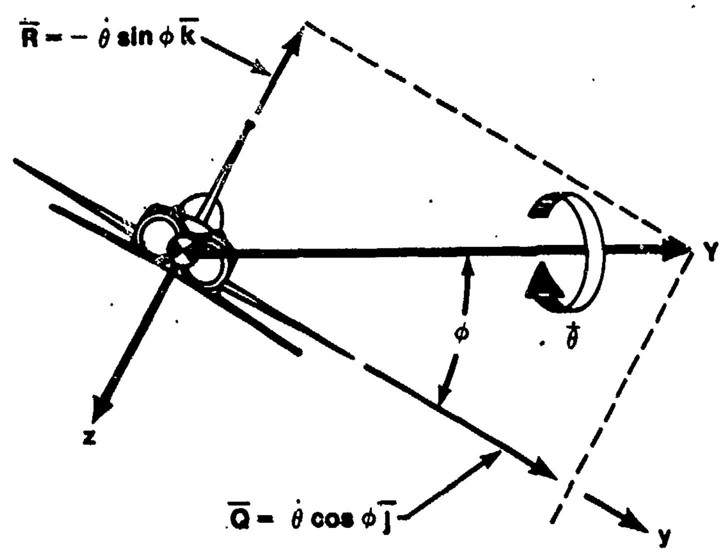


FIGURE 4.14. CONTRIBUTION OF THE EULER PITCH ANGLE RATE TO AIRCRAFT ANGULAR VELOCITIES (θ ROTATION)

It can be seen from Figure 4.14 that the components of $\bar{\dot{\theta}}$ on the body axes are

$$Q = \dot{\theta}\cos\phi$$

$$R = -\dot{\theta}\sin\phi$$

Notice that P is not affected by θ since by definition θ is measured on an axis perpendicular to the x body axis.

<u>Step 3</u> - Resolve the components of $\bar{\dot{\Phi}}$ along the body axes.

This one is easy since by definition $\bar{\dot{\phi}}$ is measured along the x body axis. Therefore, $\bar{\dot{\phi}}$ affects the value of P only, or

$$P = \dot{\phi}$$

The components of $\dot{\overline{\psi}}$, $\dot{\overline{\theta}}$, and $\dot{\overline{\phi}}$ along the x, y, and z body axes for any aircraft attitude have been derived. These can now be summed to give the transformation equations.

$$\overline{P} = (\dot{\phi} - \psi \sin \theta)\overline{i}$$

$$\overline{Q} = (\dot{\theta} \cos \phi + \dot{\psi} \sin \phi \cos \theta)\overline{j}$$

$$\overline{R} = (\dot{\psi} \cos \phi \cos \theta - \dot{\theta} \sin \phi)\overline{k}$$

4.8 ASSUMPTIONS

The following assumptions will be made to simplify the derivation of the equations of motion. The reasons for these assumptions will become obvious as the equations are derived.

Rigid Body - Aeroelastic effects must be considered separately.

Earth and Atmosphere are Assumed Fixed - Allows use of Moving Earth Axis System as an "inertial reference" so that Newton's Law can be applied.

Constant Mass - Most motion of interest in stability and control takes place in a relatively short time.

The x-z plane is a Plane of Symmetry - This restriction is made to simplify the RHS of the equation. This causes two products of inertia, I_{xy}, and I_{yz} to be zero. It allows the cancellation of terms containing these products of inertia. The restriction can easily be removed by including these terms.

4.9 RIGHT-HAND SIDE OF EQUATION

The RHS of the equation represents the aircraft response to forces or moments. Through the application of Newton's Second Law, two vector relations can be used to derive the six required equations, three translational and three rotational.

4.9.1 Linear Force Relation

The vector equation for response to an applied linear force is

$$\bar{F} = \left.\frac{d(m\bar{V})}{dt}\right|_{XYZ} \tag{4.10}$$

or the change in linear momentum of an object is equal to the force applied to it.

This applies only with respect to inertial space. Therefore, the motion of a body is determined by all the forces applied to it including gravitational attraction of the earth, moon, sun, and even the stars. For a great majority of dynamics problems on the earth and in the lower atmosphere, the effects of the sun, moon, and distant stars as well as the spin of the earth and movement of the atmosphere are dynamically inconsequential. When considering the forces on an aircraft, the motion of the earth and atmosphere can then be disregarded since forces resulting from the earth's rotation and Coriolis effects are negligible when compared with the large aerodynamic and gravitational forces involved. This simplifies the derivation considerably. The equations can now be derived using either a fixed or moving earth axis system. For graphical clarity consider a fixed earth axis system. The vehicle represented by the center of gravity symbol has a total velocity vector that is changing in both magnitude and direction.

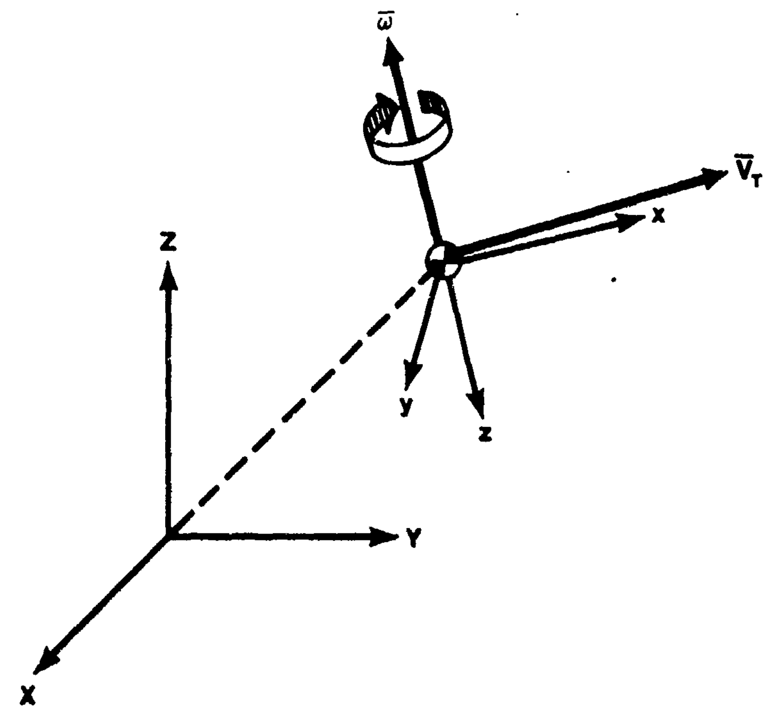


FIGURE 4.15. DERIVATIVE OF A VECTOR IN A ROTATING REFERENCE FRAME

From vector analysis (Appendix A), the acceleration of a rigid axis system, translating at a velocity $\bar{V}_T$ (corresponding to the velocity of its origin) and rotating at $\bar{\omega}$ about an axis of rotation through the origin, (Figure 4.15) will transform to another axis system (x, y, z) by the following relationship:

$$\left.\frac{dV_T}{dt}\right|_{XYZ} = \left.\frac{d\bar{V}_T}{dt}\right|_{xyz} + \bar{\omega} \times \bar{V}_T$$

Substituting this into Equation 4.10, and assuming mass is constant, the applied force is

$$\bar{F} = m\left[\left.\frac{d\bar{V}_T}{dt}\right|_{xyz} + \bar{\omega} \times \bar{V}_T\right]$$

which in component form is

$$\bar{F} = m\left[\dot{U}\bar{i} + \dot{V}\bar{j} + \dot{W}\bar{k} + \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ P & Q & R \\ U & V & W \end{vmatrix}\right]$$

Expanding

$$\bar{F} = m\left[\dot{U}\bar{i} + \dot{V}\bar{j} + \dot{W}\bar{k} + (QW - RV)\bar{i} - (PW - RU)\bar{j} + (PV - QU)\bar{k}\right]$$

Rearranging

$$\bar{F} = m\left[(\dot{U} + QW - RV)\bar{i} + (\dot{V} + RU - PW)\bar{j} + (\dot{W} + PV - QU)\bar{k}\right]$$

Now since

$$\bar{F} = F_x\bar{i} + F_y\bar{j} + F_z\bar{k}$$

these three component translational equations result:

$$F_x = m\,(\dot{U} + QW - RV) \tag{4.11}$$

$$F_y = m\,(\dot{V} + RU - PW) \tag{4.12}$$

$$F_z = m\,(\dot{W} + PV - QU) \tag{4.13}$$

4.9.2 Moment Equations

Once again from Newton's Second Law,

$$\bar{G} = \left.\frac{d(\bar{H})}{dt}\right|_{XYZ} \tag{4.14}$$

or the change in angular momentum is equal to the total applied moment.

4.9.3 Angular Momentum

Angular momentum should not be as difficult to understand as some people would like to make it. It can be thought of as linear momentum with a moment arm included.

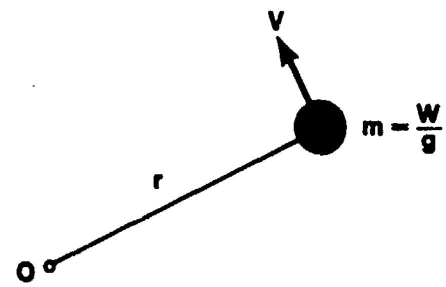


FIGURE 4.16. ANGULAR MOMENTUM

Consider a ball swinging on the end of a string, at any instant of time, as shown in Figure 4.16.

$$\text{Linear momentum} = m\bar{V}$$

and

$$\text{Angular momentum} = mr\bar{V} \text{ (axis of rotation must be specified).}$$

Therefore, they are related in the same manner that forces are related to moments.

$$\text{Moment} = \bar{r} \times \text{Force}$$

$$\text{Angular Momentum} = \bar{r} \times \text{Linear Momentum}$$

and just as a force changes linear momentum, a moment will change angular momentum.

4.9.4 Angular Momentum Of An Aircraft

Consider a small element of mass m_1, somewhere in the aircraft, a distance $\bar{r}_1$ from the cg (Figure 4.17).

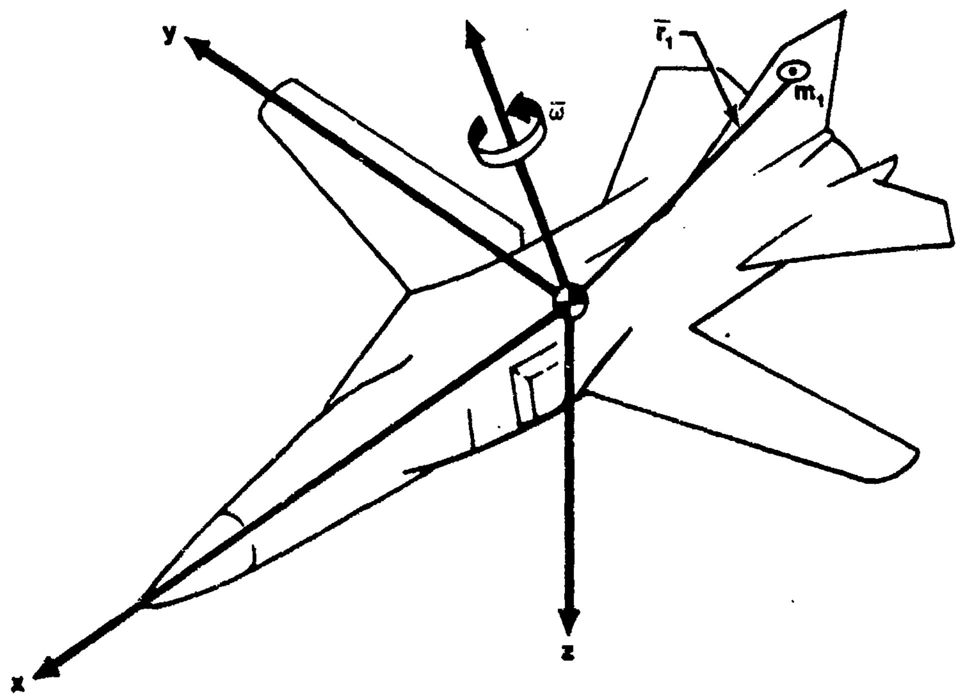


FIGURE 4.17. ELEMENTAL DEVELOPMENT OF RIGID BODY ANGULAR MOMENTUM

The airplane is rotating about all three axes so that

$$\bar{\omega} = P\bar{i} + Q\bar{j} + R\bar{k} \tag{4.15}$$

and

$$\bar{r}_1 = x_1\bar{i} + y_1\bar{j} + z_1\bar{k} \tag{4.16}$$

The angular momentum of m_1 is

$$\bar{H}_{m_1} = m_1 (\bar{r}_1 \times \bar{V}_1) \tag{4.17}$$

and

$$\bar{V}_1 = \left.\frac{d\bar{r}_1}{dt}\right|_{XYZ} \quad \text{(i.e., in the inertial frame)}$$

Again from vector analysis, the radius vector r can be related to the moving earth axis system (XYZ) by

No Aeroelasticity

$$\left.\frac{d\bar{r}_1}{dt}\right|_{XYZ} = \cancel{\left.\frac{d\bar{r}_1}{dt}\right|_{xyz}} + \bar{\omega} \times \bar{r}_1 \tag{4.18}$$

Since the airplane is a rigid body r_1 does not change with time. Therefore the first term can be excluded, and the inertial velocity of the element m_1 is

$$\bar{V}_1 = \bar{\omega} \times \bar{r}_1 \tag{4.19}$$

Substituting this into Equation 4.17

$$\bar{H}_{m_1} = m_1 (\bar{r}_1 \times \bar{\omega} \times \bar{r}_1) \tag{4.20}$$

This is the angular momentum of the elemental mass m_1. In order to find the angular momentum of the whole airplane, take the sum of all the elements. Using notation in which the i subscript indicates any particular element and n the total number of elements in the airplane,

$$\bar{H} = \sum_{i=1}^{n} m_i \left[\bar{r}_i \times \bar{\omega} \times \bar{r}_i\right] \tag{4.21}$$

where

$$m_i = \text{scalar}$$

$$\bar{r}_i = x_i\bar{i} + y_i\bar{j} + z_i\bar{k} \tag{4.22}$$

then

$$\bar{\omega} \times \bar{r}_i = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ P & Q & R \\ x_i & y_i & z_i \end{vmatrix} \tag{4.23}$$

In an effort to reduce the clutter, the subscripts will be left off. The determinant can be expanded to give,

$$\bar{\omega} \times \bar{r} = (Qz - Ry)\bar{i} + (Rx - Pz)\bar{j} + (Py - Qx)\bar{k} \tag{4.24}$$

therefore, Equation 25 becomes

$$\bar{H} = \sum m \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ x & y & z \\ (Qz-Ry) & (Rx-Pz) & (Py-Qx) \end{vmatrix} \tag{4.25}$$

So the components of $\bar{H}$ are

$$H_x = \sum my\ (Py - Qx) - \sum mz\ (Rx - Pz) \tag{4.26}$$

$$H_y = \sum mz\ (Qz - Ry) - \sum mx\ (Py - Qx) \tag{4.27}$$

$$H_z = \sum mx\ (Rx - Pz) - \sum my\ (Qz - Ry) \tag{4.28}$$

Rearranging the equations

$$H_x = P\sum m\ (y^2 + z^2) - Q\sum mxy - R\sum mxz \tag{4.29}$$

$$H_y = Q\sum m\ (z^2 + x^2) - R\sum myz - P\sum mxy \tag{4.30}$$

$$H_z = R\sum m\ (x^2 + y^2) - P\sum mxz - Q\sum myz \tag{4.31}$$

Define moments of inertia as $I = md^2$, where d^2 is measured in a plane normal to the axis of rotation (Figure 4.18). Therefore,

$$I_x = \sum m\ (y^2 + z^2)$$

$$I_y = \sum m\ (x^2 + z^2)$$

$$I_z = \sum m\ (x^2 + y^2)$$

These are a measure of resistance to rotation - they are never zero.

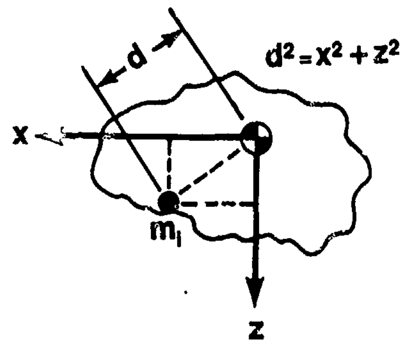


FIGURE 4.18. MOMENT OF INERTIA

Define products of inertia (Figure 4.19)

$$I_{xy} = \sum mxy$$

$$I_{yz} = \sum myz$$

$$I_{xz} = \sum mxz$$

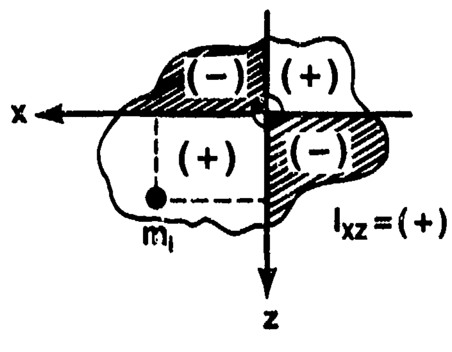


FIGURE 4.19. PRODUCT OF INERTIA

These are a measure of symmetry. They are zero for views having a plane of symmetry.

The angular momentum of a rigid body is therefore

$$\bar{H} = H_x\bar{i} + H_y\bar{j} + H_z\bar{k} \tag{4.32}$$

So that

$$H_x = PI_x - QI_{xy} - RI_{xz} \tag{4.33}$$

$$H_y = QI_y - RI_{yz} - PI_{xy} \tag{4.34}$$

$$H_z = RI_z - PI_{xz} - QI_{yz} \tag{4.35}$$

4.9.5. Simplification Of Angular Moment Equation For Symmetric Aircraft

A symmetric aircraft has two views that contain a line of symmetry and hence two products of inertia that are zero (see Figure 4.20). The angular momentum of a symmetric aircraft therefore simplifies to

$$\bar{H} = (PI_x - RI_{xz})\ \bar{i} + QI_y\bar{j} + (RI_z - PI_{xz})\ \bar{k} \tag{4.36}$$

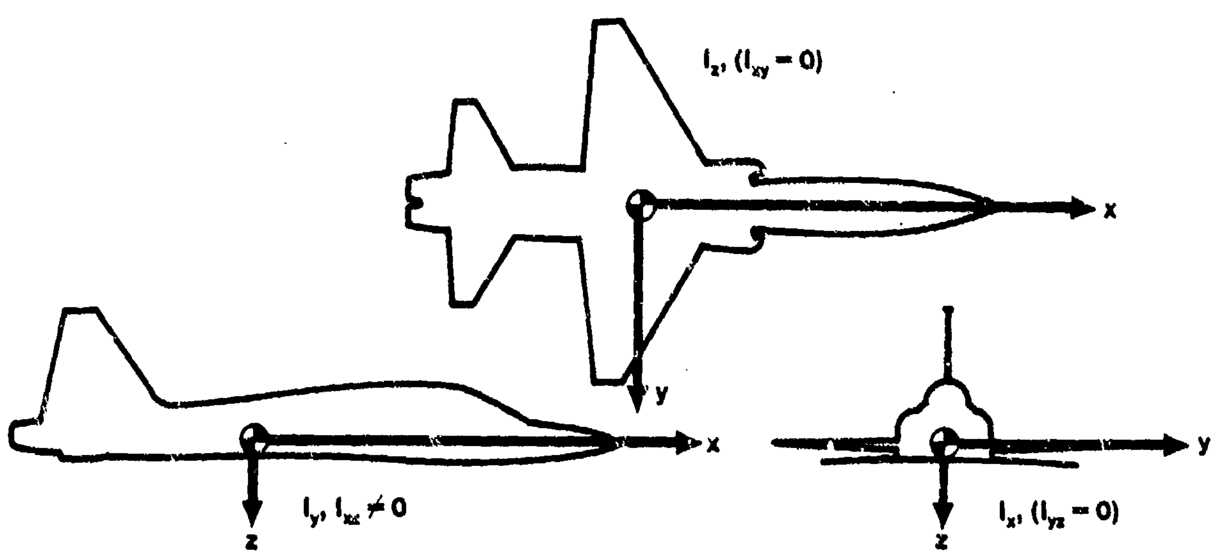


FIGURE 4.20. AIRCRAFT INERTIAL PROPERTIES WITH AN x-z PLANE OF SYMMETRY

4.9.6 Derivation Of The Three Rotational Equations

The equation for angular momentum can now be substituted into the moment equation. Remember

$$\bar{G} = \left.\frac{d\bar{H}}{dt}\right|_{XYZ} \tag{4.37}$$

applies only with respect to inertial space. Expressed in the fixed body axis system, the equation becomes:

$$\bar{G} = \left.\frac{d\bar{H}}{dt}\right|_{xyz} + \bar{\omega} \times \bar{H} \tag{4.38}$$

which is

$$\bar{G} = \dot{H}_x\bar{i} + \dot{H}_y\bar{j} + \dot{H}_z\bar{k} + \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ P & Q & R \\ H_x & H_y & H_z \end{vmatrix} \tag{4.39}$$

Remember, for a symmetric aircraft,

$$\bar{H} = (PI_x - RI_{xz})\,\bar{i} + QI_y\bar{j} + (RI_z - PI_{xz})\,\bar{k} \tag{4.40}$$

Since the body axis system is used, the moments of inertia and the products of inertia are constant. Therefore, by differentiating and substituting, the moment equation becomes

$$\bar{G} = (\dot{P}I_x - \dot{R}I_{xz})\,\bar{i} + \dot{Q}I_y\bar{j} + (\dot{R}I_z - \dot{P}I_{xz})\,\bar{k} + \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ P & Q & R \\ (PI_x - RI_{xz}) & QI_y & (RI_z - PI_{xz}) \end{vmatrix} \tag{4.41}$$

Therefore, the rotational component equations are,

$$G_x = \dot{P}I_x + QR\,(I_z - I_y) - (\dot{R} + PQ)\,I_{xz} \tag{4.42}$$

$$G_y = \dot{Q}I_y - PR\,(I_z - I_x) + (P^2 - R^2)\,I_{xz} \tag{4.43}$$

$$G_z = \dot{R}I_z + PQ\,(I_y - I_x) + (QR - \dot{P})\,I_{xz} \tag{4.44}$$

This completes the development of the RHS of the six equations. Remember the RHS is the aircraft response or the motion of the aircraft that would result from the application of a force or a moment. The LHS of the equation represents these applied forces or moments.

4.10 LEFT-HAND SIDE OF EQUATION

4.10.1 Terminology

Before launching into the development of the LHS, it will help to clarify some of the terms used to describe the motion of the aircraft.

Steady Flight. Motion with zero rates of change of the linear and angular velocity components, i.e.,

$$\dot{U} = \dot{V} = \dot{W} = \dot{P} = \dot{Q} = \dot{R} = 0.$$

Straight Flight. Motion with zero angular velocity components, P, Q, and R = 0.

Symmetric Flight. Motion in which the vehicle plane of symmetry remains fixed in space throughout the maneuver. The unsymmetric variables P, R, V, ϕ, and β are all zero in symmetric flight. Some symmetric flight conditions are wings-level dives, climbs, and pull-ups with no sideslip.

Unsymmetric Flight. Motion in which any or all of the above unsymmetric variables may have non-zero values. Sideslips, rolls, and turns are typical unsymmetric flight conditions.

4.10.2 Some Special-Case Vehicle Motions.

Unaccelerated Flight. (Also called straight flight or equilibrium flight.)

$$F_x = 0; \quad F_y = 0; \quad F_z = 0$$

Hence, the cg travels a straight path at constant speed. Note that equilibrium does not mean steady state. For example,

$$F_x = m(\dot{U} + QW - RV) = 0$$

could be maintained zero by fluctuation of the three terms on the right in an unsteady manner. In practice, however, it is difficult to predict that non-steady motion will remain unaccelerated; hence, the straight motions most often discussed are also steady state.

<u>Steady Straight Flight</u>

$$\underbrace{\begin{array}{ll} F_x = 0 & F_y = 0 \\ F_z = 0 & G_x = 0 \\ G_y = 0 & G_z = 0 \end{array}}_{\text{On the average}} \quad \underbrace{P = Q = R = 0}_{\text{Excluded by custom}}$$

Trim Points, Stabilized Points

<u>Steady Rolls or Spins</u>

$$\underbrace{\begin{array}{ll} F_x = 0 & F_y = 0 \\ F_z = 0 & G_x = 0 \\ G_y = 0 & G_z = 0 \end{array}}_{\text{On the average}}$$

By custom this is not called straight flight even though the cg may be traveling a straight path

Steady Developed Spins

4.10.3 Accelerated Flight (Non-Equilibrium Flight).

One or more of the linear equations is not zero. Again the steady cases are of most interest.

Steady Turns

An unbalanced horizontal force results in the cg being constantly deflected inward toward the center of a curved path. This results in a constantly changing yaw angle. By the Euler angle transform,

$P = -\dot{\psi}\theta$ (assumes small θ)

$Q = \dot{\psi} \sin \phi \cos \theta = \dot{\psi} \sin \phi$

$R = \dot{\psi} \cos \phi \cos \theta = \dot{\psi} \cos \phi$

and hence

$F_y = m\,(\dot{\psi} \cos \phi)U$

$F_z = -m\,(\dot{\psi} \sin \phi)U$ (assumes $\dot{\psi}\theta$ is very, very small)

Includes moderate climbs and descents.

Symmetrical Pull Up

Here an unbalanced z force is constantly deflecting the cg upward.

$Q = \dot{\theta}$

$F_x \approx mQW$

and

$F_z \approx -mQU$

This is a quasi-steady motion since U an W cannot long remain zero.

4.10.4 Preparation For Expansion Of The Left-Hand Side

The Equations of Motion relate the vehicle motion to the applied forces and moments.

LHS		RHS
Applied Forces and Moments	=	Observed Vehicle Motion

$F_x = m\dot{U} + ----$

$G_x = \dot{P}I_x + ----$

etc.

The RHS of each of these six equations has been completely expanded in terms of easily measured quantities. The LHS must also be expanded in terms of convenient variables to include Stability Parameters and Derivatives.

4.10.5 Initial Breakdown of the Left-Hand Side

In general, the applied forces and moments can be broken up according to the sources shown below.

		SOURCE					
		Aero-dynamic	Direct Thrust	Gravity	Gyro-Scopic	Other	
LONGITUDINAL	F_x	X	X_T	X_g	0	X_{oth}	$= m\dot{U} + \cdots$ (4.45)
	F_z	Z	Z_T	Z_g	0	Z_{oth}	$= m\dot{W} + \cdots$ (4.46)
	G_y	$\mathcal{M}$	M_T	0	M_{gyro}	M_{oth}	$= \dot{Q}I_y + \cdots$ (4.47)
LATERAL-DIRECTIONAL	F_y	Y	Y_T	Y_g	0	Y_{oth}	$= m\dot{V} + \cdots$ (4.48)
	G_x	$\mathcal{L}$	L_T	0	L_{gyro}	L_{oth}	$= \dot{P}I_x + \cdots$ (4.49)
	G_z	$\mathcal{N}$	N_T	0	N_{gyro}	N_{oth}	$= \dot{R}I_z + \cdots$ (4.50)

1. Gravity Forces - These vary with orientation of the weight vector.

   $X_g = -mg\sin\theta \quad Y_g = mg\cos\theta\sin\phi \quad Z_g = mg\cos\theta\cos\phi$

2. Gyroscopic Moments - These occur as a result of large rotating masses such as engines and props.

3. Direct Thrust Forces and Moments - These terms include the effect of the thrust vector itself - they usually do not include the indirect or induced effects of jet flow or running propellers.

4. Aerodynamic Forces and Moments - These will be further expanded into Stability Parameters and Derivatives.

5. Other Sources - These include spin chutes, reaction controls, etc.

4.10.6 Aerodynamic Forces And Moments

By far the most important forces and moments on the LHS of the equation are the aerodynamic terms. Unfortunately, they are also the most complex. As a result, certain simplifying assumptions are made, and several of the smaller terms are arbitrarily excluded to simplify the analysis. Remember we are not trying to design an airplane around some critical criteria. We are only trying to derive a set of equations that will help us analyze the important factors affecting aircraft stability and control.

4.10.6.1 Choice Of Axis System. Consider only the aerodynamic forces on an airplane. Summing forces along the x body axis (Figure 4.21)

$$X = L \sin \alpha - D \cos \alpha \tag{4.51}$$

Notice that if the forces were summed along the x stability axis (Figure 4.21), it would be

$$X = -D \tag{4.52}$$

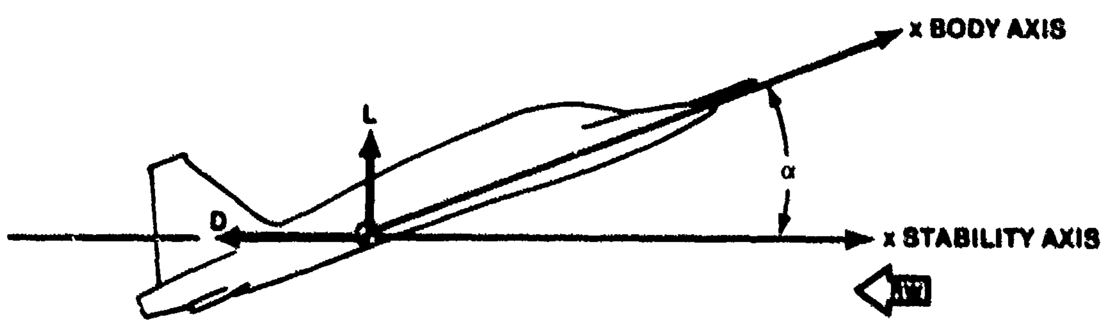


FIGURE 4.21. CHOICE OF AXIS SYSTEM

It would simplify things if the stability axes were used for development of the aerodynamic forces. A small angle assumption enables us to do this:

$$\cos \alpha \cong 1$$

$$\sin \alpha \cong 0$$

Using this assumption, Equation 4.51 reduces to Equation 4.52. Whether it be thought of as a small angle assumption or as an arbitrary choice of the stability axis system, the result is less complexity. This would not be done for preliminary design analyses; however, for the purpose of deriving a set of equations to be used as an analytical tool in determining handling qualities, the assumption is perfectly valid, and is surprisingly accurate for relatively large values of α. It should be noted that lift and drag are defined to be positive as illustrated. Thus these quantities have a negative sense with respect to the stability axis system.

The aerodynamic terms will be developed using the stability axis system so that the equations assume the form,

$$\text{"DRAG"} \quad -D + X_T + X_g + X_{oth} = m\dot{U} + ---- \tag{4.53}$$

$$\text{"LIFT"} \quad -L + Z_T + Z_g + Z_{oth} = m\dot{W} + ---- \tag{4.54}$$

$$\text{"PITCH"} \quad \mathcal{M} + M_T + M_{gyro} + M_{oth} = \dot{Q}I_y + ---- \tag{4.55}$$

$$\text{"SIDE"} \quad Y + Y_T + Y_g + Y_{oth} = m\dot{V} + ---- \tag{4.56}$$

$$\text{"ROLL"} \quad \mathcal{L} + L_T + L_{gyro} + L_{oth} = \dot{P}I_x + ---- \tag{4.57}$$

$$\text{"YAW"} \quad \mathcal{N} + N_T + N_{gyro} + N_{oth} = \dot{R}I_z + ---- \tag{4.58}$$

4.10.6.2 Expansion of Aerodynamic Terms. A stability and control analysis is concerned with how a vehicle responds to perturbation inputs. For instance, up elevator should cause the nose to come up; or for turbulence caused sideslip, the airplane should realign itself with the relative wind. Intuitively, the aerodynamic terms have the most effect on the resulting motion of the aircraft. Unfortunately, the equations that result from summing forces and moments are non-linear, and exact solutions are impossible. In view of the complexity of the problem, linearization of the equations brings about especially desirable simplifications. The linearized model is based on the assumption of small disturbances and the small perturbation theory. This model, nonetheless, gives quite adequate results for engineering purposes over a wide range of applications; because the major aerodynamic effects are nearly linear functions of the variables of interest, and because quite large disturbances in flight may correspond to relatively small disturbances in the linear and angular velocities.

4.10.6.3 Small Perturbation Theory. The small perturbation theory is based on a simple technique used for linearizing a set of differential equations. In aircraft flight dynamics, the aerodynamic forces and moments are assumed to be functions of the instantaneous values of the perturbation velocities, control deflections, and of their derivatives. They are obtained in the form of a Taylor series in these variables, and the expressions are linearized by excluding all higher-order terms. To fully understand the derivation, some assumptions and definitions must first be established.

4.10.6.4 The Small Disturbance Assumption. A summary of the major variables that affect the aerodynamic characteristics of a rigid body or a vehicle is given below.

1. Velocity, temperature, and altitude may be considered directly or indirectly as Mach, Reynolds numbers, and dynamic pressures. Velocity may be resolved into components U, V, and W along the vehicle body axes.

2. Angle of attack, α, and angle of sideslip, β, may be used with the magnitude of the total velocity, V_T, to express the orthogonal velocity components U, V, and W. It is more convenient to express variation of force and moment characteristics with these angles as independent variables rather than the velocity components.

3. Angular velocity is usually resolved into components, P, Q, and R about the vehicle body axes.

4. Control surface deflections are used primarily to change or balance aerodynamic forces and moments, and are accounted for by δ_e, δ_a, δ_r.

Because air has mass, the flow field cannot adjust instantaneously to sudden changes in these variables, and transient conditions exist. In some cases, these transient effects become significant. Analysis of certain unsteady motions may therefore require consideration of the time derivatives of the variables listed above.

D, L, $\mathcal{M}$, Y, $\mathcal{L}$, $\mathcal{N}$ Are a Function of

VARIABLE	FIRST DERIVATIVE	SECOND DERIVATIVE
$U \quad \alpha \quad \beta$	$\dot{U} \quad \dot{\alpha} \quad \dot{\beta}$	$\ddot{U}$ - - - -
$P \quad Q \quad R$	$\dot{P}$ - -	$\ddot{P}$ - - - -
$\delta_e \quad \delta_a \quad \delta_r$	$\dot{\delta}_e$ - -	- - -
$\rho \quad M \quad R_e$	→ assumed constant	

This rather formidable list can be reduced to workable proportions by assuming that the vehicle motion consists only of small deviations from some initial reference condition. Fortunately, this small disturbance assumption applies to many cases of practical interest, and as a bonus, stability parameters and derivatives derived under this assumption continue to give good results for motions somewhat larger.

The variables are considered to consist of some equilibrium value plus an incremental change, called the "perturbed value". The notation for these perturbed values is usually lower case.

$$P = P_0 + p$$

$$U = U_0 + u$$

It has been found from experience that when operating under the small disturbance assumption, the vehicle motion can be thought of as two independent motions, each of which is a function only of the variables shown below.

1. Longitudinal Motion

$$(D, L, \mathcal{M}) = f\,(U, \alpha, \dot{\alpha}, Q, \delta_e) \qquad (4.59)$$

2. Lateral-Directional Motion

$$(Y, \mathcal{L}, \mathcal{N}) = f\,(\beta, \dot{\beta}, P, R, \delta_a, \delta_r) \qquad (4.60)$$

The equations are grouped and named in the above manner because the state variables of the first group U, α, $\dot{\alpha}$, Q, δ_e are known as the longitudinal variables and those of the second group β, $\dot{\beta}$, P, R, δ_a, δ_r are known as the lateral-directional variables. With the conventional simplifying assumptions, the longitudinal and lateral-directional variables will appear explicitly only in their respective group. This separation will also be displayed in the aerodynamic force and moment terms and the equations will completely decouple into two independent sets.

4.10.6.5 Initial Conditions. It will be assumed that the motion consists of small perturbations about some equilibrium condition of steady straight symmetrical flight. The airplane is assumed to be flying wings level with all components of velocity zero except U_0 and W_0. Therefore, with reference to the body axis

$$V_T = U_0 + W_0 \approx U_0 = \text{constant}$$

$$W_0 = \text{small constant} \therefore \alpha_0 = \text{small constant}$$

$$V_0 = 0 \therefore \beta_0 = 0$$

$$P_0 = Q_0 = R_0 = 0$$

We have found that the Equations of Motion simplify considerably when the stability axis is used as the reference axis. This idea will again be employed and the final set of boundary conditions will result.

$$V_T \approx U_0 = \text{constant}$$

$$W_0 = 0 \therefore \alpha_0 = 0$$

$$V_0 = 0 \therefore \beta_0 = 0$$

$$P_0 = Q_0 = R_0 = 0$$

$$(\rho,\ M,\ R_e,\ \text{aircraft configuration}) = \text{constant}$$

4.10.6.6 Expansion By Taylor Series. The equations resulting from summing forces and moments are nonlinear and the exact solutions are not obtainable. An approximate solution is found by linearizing these equations using a Taylor Series expansion and neglecting higher ordered terms.

As an introduction to this technique, assume some arbitrary non-linear function, f(U), having the graphical representation shown in Figure 4.22.

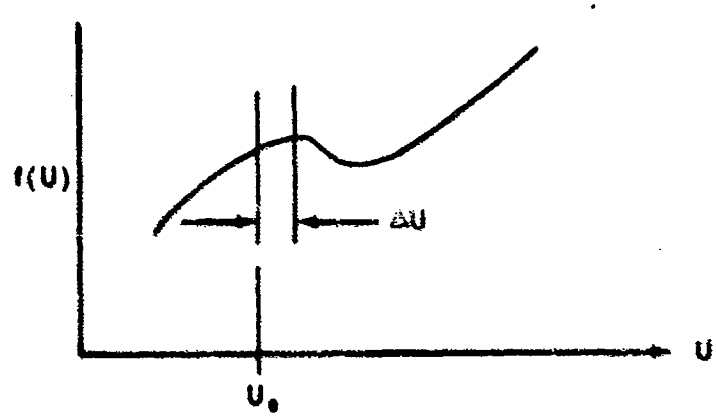
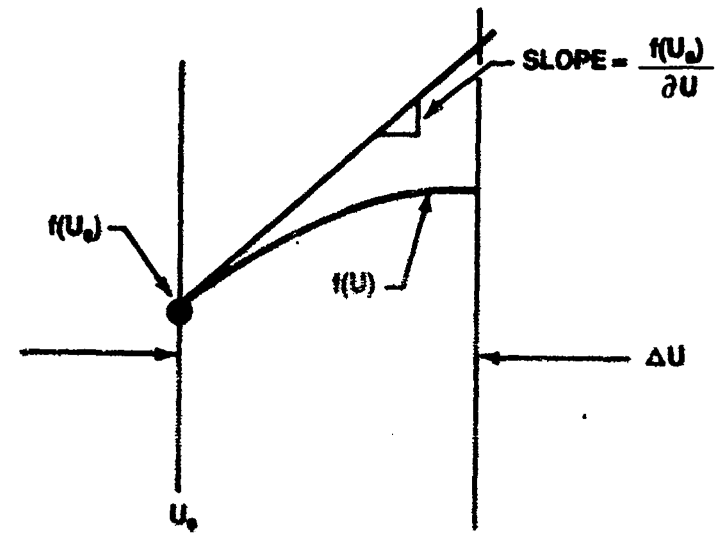


FIGURE 4.22. APPROXIMATION OF AN ARBITRARY FUNCTION BY TAYLOR SERIES

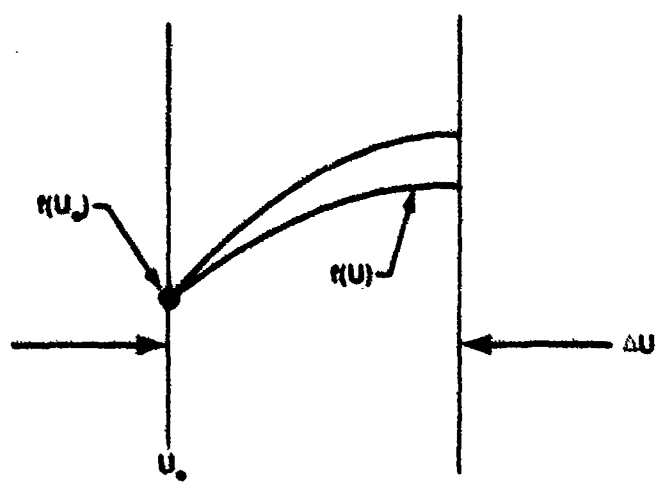
A Taylor Series expansion will approximate the curve over a short span, ΔU. The first derivative assumes the function between ΔU to be a straight line with slope $\partial f(U_0)/\partial U$. This approximation is illustrated in Figure 4.23.



$$f(U)\Big|_{U_o + \Delta U} = f(U_o) + \frac{f(U_o)}{U}\Delta U$$

FIGURE 4.23. FIRST ORDER APPROXIMATION BY TAYLOR SERIES

To refine the accuracy of the approximation, a second derivative term is added. Second order approximation is shown in Figure 4.24.



$$f(U)\Big|_{U_o + \Delta U} = f(U_o) + \frac{f(U_o)}{U}\Delta U + \frac{1}{2!}\frac{\partial^2 f(U_o)}{\partial U^2}(\Delta U)^2$$

FIGURE 4.24. SECOND ORDER APPROXIMATION BY TAYLOR SERIES

Additional accuracy can be further obtained by adding higher order derivatives. The resulting Taylor Series expansion has the form

$$f(U)\Big|_{U_0 + \Delta U} = f(U_0) + \frac{\partial f\ (U_0)}{\partial U}\,\Delta U + \frac{1}{2!}\,\frac{\partial^2 f\ (U_0)}{\partial U^2}\,(\Delta U)^2 + \frac{1}{3!}\,\frac{\partial^3 f\ (U_0)}{\partial U^3}\,(\Delta U)^3$$

$$+ \ldots + \frac{1}{n!}\,\frac{\partial^n f\ (U_0)}{\partial U^n}\,(\Delta U)^n \quad .$$

Reasonably, if we make ΔU smaller, our accuracy will increase and higher order terms can be neglected without significant error. Also since ΔU is small, $(\Delta U)^2$, $(\Delta U)^3$, $(\Delta U)^n$ are very small. Therefore, for small perturbed values of ΔU, the function can be accurately approximated by

$$f(U)\Big|_{U_0 + \Delta U} = f\ (U_0) + \frac{\partial f\ (U_0)}{\partial U}\,\Delta U$$

We can now linearize the aerodynamic forces and moments using this technique. To illustrate, recall the lift term from the longitudinal set of equations. From Equation 4.59 we saw that lift was a function of U, α, α, Q, δ_e. The Taylor Series expansion for lift is therefore

$$L = \left[\begin{array}{lll}
L_0 + \frac{\partial L}{\partial U}\,\Delta U & + \frac{1}{2}\,\frac{\partial^2 L}{\partial U^2}\,\Delta U^2 & + - - - \\[2ex]
+ \frac{\partial L}{\partial \alpha}\,\Delta\alpha & + \frac{1}{2}\,\frac{\partial^2 L}{\partial \alpha^2}\,\Delta\alpha^2 & + - - - \\[2ex]
+ \frac{\partial L}{\partial \alpha}\,\Delta\alpha & + - - - - & \\[2ex]
+ \frac{\partial L}{\partial Q}\,\Delta Q & + - - - - & \\[2ex]
+ \frac{\partial L}{\partial \delta_e}\,\Delta\delta_e & + - - - - &
\end{array}\right]$$

where $L_0 \;=\; L\left(U_0,\ \alpha_0,\ \dot{\alpha}_0,\ Q_0,\ \delta_{e_0}\right)$

Each of the variables are then expressed as the sum of an initial value plus a small perturbated value.

$$U \;=\; U_0 + u \text{ where } u \;=\; \Delta U = U - U_0 \tag{4.62}$$

and

$$\frac{\partial u}{\partial U} \;=\; \frac{\partial\,(U-U_0)}{\partial U} \;=\; \frac{\partial U}{\partial U} - \frac{\partial U_0}{\partial U} \;=\; 1$$

Therefore

$$\frac{\partial L}{\partial U} \;=\; \frac{\partial L}{\partial u}\frac{\partial u}{\partial U} \;=\; \frac{\partial L}{\partial u} \tag{4.63}$$

and

$$\Delta U \;=\; u$$

The first term of the expression then becomes

$$\frac{\partial L}{\partial U}\,\Delta U \;=\; \frac{\partial L}{\partial u}\,u \tag{4.64}$$

Similarly

$$\frac{\partial L}{\partial Q}\,\Delta Q \;=\; \frac{\partial L}{\partial q}\,q \tag{4.65}$$

And all other terms follow.

We also elect to let $\alpha \;=\; \Delta\alpha$, $\dot{\alpha} \;=\; \Delta\dot{\alpha}$ and $\delta_e \;=\; \Delta\delta_e$.

Dropping higher order terms involving u^2, q^2, etc., Equation 4.61 now becomes

$$L \;=\; L_o + \frac{\partial L}{\partial u}\,u + \frac{\partial L}{\partial \alpha}\,\alpha + \frac{\partial L}{\partial \dot{\alpha}}\,\dot{\alpha} + \frac{\partial L}{\partial q}\,q + \frac{\partial L}{\partial \delta_e}\,\delta_e \tag{4.66}$$

The lateral-directional motion is a function of β, $\dot{\beta}$, P, R, δ_a, δ_r and can be handled in a similar manner. For example, the aerodynamic terms for rolling moment become

$$\mathcal{L} = \mathcal{L}_0 + \frac{\partial \mathcal{L}}{\partial \beta}\beta + \frac{\partial \mathcal{L}}{\partial \dot{\beta}}\dot{\beta} + \frac{\partial \mathcal{L}}{\partial p}p + \frac{\partial \mathcal{L}}{\partial r}r + \frac{\partial \mathcal{L}}{\partial \delta_a}\delta_a + \frac{\partial \mathcal{L}}{\partial \delta_r}\delta_r \qquad (4.67)$$

This development can be applied to all of the aerodynamic forces and moments. The equations are linear and account for all variables that have a significant effect on the aerodynamic forces and moments on an aircraft.

The equations resulting from this development can now be substituted into the LHS of the equations of motion.

4.10.7 Effects of Weight

The weight acts through the cg of an airplane and, as a result, has no effect on the aircraft moments. It does affect the force equations as shown in Figure 4.25.

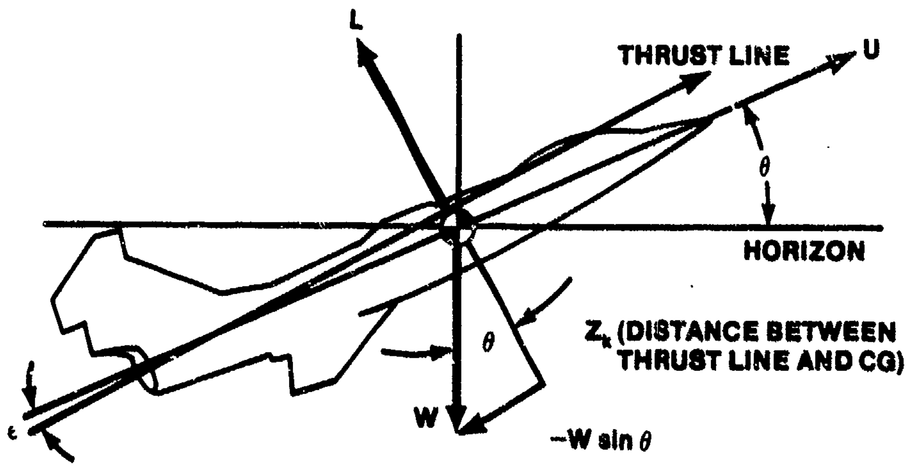


FIGURE 4.25. ORIGIN OF WEIGHT AND THRUST EFFECTS ON FORCES AND MOMENTS

The same "small perturbation" technique can be used to analyze the effects of weight. For longitudinal motion, the only variable to consider is θ. For example, consider the effect of weight on the x-axis.

$$X_g = -W \sin \theta \qquad (4.68)$$

Since weight is considered constant, θ is the only pertinent variable. Therefore, the expansion of the gravity term (X_g) can be expressed as

$$X_g = X_{g_0} + \frac{\partial X_g}{\partial \theta}\theta \qquad X_{g_0} = \text{equilibrium condition of } X_g \tag{4.69}$$

For simplification, the term X_g will be referred to as drag due to weight, (D_{wt}). This incorporates the small angle assumption that was made in development of the aerodynamic terms; however, the effect is negligible. Therefore, Equation 4.69 becomes

$$D_{wt} = D_{0_{wt}} + \frac{\partial D_{wt}}{\partial \theta}\theta$$

Likewise the Z-force can be expressed as negative lift due to weight (L_{wt}), and the expanded term becomes

$$L_{wt} = L_{0_{wt}} + \frac{\partial L_{wt}}{\partial \theta}\theta$$

The effect of weight on side force depends solely on bank angle (ϕ), assuming small θ. Therefore,

$$Y_{wt} = Y_{0_{wt}} + \frac{\partial Y_{wt}}{\partial \phi}\phi$$

These component equations relate the effects of gravity to the equations of motion and can be substituted into the LHS of the equations.

4.10.8 Effects of Thrust

The thrust vector can be considered in the same way. Since thrust does not always pass through the cg, its effect on the moment equation must be considered (Figure 4.25). The component of the thrust vector along the x-axis is

$$X_T = T\cos\epsilon$$

The component of the thrust vector along the z-axis is

$$Z_T = -T \sin \varepsilon$$

The pitching moment component is,

$$M_T = T\,(-Z_k) = -T\,Z_k.$$

where Z_k is the perpendicular distance from the thrust line to the cg and ε is the thrust angle. For small disturbances, changes in thrust depend only upon the change in forward speed and engine RPM. Therefore, by a small perturbation analysis

$$T = T\,(U,\ \delta_{RPM})$$

$$T = T_0 + \frac{\partial T}{\partial u}\,u + \frac{\partial T}{\partial \delta_{RPM}}\,\delta_{RPM} \tag{4.70}$$

Thrust effects will be considered in the longitudinal equations only since the thrust vector is normally in the vertical plane of symmetry and does not affect the lateral-directional motion. When considering engine-out characteristics in multi-engine aircraft, however, the asymmetric thrust effects must be considered. Once again, for clarity, X_T and Z_T will be referred to as "drag due to thrust" and "lift due to thrust" and are components of thrust in the drag (x) and lift (z) directions.
Thus:

$$X_{THRUST} = \left(T_0 + \frac{\partial T}{\partial u}\,u + \frac{\partial T}{\partial \delta_{RPM}}\,\delta_{RPM}\right)\ (\cos \varepsilon) \tag{4.71}$$

$$Z_{THRUST} = -\left(T_0 + \frac{\partial T}{\partial u}\,u + \frac{\partial T}{\partial \delta_{RPM}}\,\delta_{RPM}\right)\ (\sin \varepsilon) \tag{4.72}$$

$$M_{THRUST} = -\left(T_0 + \frac{\partial T}{\partial u}\,u + \frac{\partial T}{\partial \delta_{RPM}}\,\delta_{RPM}\right)\ (Z_k) \tag{4.73}$$

4.10.9 Gyroscopic Effects

Gyroscopic effects are insignificant for most static and dynamic analyses since angular rates are not considered large. They begin to become important as angular rates increase (i.e., P, Q, and R become large). For spin and roll coupling analyses, they are large and gyroscopic effects will be considered. However, in the basic development of the equations of motion, they will not be considered. See Appendix A for a complete set of equations.

4.11 RHS IN TERMS OF SMALL PERTURBATIONS

To conform with the Taylor Series expansion of the LHS, the RHS must also be expressed in terms of small perturbations. Recall that each variable is expressed as the sum of an equilibrium value plus a small perturbed value (i.e., $U = U_0 + u$, $Q = Q_0 + q$, etc.). These expressions can be substituted directly into the full set of the RHS equations (Equations 4.1 - 4.6). As an example, the lift equation (z-direction of longitudinal equations) will be expanded. Start with the RHS of Equation 4.2

$$m\ (\dot{W} + PV - QU)$$

Substitute the initial plus perturbed values for each variable.

$$m\ [\dot{W}_0 + \dot{w} + (P_0 + p)\ (V_0 + v) - (Q_0 + q)\ (U_0 + u)]$$

Multiplying out each term yields:

$$m\ [\dot{W}_0 + \dot{w} + P_0\ V_0 + p\ V_0 + P_0 v + pv - Q_0\ U_0 - q\ U_0 - Q_0 u - qu]$$

Applying the boundary conditions simplifies the equation to

$$m\ [\dot{w} + pv - qU_0 - qu]$$

$$m\ [\dot{w} + pv - q\ (U_0 + u)]$$

or

$$m\,[\dot{w} + pv - qU]$$

The complete set of RHS equations are:

"DRAG": $m\,(\dot{u} + qw - rv)$

"LIFT": $m\,(\dot{w} + pv - qU)$

"PITCH": $\dot{q}\,I_y - pr\,(I_z - I_x) + (p^2 - r^2)\,I_{xz}$

"SIDE": $m\,(\dot{v} + rU - pw)$

"ROLL": $\dot{p}\,I_x + qr\,(I_z - I_y) - (\dot{r} + pq)\,I_{xz}$

"YAW": $\dot{r}\,I_z + pq\,(I_y - I_x) + (qr - \dot{p}\,)\,I_{xz}$

4.12 REDUCTION OF EQUATIONS TO A USABLE FORM

4.12.1 Normalization Of Equations

To put the linearized expressions into a more usable form, each equation is multiplied by a "normalization factor." This factor is different for each equation and is picked to simplify the first term on the RHS of the equation. It is desirable to have the first term of the RHS be either a pure acceleration, $\dot{\alpha}$, or $\dot{\beta}$ and these terms were previously identified in Equations 4.59 and 4.60 as the longitudinal or lateral-directional variables. The following table presents the normalizing factor for each equation:

TABLE 4.1

NORMALIZING FACTORS

Equation	Normalizing Factor	First Term is Now Pure Accel, $\dot{\alpha}$, or $\dot{\beta}$	Units	
"DRAG"	$\frac{1}{m}$	$-\frac{D}{m} + \frac{X_T}{m} + - - - = \dot{u}$ —	$\left[\frac{ft}{sec^2}\right]$	(4.74)
"LIFT"	$\frac{1}{mU_0}$	$-\frac{L}{mU_0} + \frac{Z_T}{mU_0} + - - - = \frac{\dot{w}}{U_0} +$ —	$\left[\frac{rad}{sec}\right]$	(4.75)
"PITCH"	$\frac{1}{I_Y}$	$\frac{\mathcal{M}}{I_Y} + \frac{M_T}{I_Y} + - - - = \dot{q}$ —	$\left[\frac{rad}{sec^2}\right]$	(4.76)
"SIDE"	$\frac{1}{mU_0}$	$\frac{Y}{mU_0} + \frac{Y_T}{mU_0} + - - - = \frac{\dot{v}}{U_0} +$ —	$\left[\frac{rad}{sec}\right]$	(4.77)
"ROLL"	$\frac{1}{I_x}$	$\frac{\mathcal{L}}{I_x} + \frac{L_T}{I_x} + - - - = \dot{p} +$ —	$\left[\frac{rad}{sec^2}\right]$	(4.78)
"YAW"	$\frac{1}{I_z}$	$\frac{\mathcal{N}}{I_z} + \frac{N_T}{I_z} + - - - = \dot{r} +$ —	$\left[\frac{rad}{sec^2}\right]$	(4.79)

4.12.2 Stability Parameters

Stability parameters used in this text are simply the partial coefficients ($\partial L/\partial u$, etc.) multiplied by their respective normalizing factors. They express the variation of forces or moments caused by a disturbance from steady state. Stability parameters are important because they can be used directly as numerical coefficients in a set of simultaneous differential equations describing the dynamics of an airframe. To demonstrate their development, consider the aerodynamic terms of the lift equation.

By multiplying Equation 4.75 by the normalizing factor $1/mU_0$, we get

$$\frac{L}{mU_0} = \frac{L_0}{mU_0} + \underbrace{\frac{1}{mU_0}\frac{\partial L}{\partial u}}_{L_u} u + \underbrace{\frac{1}{mU_0}\frac{\partial L}{\partial \alpha}}_{L_\alpha} \alpha + \underbrace{\frac{1}{mU_0}\frac{\partial L}{\partial \dot{\alpha}}}_{L_{\dot{\alpha}}} \dot{\alpha} + \underbrace{\frac{1}{mU_0}\frac{\partial L}{\partial q}}_{L_q} q + \underbrace{\frac{1}{mU_0}\frac{\partial L}{\partial \delta_e}}_{L_{\delta_e}} \delta_e \left[\frac{rad}{sec}\right] \quad (4.80)$$

The indicated quantities are defined as stability parameters and the equation becomes

$$\frac{L}{mU_0} = \frac{L_0}{mU_0} + L_u u + L_{\dot{\alpha}}\dot{\alpha} + L_\alpha \alpha + L_q q + L_{\delta_e}\delta_e \left[\frac{\text{rad}}{\text{sec}}\right] \qquad (4.81)$$

Stability parameters have various dimensions depending on whether they are multiplied by a linear velocity, an angle, or an angular rate

$$L_u \left[\frac{1}{\text{ft}}\right] u \left[\frac{\text{ft}}{\text{sec}}\right] = \left[\frac{\text{rad}}{\text{sec}}\right], \; L_\alpha \left[\frac{1}{\text{sec}}\right] \alpha \; [\text{rad}] = \left[\frac{\text{rad}}{\text{sec}}\right],$$

$$L_{\dot{\alpha}} \; [\text{none}] \; \dot{\alpha} \left[\frac{\text{rad}}{\text{sec}}\right] = \left[\frac{\text{rad}}{\text{sec}}\right]$$

The lateral-directional motion can be handled in a similar manner. For example, the normalized aerodynamic rolling moment becomes

$$\frac{\mathcal{L}}{I_x} = \frac{\mathcal{L}_0}{I_x} + \mathcal{L}_\beta \beta + \mathcal{L}_{\dot{\beta}}\dot{\beta} + \mathcal{L}_p p + \mathcal{L}_r r + \mathcal{L}_{\delta_a}\delta_a + \mathcal{L}_{\delta_r}\delta_r \left[\frac{\text{rad}}{\text{sec}^2}\right] \qquad (4.82)$$

where

$$\mathcal{L}_\beta = \frac{1}{I_x}\frac{\partial \mathcal{L}}{\partial \beta} \left[\frac{\text{rad}}{\text{sec}^2}\right], \text{ etc.}$$

These stability parameters are sometimes called "dimensional derivatives" or "stability derivative parameters," but we will reserve the word "derivative" to indicate the nondimensional form which can be obtained by rearrangement. This will be developed later in the chapter. See Appendix A for a complete set of equations in stability parameter form.

4.12.3 Simplification Of The Equations

By combining all of the terms derived so far, the resulting equations are somewhat lengthy. In order to economize on effort, several simplifications can be made. For one, all "small effect" terms can be disregarded. Normally these terms are an order of magnitude less than the more predominant terms.

These and other simplifications will help derive a concise and workable set of equations.

4.12.4 Longitudinal Equations

4.12.4.1 Drag Equation. The complete normalized drag equation is

$$\overbrace{-\left[\frac{D_0}{m} + D_\alpha \alpha + D_{\dot\alpha}\dot\alpha + D_u u + D_q q + D_{\delta_e}\delta_e\right]}^{\text{Aero Terms}} \quad - \quad \overbrace{\left[\frac{D_{0_{wt}}}{m} + D_\theta \theta\right]}^{\text{Gravity Terms}}$$

$$\overbrace{-\frac{1}{m}\left[T_0 + \frac{\partial T}{\partial u}u + \frac{\partial T}{\partial \delta_{RPM}}\delta_{RPM}\right](\cos\varepsilon)}^{\text{Thrust Terms}} = \dot u + qw - rv \tag{4.83}$$

Simplifying assumptions

1. $\frac{T_0}{m}\cos\varepsilon - \frac{D_0}{m} - \frac{D_{0_{wt}}}{m} = 0$ (Steady State, Sum to Zero)

2. $\left(\frac{\partial T}{\partial u}u + \frac{\partial T}{\partial \delta_{RPM}}\delta_{RPM}\right)\cos\varepsilon = 0$ (Constant RPM, $\frac{\partial T}{\partial u}$ is small)

3. $rv = 0$ (No lat-dir motion) The small perturbation assumption allows us to analyze the longitudinal motion independent of lateral-directional motion.

4. $qw \approx 0$ (Order of magnitude)

5. $D_{\dot\alpha}$, D_q, and D_{δ_e} are all very small, essentially zero.

The resulting equation is

$$-[D_\alpha \alpha + D_u u + D_\theta \theta] = \dot u \tag{4.84}$$

Rearranging

$$\dot{u} + D_u u + D_\alpha \alpha + D_\theta \theta = 0 \qquad (4.85)$$

4.12.4.2 Lift Equation. The complete life equation is

$$-\overbrace{\left[\frac{L_0}{mU_0} + L_\alpha \alpha + L_{\dot{\alpha}} \dot{\alpha} + L_u u + L_q q + L_{\delta_e} \delta_e\right]}^{\text{Aero Terms}} + \overbrace{\frac{L_{0_{wt}}}{mU_0} + L_\theta \theta}^{\text{Gravity Terms}}$$

$$\overbrace{\left[-\frac{1}{mU_0} \ T_0 + \frac{\partial T}{\partial u} u + \frac{\partial T}{\partial \delta_{RPM}} \delta_{RPM}\right] (\sin \epsilon)}^{\text{Thrust Terms}} = \frac{\dot{w} + pv - qU}{U_0}$$

Simplifying assumptions

1. $-\frac{L_0}{mU_0} + \frac{L_{0_{wt}}}{mU_0} - \frac{T_0}{mU_0} \sin \epsilon = 0$ (Steady State)

2. $\frac{\partial T}{\partial u} u + \frac{\partial T}{\partial \delta_{RPM}} \delta_{RPM} = 0$ (Constant RPM, $\frac{\partial T}{\partial u}$ is small)

3. $L_\theta \theta \approx 0$ (Order of magnitude, for small θ)

4. $\frac{\dot{w}}{U_0} = \dot{\alpha}$ (α is small)

5. $pv = 0$ (No lat-dir motion)

6. $\frac{qU}{U_0} \approx q$ ($U \approx U_0$)

Thus

$$-L_\alpha \alpha - L_{\dot{\alpha}} \dot{\alpha} - L_u u - L_q q - L_{\delta_e} \delta_e = \dot{\alpha} - q \qquad (4.86)$$

Rearranging

$$-L_{\alpha}\alpha - (1 + L_{\dot{\alpha}})\,\dot{\alpha} - L_{u}u + (1 - L_{q})\,q = L_{\delta_e}\,\delta_e \tag{4.87}$$

4.12.4.3 <u>Pitch Moment Equation.</u>

$$\frac{\mathcal{M}_0}{I_y} + \mathcal{M}_{\alpha}\alpha + \mathcal{M}_{\dot{\alpha}}\dot{\alpha} + \mathcal{M}_{u}u + \mathcal{M}_{\delta_e}\delta_e + \mathcal{M}_{q}q - \text{Thrust Terms}$$

$$= \dot{q} - pr\,\frac{(I_z - I_x)}{I_y} + \frac{(p^2 - r^2)}{I_y}\,I_{xz} \tag{4.88}$$

This can be simplified as before. Thus

$$\dot{q} - \mathcal{M}_{q}q - \mathcal{M}_{\alpha}\alpha - \mathcal{M}_{\dot{\alpha}}\dot{\alpha} - \mathcal{M}_{u}u = \mathcal{M}_{\delta_e}\,\delta_e \tag{4.89}$$

We now have three longitudinal equations that are easy to work with. Notice that there are four variables, θ, α, u, and q, but only three equations. To solve this problem, θ can be substituted for q.

$$q = \dot{\theta} \text{ and } \dot{q} = \ddot{\theta}$$

This can be verified from the Euler angle transformation for pitch rate where the roll angle, ϕ, is zero.

$$q = \dot{\theta}\,\cancelto{1}{\cos}\,\phi + \phi\,\cancelto{0}{\sin}\,\phi\,\cos\theta$$

$$\therefore q = \theta$$

Therefore the longitudinal equations become

	(θ)	(u)	(a)		
DRAG	$D_\theta \theta$	$+ \dot{u} + D_u u$	$+ D_\alpha \alpha$	$= 0$	(4.90)
LIFT	$(1-L_q)\ \dot{\theta}$	$- L_u u$	$- (1 + L_{\dot{\alpha}})\ \dot{\alpha} - L_\alpha \alpha$	$= L_{\delta_e}\ \delta_e$	(4.91)
PITCH	$\theta - \mathcal{M}_q \dot{\theta}$	$- \mathcal{M}_u u$	$- \mathcal{M}_{\dot{\alpha}} \dot{\alpha} - \mathcal{M}_\alpha \alpha$	$= \mathcal{M}_{\delta_e}\ \delta_e$	(4.92)

There are now three independent equations with three variables. The terms on the RHS are now the inputs or "forcing functions." Therefore, for any input δ_e, the equations can be solved to get θ, u and α at any time.

4.12.5 Lateral-Directional Equations

The complete lateral-directional equations are as follows:

4.12.5.1 Side Force.

$$\frac{Y_0}{mU_0} + Y_\beta \beta + Y_{\dot{\beta}} \dot{\beta} + Y_p p + Y_r r + Y_{\delta_a}\ \delta_a + Y_{\delta_r}\ \delta_r$$

$$+ \frac{Y_{0_{wt}}}{mU_0} + Y_\phi \phi \;=\; \frac{\dot{v} + rU - pw}{U_0} \qquad (4.93)$$

4.12.5.2 Rolling Moment.

$$\frac{0}{I_x} + \mathcal{L}_\beta \beta + \mathcal{L}_{\dot{\beta}} \dot{\beta} + \mathcal{L}_p p + \mathcal{L}_r r + \mathcal{L}_{\delta_a}\ \delta_a + \mathcal{L}_{\delta_r} \delta_r$$

$$= \dot{p} + qr\ \frac{I_z - I_y}{I_x} - (\dot{r} - pq)\ \frac{I_{xz}}{I_x} \qquad (4.94)$$

4.12.5.3 Yawing Moment.

$$\frac{\mathcal{N}_0}{I_z} + \mathcal{N}_\beta \beta + \mathcal{N}_{\dot{\beta}} \dot{\beta} + \mathcal{N}_p p + \mathcal{N}_r r + \mathcal{N}_{\delta_a} \delta_a + \mathcal{N}_{\delta_r} \delta_r$$

$$= \dot{r} + pq\left(\frac{I_y - I_x}{I_z}\right) + (qr - \dot{p})\,\frac{I_{xz}}{I_z}\,. \qquad (4.95)$$

In order to simplify the equations, the following assumptions are made:

1. A wings level steady state condition exists initially. Therefore, $\mathcal{L}_0, \mathcal{N}_0, Y_0$, and $Y_{0_{wt}}$ are zero.

2. $p = \dot{\phi},\ \dot{p} = \ddot{\phi}$ ($\theta \approx 0$, see Euler angle transformations for roll rate, Equation 4.7)

3. The terms $Y_{\dot{\beta}}\dot{\beta}, \mathcal{N}_{\dot{\beta}}\dot{\beta}, \mathcal{L}_{\dot{\beta}}\dot{\beta}$, and $Y_{\delta_a}\delta_a$ are all small, essentially zero.

4. $\dfrac{\dot{v}}{U_0} = \dot{\beta}$ (β is small)

5. $\dfrac{rU}{U_0} = r$ ($U \approx U_0$)

6. $q = 0$ (no pitching rate in the Lateral-Direction equations)

7. $w = 0$ (no longitudinal motion)

Using these assumptions, the lateral-directional equations reduce to

	(β)	(ϕ)	(r)		
SIDE FORCE	$\dot{\beta}-Y_\beta\beta$	$-Y_p\dot{\phi}-Y_\phi\phi$	$+(1-Y_r)\,r$	$=Y_{\delta_r}\delta_r$	(4.96)
ROLLING MOMENT	$-\mathcal{L}_\beta\beta$	$+\ddot{\phi}-\mathcal{L}_p\dot{\phi}$	$-\frac{I_{xz}}{I_x}\dot{r}-\mathcal{L}_r r$	$=\mathcal{L}_{\delta_a}\delta_a+\mathcal{L}_{\delta_r}\delta_r$	(4.97)
YAWING	$-\mathcal{N}_\beta\beta$	$-\frac{I_{xz}}{I_z}\ddot{\phi}-\mathcal{N}_p\dot{\phi}$	$+\dot{r}-\mathcal{N}_r r$	$=\mathcal{N}_{\delta_a}\delta_a+\mathcal{N}_{\delta_r}\delta_r$	(4.98)

Once again, there are three unknowns and three equations. These equations may be used to analyze the lateral-directional motion of the aircraft.

4.13 STABILITY DERIVATIVES

The parametric equations give all the information necessary to describe the motion of any particular airplane. There is only one problem. When using a wind tunnel model for verification, a scaling factor must be used to find the values for the aircraft. In order to eliminate this requirement, a set of nondimensional equations must be derived. This can be illustrated best by the following example:

Given the parametric equation for pitching moment,

$$\ddot{\theta}-\mathcal{M}_q\dot{\theta}-\mathcal{M}_u u-\mathcal{M}_{\dot{\alpha}}\dot{\alpha}-\mathcal{M}_\alpha\alpha=\mathcal{M}_{\delta_e}\delta_e$$

Derive an equation in which all terms are NONDIMENSIONAL.

The steps in this process are:

1. Take each stability parameter and substitute its coefficient relation and take the derivative at the initial condition, i.e.,

$$\mathcal{M}_q=\frac{1}{I_y}\frac{\partial\mathcal{M}}{\partial q}=\frac{1}{I_y}\left.\frac{\partial\left(C_m\frac{1}{2}\rho U^2 Sc\right)}{\partial q}\right|_0 \tag{4.99}$$

C_m is the only variable that is dependent on q, therefore,

$$\mathcal{M}_q = \frac{\rho U_0^2 Sc}{2I_y} \frac{\partial C_m}{\partial q} \tag{4.100}$$

2. Nondimensionalize the partial term, i.e.,

$$\frac{\partial C_m}{\partial q} \text{ has dimensions } = \frac{\text{dimensionless}}{\text{rad/sec}} = \text{sec}$$

To nondimensionalize the partial terms, there exist certain compensating factors that will be shown later. In this case, the compensating factor is

$$\frac{c}{2U_0} \frac{[\text{ft}]}{[\text{ft/sec}]} = \text{sec}$$

Multiply and divide Equation (4.100) by the compensating factor and get

$$\mathcal{M}_q = \frac{\rho U_0^2 Sc}{2I_y} \cdot \frac{c}{2U_0} \cdot \boxed{\frac{\frac{\partial C_m}{\partial q}}{\frac{c}{2U_0}}}$$

This term is now dimensionless.

Check

$$\frac{\partial m}{\partial\left(\frac{cq}{2U_0}\right)} = \frac{\text{dimensionless}}{\frac{\text{ft/sec}}{\text{ft/sec}}} = \text{dimensionless}$$

This is called a stability derivative and is written

$$C_{m_q} = \frac{\partial C_m}{\partial\left(\frac{cq}{2U_0}\right)}$$

The basic nondimensional form $\left(C_{m_q}\right)$ is important because correlation between geometrically similar airframes or the same airframe at different flight conditions is easily attained with stability derivatives. Additionally, aerodynamic stability derivative data from wind tunnel tests, flight tests, and theoretical analyses are usually presented in nondimensional form. Stability derivatives generally fall into two classes: static and dynamic.

Static derivatives arise from the position of the airframe with respect to the relative wind $\left(\text{i.e., } C_{L_\alpha}, C_{m_\alpha}, C_{n_\beta}, C_{\ell_\beta}\right)$. Whereas, dynamic derivatives arise from the motion (velocities) of the airframe $\left(\text{i.e., } C_{L_{\dot\alpha}}, C_{m_q}, C_{n_r}, C_{\ell_p}\right)$.

3. When the entire term as originally derived is considered, i.e.,

$$\mathcal{M}_q q = \frac{\rho U_0^{\,2}\, Sc}{2I_y} \cdot \frac{c}{2U_0} \cdot C_{m_q} \cdot q$$

It can be rearranged so that

$$\mathcal{M}_q q = \frac{\rho U_0^{\,2}\, Sc}{2I_y} \cdot C_{m_q} \cdot \frac{cq}{2U_0}$$

Define

$$\hat{q} = \frac{cq}{2U_0} \quad \frac{[\text{ft/sec}]}{[\text{ft/sec}]} \quad \text{dimensionless}$$

∴ The term becomes

$$\mathcal{M}_q q = \underbrace{\frac{\rho U_0^{\,2}\, Sc}{2I_y}}_{\text{Constants}} \; \underbrace{C_{m_q}}_{\text{Dimensionless stability derivative}} \; \underbrace{\hat{q}}_{\text{Dimensionless variable}}$$

But q is expressed as $\dot\theta$ in the equation. To convert this, substitute $d\theta/dt$ for q.

$$\hat{q} = \frac{cq}{2U_0} = \frac{c}{2U_0}\frac{d(\theta)}{dt}$$

Then,

$$\text{Define} \quad \nabla = \frac{c}{2U_0}\frac{d(\;)}{dt}$$

∇ can be considered to be a dimensionless derivative with respect to time and acts like an operator.

Therefore,

$$q = \nabla\theta = \frac{c}{2U_0}\frac{d\theta}{dt} \text{ (i.e., dimensionless derivative of } \theta\text{)}$$

4. Do the same for each term in the parametric equation.

$$\mathcal{M}_u = \frac{1}{I_y}\frac{\partial \mathcal{M}}{\partial u} = \frac{1}{I_y}\left.\frac{\partial\left(C_m \frac{1}{2}\rho U^2 Sc\right)}{\partial u}\right|_0$$

Since both C_m and U are functions of u, then

$$\mathcal{M}_u = \frac{\rho Sc}{2I_y}\left.\frac{\partial\left(C_m U^2\right)}{\partial u}\right|_0$$

$$\mathcal{M}_u = \frac{\rho Sc}{2I_y}\left[U_0^2\frac{\partial C_m}{\partial u} + 2C_{m_0} U_0\right]$$

$$\therefore \mathcal{M}_u = \frac{\rho U_0{}^2 Sc}{2I_y}\left(\frac{\partial C_m}{\partial u} + \frac{2C_{m_0}}{U_0}\right)$$

but $C_{m_0} = 0$ since initial conditions are steady state. The compensating factor for this case is $1/U_0$

$$\therefore \mathcal{M}_u u = \frac{\rho U_0{}^2 Sc}{2I_y} \cdot \underbrace{\left(\frac{\partial C_m}{\partial \frac{u}{U_0}}\right)}_{C_{m_u}} \cdot \underbrace{\left(\frac{u}{U_0}\right)}_{\hat{u}}$$

5. Once all of the terms have been derived, they are substituted into the original equation, and multiplied by

$$\frac{2I_y}{\rho U_0{}^2 Sc}$$

which gives

$$\frac{2I_y}{\rho U_0^{\,2}\ Sc}\ddot{\theta} - C_{m_q}\nabla\theta - C_{m_u}\hat{u} - C_{m_\alpha}\nabla\alpha - C_{m_\alpha}\alpha = C_{m_{\delta_e}}\delta_e$$

The first term is nondimensional, however, it can be changed to a more convenient form. Multiply and divide by

$$c^2/4U_0^{\,2}$$

$$\frac{2I_y}{\rho U_0^{\,2}\ Sc}\ \frac{4U_0^2}{c^2} \qquad\qquad \frac{c^2}{4U_0^2}\ddot{\theta}$$

Define $\qquad i_y = \dfrac{8I_y}{\rho S c^3}$

$$\frac{c^2}{4U_0^{\,2}}\ddot{\theta} = \frac{c}{2U_0}\,\frac{d\left(\frac{c}{2U_0}\ \frac{d(\theta)}{dt}\right)}{dt}$$

$$= \nabla^2\,\theta$$

Therefore, the term becomes,

$$i_y\,\nabla^2\,\theta$$

and the final equation is,

$$\left(i_y\,\nabla^2 - C_{m_q}\nabla\right)\theta - C_{m_u}\hat{u} - \left(C_{m_{\dot\alpha}}\nabla - C_{m_\alpha}\right)\alpha = C_{m_{\delta_e}}\delta_e$$

The compensating factors for all of the variables are listed in Table 4.2.

TABLE 4.2

COMPENSATING FACTORS

Angular Rates	Compensating Factor	Nondimensional Angular Rates
p = rad/sec	$\frac{b}{2U_0}$	$\hat{p} = \frac{pb}{2U_0} = \nabla\, \phi$
q = rad/sec	$\frac{c}{2U_0}$	$\hat{q} = \frac{qc}{2U_0} = \nabla\, \theta$
r = rad/sec	$\frac{b}{2U_0}$	$\hat{r} = \frac{rb}{2U_0} = \nabla\, \phi$
$\dot{\beta}$ = rad/sec	$\frac{b}{2U_0}$	$\hat{\dot{\beta}} = \frac{\dot{\beta} b}{2U_0} = \nabla\, \beta$
$\dot{\alpha}$ = rad/sec	$\frac{c}{2U_0}$	$\hat{\dot{\alpha}} = \frac{\dot{\alpha} c}{2U_0} = \nabla\, \alpha$
u = ft/sec	$\frac{1}{U_0}$	$\hat{u} = \frac{u}{U_0}$

α - no change

β - no change

These derivations have been presented to give an understanding of their origin and what they represent. It is not necessary to be able to derive each and every one of the equations. It is important, however, to understand several facts about the nondimensional equations.

1. Since these equations are nondimensional, they can be used to compare aircraft characteristics of geometrically similar airframes.

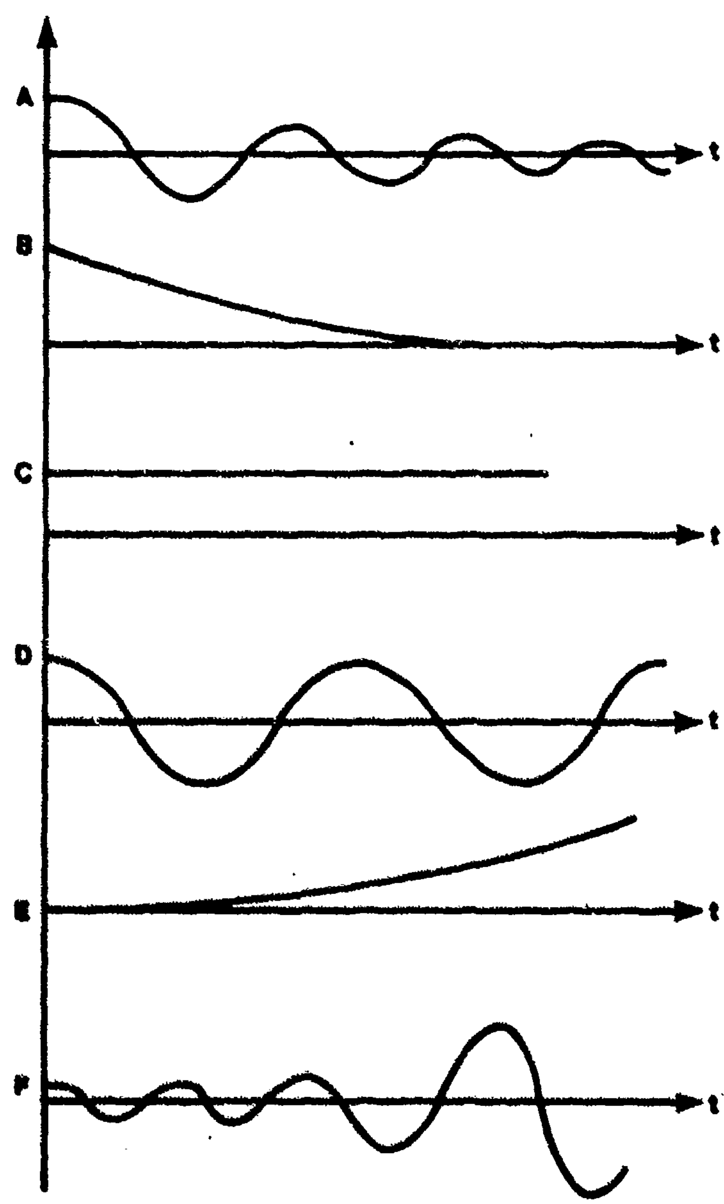
2. Stability derivatives can be thought of as if they were stability parameters. Therefore, C_{m_α} refers to the same aerodynamic characteristics as M_α, only it is in a nondimensional form.

3. Most aircraft designers and builders are accustomed to speaking in terms of stability derivatives. Therefore, it is a good idea to develop a "feel" for all of the important ones.

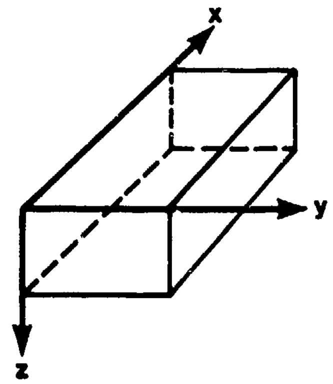
4. These equations as well as the parametric equations describe the complete motion of an aircraft. They can be programmed directly into a computer and connected to a flight simulator. They may also be used in cursory design analyses. Due to their simplicity, they are especially useful as an analytical tool to investigate aircraft handling qualities and determine the effect of changes in aircraft design.

PROBLEMS

4.1. What type of stability is depicted by the following time histories?



4.2. Draw in the vectors $\overline{V}_T$, $\overline{U}$, $\overline{V}$, $\overline{W}$, and show the angles α and β.



Derive $\alpha \cong W/V_T$ and $\beta \cong V/V_T$ using the small angle assumption.

4.3. Define "Right Hand" and "Orthogonal" with reference to a coordinate system.

4.4. Define "Moving" earth and "Fixed" earth coordinate systems.

4.5. Describe the "Body" and Stability" axes systems.

4.6. Define:

a. $\mathcal{L}$
b. $\mathcal{M}$
c. $\mathcal{N}$
d. P
e. Q
f. R

4.7. Define ψ, ϕ, θ. What are they used for? In what sequence must they be used? Explain the difference between ψ and β.

4.8. What are the expressions for P, Q, R, in terms of Euler angles?

4.9. Given $F = d/dt\left(m\bar{V}_T\right)$, $\bar{F}$ is a force vector, m is a constant mass, and $\bar{V}_T$ is the velocity vector of the mass center. Find F_x, F_y, and F_z (if $\bar{V}_T = U\bar{i} + V\bar{j} + W\bar{k}$ and $\bar{\omega} = P\bar{i} + Q\bar{j} + R\bar{k}$) with respect to the fixed earth axis system.

4.10. Given $\bar{H} = \sum_{i=1}^{n} m_i \left(\bar{r}_i \times \bar{V}_i\right)$ where m_i is the mass of the i^{th} particle, $\bar{r}_i$ is the radius vector from the cg to the i^{th} particle, and $\bar{V}_i$ is the velocity, with respect to the cg, of the i^{th} particle and n is the number of particles. Find H_x with respect to the fixed earth axis system.

4.11. Write H_x in terms of I_x, I_{xy}, and I_{xz}, given:

$$I_x = \sum_{i=1}^{n} m_i \left(y_i^2 + z_i^2\right)$$

$$I_{xy} = \sum_{i=1}^{n} m_i \left(x_i y_i\right)$$

$$I_{xz} = \sum_{i=1}^{n} m_i \left(x_i z_i\right)$$

4.12. Draw the three views of a symmetric aircraft and explain why $I_{xy} = 0$, $I_{yz} = 0$, and $I_{xz} = 0$. What is the aircraft's plane of symmetry?

4.13. Using the results from Problems 4.11 and 4.12, simplify the H_x equation.

4.14. If $\overline{G} = d\overline{H}/dt \Big|_{XYZ}$, use the following:

$$H_x = (PI_x - RI_{xz})$$

$$H_y = QI_y$$

$$H_z = (RI_z - PI_{xz}) \quad .$$

to derive G_x, G_y, and G_z.

4.15. What is the difference between straight flight and steady straight flight?

4.16. Make a chart that has 5 columns and 6 rows. The columns should contain the terms of the left hand side of the equations of motion. Also, name each equation. (Pitch, drag, etc.)

4.17. $D, L, \mathcal{M} = f(\quad , \quad , \quad , \quad , \quad)$
$Y, \mathcal{L}, \mathcal{N} = f(\quad , \quad , \quad , \quad , \quad , \quad)$

4.18. Write $\Delta L/mU_0$ in terms of the stability parameters. Define L_u, L_α, $L_{\dot{\alpha}}$, L_q, L_{δ_e}.

4.19. Repeat 4.18 above for the other 5 equations.

4.20. Given

$$\frac{\Delta\mathcal{M}}{I_y} + \mathcal{M}_u u + \mathcal{M}_\alpha \alpha + \mathcal{M}_{\dot{\alpha}} \dot{\alpha} + \mathcal{M}_{\delta_e} \delta_e + \mathcal{M}_q q$$

Where

$$\mathcal{M}_u = \frac{1}{I_y} \frac{\partial \mathcal{M}}{\partial u}, \quad \mathcal{M}_\alpha = \frac{1}{I_y} \frac{\partial \mathcal{M}}{\partial \alpha}, \quad \mathcal{M}_{\dot{\alpha}} = \frac{1}{I_y} \frac{\partial \mathcal{M}}{\partial \dot{\alpha}}$$

$$\mathcal{M}_{\delta_e} = \frac{1}{I_y} \frac{\partial \mathcal{M}}{\partial \delta_e}, \quad \mathcal{M}_q = \frac{1}{I_y} \frac{\partial \mathcal{M}}{\partial q}$$

Find: C_{M_u}, C_{M_α}, C_{M_α}, $C_{M_{\delta_e}}$, and C_{M_q};

if the compensating factor for u is $1/U_0$, α and δ_e's compensating factor is 1, and α and q's compensating factor is $c/2U_0$.

4.21. Repeat Problem 4.20 for

(a) $\dfrac{\Delta \mathcal{L}}{I_x}$

(b) $\dfrac{\Delta \mathcal{N}}{I_z}$

<u>Note</u>: The compensating factor for β, δ_e, and δ_r is 1. The compensating factor for $\dot{\beta}$, p, r is $b/2U_0$.

4.22. Given: $\hat{p} = \dfrac{pb}{2U_0}$ $\quad \hat{q} = \dfrac{qc}{2U_0}$ $\quad \hat{r} = \dfrac{rb}{2U_0}$

$$\hat{\beta} = \frac{\dot{\beta} b}{2U_0} \qquad \hat{\alpha} = \frac{\dot{\alpha} c}{2U_0}$$

(a) Show that $C_\ell = C_{\ell_0} + C_{\ell_\beta}\beta + C_{\ell_{\dot{\beta}}}\hat{\beta} + C_{\ell_p}\hat{p} + C_{\ell_r}\hat{r} +$

$$C_{\ell_{\delta_a}}\delta_a + C_{\ell_{\delta_r}}\delta_r$$

(b) Show that $C_n = C_{n_0} + C_{n_\beta}\beta + C_{n_{\dot{\beta}}}\hat{\beta} + C_{n_p}\hat{p} + C_{n_r}\hat{r} +$

$$C_{n_{\delta_a}}\delta_a + C_{n_{\delta_r}}\delta r$$

CHAPTER 5

LONGITUDINAL STATIC STABILITY

5.1 DEFINITION OF LONGITUDINAL STATIC STABILITY

Static stability is the reaction of a body to a disturbance from equilibrium. To determine the static stability of a body, the body must be initially disturbed from its equilibrium state. If, when disturbed from equilibrium, the initial tendency of the body is to return to its original equilibrium position, the body displays positive static stability or is stable. If the initial tendency of the body is to remain in the disturbed position, the body is said to be neutrally stable. However, should the body, when disturbed, initially tend to continue to displace from equilibrium, the body has negative static stability or is unstable.

Longitudinal static stability or "gust stability" of an aircraft is determined in a similar manner. If an aircraft in equilibrium is momentarily disturbed by a vertical gust, the resulting change in angle of attack causes changes in lift coefficients on the aircraft (velocity is constant for this time period). The changes in lift coefficients produce additional aerodynamic forces and moments in this disturbed position. If the aerodynamic forces and moments created tend to return the aircraft to its original undisturbed condition, the aircraft possesses positive static stability or is stable. Should the aircraft tend to remain in the disturbed position, it possesses neutral stability. If the forces and moments tend to cause the aircraft to diverge further from equilibrium, the aircraft possesses negative longitudinal static stability or is unstable. Pictorial examples of static stability as related to the gust stability of an aircraft are shown in Figure 5.1.

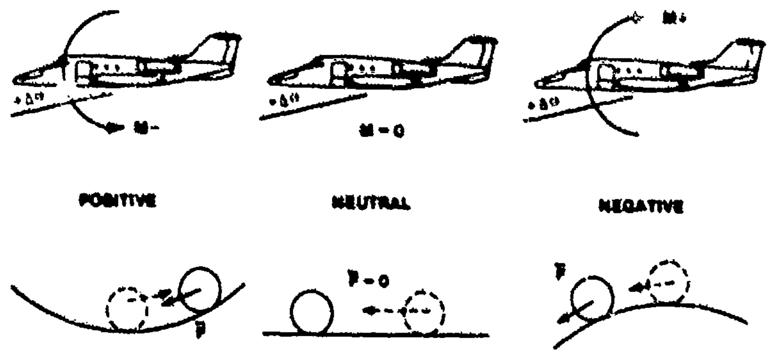


FIGURE 5.1. STATIC STABILITY AS RELATED TO GUST STABILITY OF AIRCRAFT

5.2 DEFINITIONS

Aerodynamic center. The point of action of the lift and drag forces such that the value of the moment created does not change with angle of attack.

Apparent stability. The value of dF_s/dV_e about trim velocity. Also referred to as "speed stability".

Aerodynamic balancing. Design or tailoring $C_{h_{\delta_e}}$ and $C_{h_{\alpha_t}}$ to either increase or decrease hinge moments and floating tendency.

Center of pressure. The point along the chord of an airfoil, or on an aircraft itself, where the lift and drag forces act, and there is no moment produced.

Dynamic elevator balancing. Designing $C_{h_{\alpha_t}}$ (the floating moment coefficient) to be small or zero.

Dynamic overbalancing. Designing $C_{h_{\alpha_t}}$ to be negative (tail to rear aircraft only).

Elevator effectiveness. The change in tail angle of attack per degree or change in elevator deflection. $\tau = d\alpha_t/d\delta_e$ and equals -1.0 for the all moving horizontal tail or "stabilizer".

Elevator power. A control derivative. $C_{m_{\delta_e}} = -aV_H\eta_t\tau$

Hinge moments. The moment about the hinge line of a control surface.

Longitudinal static stability (or "gust" stability). The initial tendency of an aircraft to return to trim when disturbed in pitch. $dC_m/dC_L < 0$ for the airplane to be statically stable. The airplane must also be able to trim at a useful positive C_L.

Static elevator balancing. Balancing the elevator so that the C_h contribution due to the weight of the surface is zero.

Stick-fixed neutral point. The cg location when $dC_m/dC_L = 0$ for the stick-fixed airplane.

Stick-fixed stability. The magnitude of dC_m/dC_L for the stick-fixed airplane.

Stick-fixed static margin. The distance, in percent MAC, between the cg and the stick-fixed neutral point.

Stick force gradient. The value of dF_s/dV_e about trim velocity. Also referred to as "speed stability" and "apparent stability".

Stick-free neutral point. The cg position where $dC_m/dC_L = 0$ for the stick-free airplane.

Tail efficiency factor. $\eta_t = q_t/q_w$, the ratio of tail dynamic pressure to wing dynamic pressure.

Tail volume coefficient. $V_H = l_t S_t / C_w S_w$

5.3 MAJOR ASSUMPTIONS

1. Aerodynamic characteristics are linear $\left(C_{L_\alpha},\ dC_m/dC_L,\ C_{m_{\delta_e}},\ \text{etc.}\right)$
2. The aircraft is in steady, straight ($\beta = 0^\circ$, $\phi = 0^\circ$), unaccelerated flight ($\dot{q}$, $\dot{p}$, $\dot{r}$ are all zero).
3. Power is at a constant setting.
4. Jet engine thrust does not change with velocity or angle of attack.
5. The lift curve slope of the tail is very nearly the same as the slope of the normal force curve.
6. $C_{m_\alpha} = C_{L_\alpha} (dC_m/dC_L)$ is true for rigid aircraft at low Mach when thrust effects are small.
7. X_w, Z_w, V_H, and η_t do not vary with C_L.
8. C_t may be neglected since it is 1/10 the magnitude of C_w and 1/100 the magnitude of N_w.
9. Fighter-type aircraft and most low wing, large aircraft have cg's very close to the top of the mean aerodynamic chord.
10. Elevator effectiveness and elevator power are constant.

5.4 ANALYSIS OF LONGITUDINAL STATIC STABILITY

Longitudinal static stability is only a special case for the total equations of motion of an aircraft. Of the six equations of motion, longitudinal static stability is concerned with only one, the pitch equation, describing the aircraft's motion about the y axis.

$$G_y = \dot{Q}I_y - PR(I_z - I_x) + (P^2 - R^2)I_{xz} \tag{5.1}$$

The fact that theory pertains to an aircraft in straight, steady, symmetrical flight with no unbalance of forces or moments permits longitudinal static stability motion to be independent of the lateral and directional equations of motion. This is not an oversimplification since most aircraft spend much of the flight under symmetric equilibrium conditions. Furthermore, the disturbance required for determination and the measure of the aircraft's response takes place about the axis or in the longitudinal plane. Under these conditions, Equation 5.1 reduces to:

$$G_y = 0$$

Since longitudinal static stability is concerned with resultant aircraft pitching moments caused by momentary changes in angle of attack and lift coefficients, the primary stability derivatives become C_{m_α} or $C_{m_{C_L}}$. The value of either derivative is a direct indication of the longitudinal static stability of the particular aircraft.

To determine an expression for the derivative $C_{m_{C_L}}$, an aircraft in stabilized equilibrium flight with horizontal stabilizer control surface fixed will be analyzed. A moment equation will be determined from the forces and moments acting on the aircraft. Once this equation is nondimensionalized, in moment coefficient form, the derivative with respect to C_L will be taken. This differential equation will be an expression for $C_{m_{C_L}}$ and will relate directly to the aircraft's stability. Individual term contributions to

stability will, in turn, be analyzed. A flight test relationship for determining the stability of an aircraft will be developed followed by a repeat of the entire analysis for an aircraft with a free control surface.

5.5 THE STICK FIXED STABILITY EQUATION

To derive the longitudinal pitching moment equation, refer to the aircraft in Figure 5.2. Writing the moment equation using the sign convention of pitch-up being a positive moment

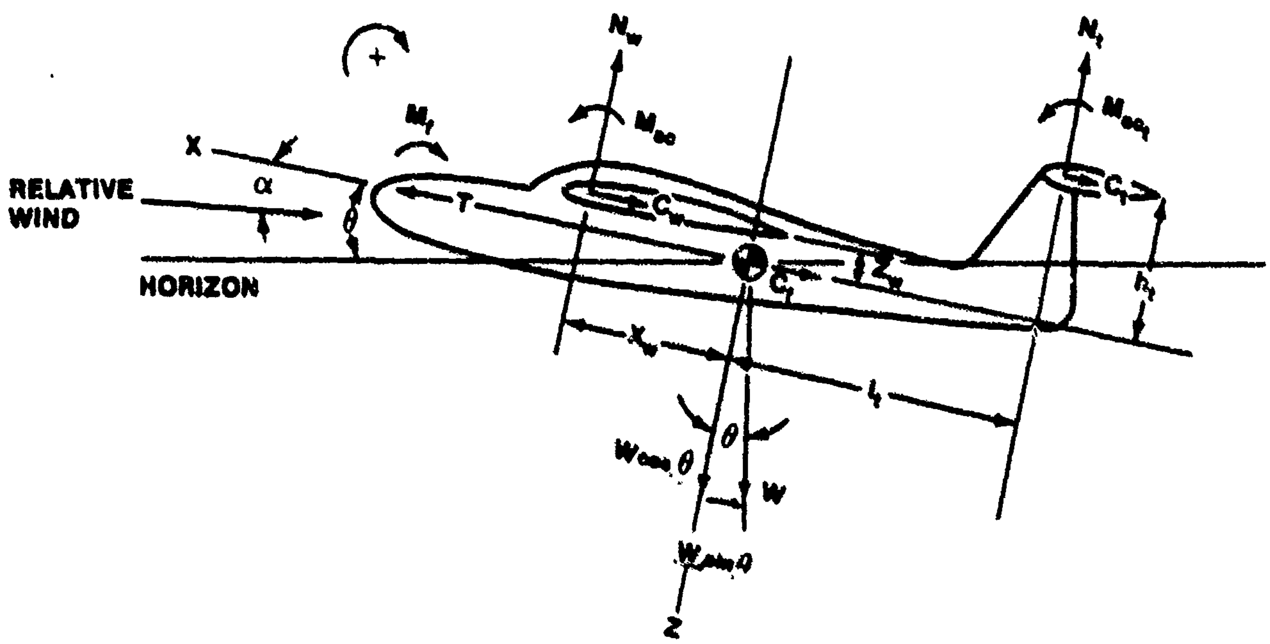


FIGURE 5.2. AIRCRAFT PITCHING MOMENTS

$$M_{cg} = N_w X_w + C_w Z_w - M_{ac} + M_f - N_t l_t + C_t h_t - M_{ac_t} \tag{5.2}$$

If an order of magnitude check is made, some of the terms can be logically eliminated because of their relative size. C_t can be omitted since

$$C_t \approx \frac{C_w}{10} \approx \frac{N_w}{100}$$

M_{ac_t} is zero for a symmetrical airfoil horizontal stabilizer section.

Rewriting the simplified equation

$$M_{cg} = N_w X_w + C_w Z_w - M_{ac} + M_f - N_t l_t \tag{5.3}$$

It is convenient to express Equation 5.3 in nondimensional coefficient form by dividing both sides of the equation by $q_w S_w c_w$

$$\frac{M_{cg}}{q_w S_w c_w} = \frac{N_w X_w}{q_w S_w c_w} + \frac{C_w Z_w}{q_w S_w c_w} - \frac{M_{ac}}{q_w S_w c_w} + \frac{M_f}{q_w S_w c_w} - \frac{N_t l_t}{q_w S_w c_w} \tag{5.4}$$

Substituting the following coefficients in Equation 5.4

$$C_{m_{cg}} = \frac{M_{cg}}{q_w S_w c_w}$$ total pitching moment coefficient about the cg

$$C_{m_{ac}} = \frac{M_{ac}}{q_w S_w c_w}$$ wing aerodynamic pitching moment coefficient

$$C_{m_f} = \frac{M_f}{q_w S_w c_w}$$ fuselage aerodynamic pitching moment coefficient

$$C_N = \frac{N_w}{q_w S_w}$$ wing aerodynamic normal force coefficient

$$C_{N_t} = \frac{N_t}{q_t S_t}$$ tail aerodynamic normal force coefficient

$$C_C = \frac{C_w}{q_w S_w}$$ wing aerodynamic chordwise force coefficient

Equation 5.4 may now be written

$$C_{m_{cg}} = C_N \frac{X_w}{c} + C_C \frac{Z_w}{c} - C_{m_{ac}} + C_{m_f} - \frac{N_t l_t}{qSc}$$

where the subscript w is dropped. (Further equations, unless subscripted, will be with reference to the wing.) To have the tail indicated in terms of a coefficient, multiply and divide by $q_t S_t$

$$\frac{N_t l_t}{q_w S_w c_w} \cdot \frac{q_t S_t}{q_t S_t}$$

Substituting tail efficiency factor $\eta_t = q_t/q_w$ and designating tail volume coefficient $V_H = l_t S_t / cS$ Equation 5.5 becomes

$$C_{m_{cg}} = C_N \frac{X_w}{c} + C_C \frac{Z_w}{c} - C_{m_{ac}} + C_{m_f} - C_{N_t} V_H \eta_t \quad \boxed{\text{BALANCE EQUATION}} \tag{5.6}$$

Equation 5.6 is referred to as the equilibrium equation in pitch. If the magnitudes of the individual terms in the above equation are adjusted to the proper value, the aircraft may be placed in equilibrium flight where $C_{m_{cg}} = 0$

Taking the derivative of Equation 5.6 with respect to C_L and assuming that X_w, Z_w, V_H and η_t do not vary with C_L,

$$\frac{dC_{m_{cg}}}{dC_L} = \underbrace{\frac{dC_N}{dC_L}\frac{X_w}{c} + \frac{dC_C}{dC_L}\frac{Z_w}{c} - \frac{dC_{m_{ac}}}{dC_L}}_{\text{WING}} + \underbrace{\frac{dC_{m_f}}{dC_L}}_{\text{FUSELAGE}} - \underbrace{\frac{dC_{N_t}}{dC_L} V_H \eta_t}_{\text{TAIL}} \quad \boxed{\text{STABILITY EQUATION}} \tag{5.7}$$

Equation 5.7 is the stability equation and is related to the stability derivative C_{m_α} by the slope of the lift curve, a. Theoretically,

$$C_{m_\alpha} = \frac{dC_m}{d\alpha} = \frac{dC_L}{d\alpha}\frac{dC_m}{dC_L} = a\frac{dC_m}{dC_L} = C_{L_\alpha}\frac{dC_m}{dC_L} \qquad (5.8)$$

Equation 5.8 is only true for a rigid aircraft at low Mach when thrust effects are small; however, this relationship does provide a useful index of stability.

Equation 5.6 and Equation 5.7 determine the two criteria necessary for longitudinal stability:

Criteria 1. The aircraft is balanced.

Criteria 2. The aircraft is stable.

The first condition is satisfied if the pitching moment equation can be forced to $C_{m_{cg}} = 0$ for useful positive values of C_L.

This condition is achieved by trimming the aircraft (adjusting elevator deflection) so that moments about the center of gravity are zero (i.e., $M_{cg} = 0$).

The second condition is satisfied if Equation 5.7 or $dC_{m_{cg}}/dC_L$ has a negative value. From Figure 5.3, a negative value for Equation 5.7 is necessary if the aircraft is to be stable. Should a gust cause an angle of attack increase (and a corresponding increase in C_L), a negative $C_{m_{cg}}$ should be produced to return the aircraft to equilibrium, or $C_{m_{cg}} = 0$. The greater the slope or the negative value, the more restoring moment is generated for an increase in C_L. The slope of dC_m/dC_L is a direct measure of the "gust stability" of the aircraft. (In further stability equations, the c.g. subscript will be dropped for ease of notation).

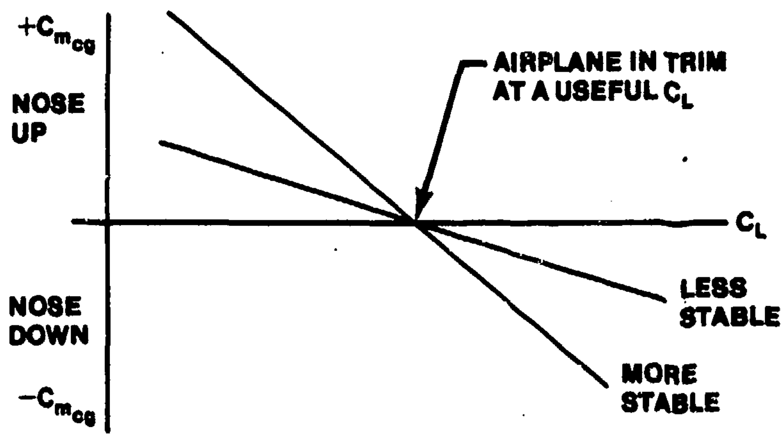


FIGURE 5.3. STATIC STABILITY

If the aircraft is retrimmed from one angle of attack to another, the basic stability of the aircraft or slope dC_m/dC_L does not change. Note Figure 5.4.

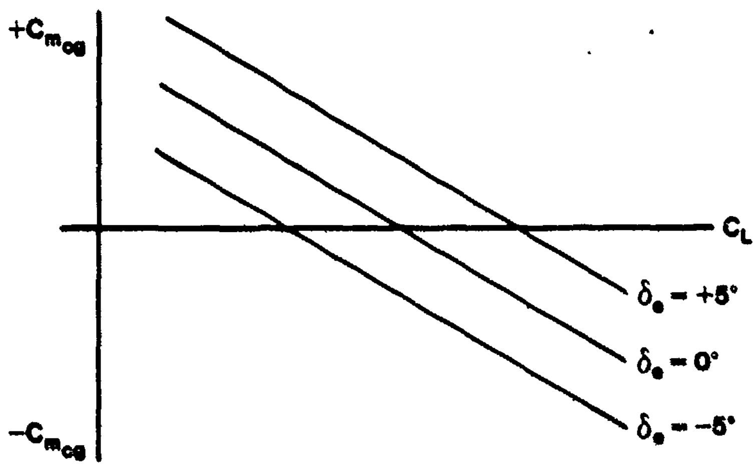


FIGURE 5.4. STATIC STABILITY WITH TRIM CHANGE

However, if moving the cg is changing the values of X_w or Z_w, or if V_H is changed, the slope or stability of the aircraft is changed. See Equation 5.7. For no change in trim setting, the stability curve may shift as in Figure 5.5.

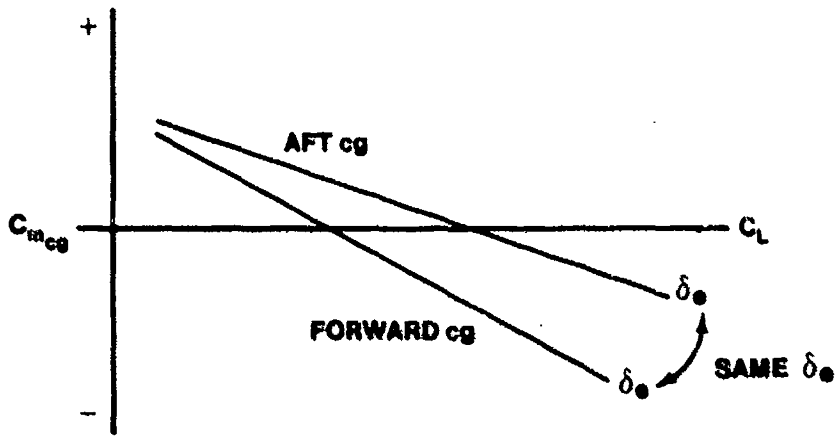


FIGURE 5.5. STATIC STABILITY CHANGE WITH CG CHANGE

5.6 AIRCRAFT COMPONENT CONTRIBUTIONS TO THE STABILITY EQUATION

5.6.1 The Wing Contribution to Stability

The lift and drag are by definition always perpendicular and parallel to the relative wind. It is therefore inconvenient to use these forces to obtain moments, for their arms to the center of gravity vary with angle of attack. For this reason, all forces are resolved into normal and chordwise forces whose axes remain fixed with the aircraft and whose arms are therefore constant.

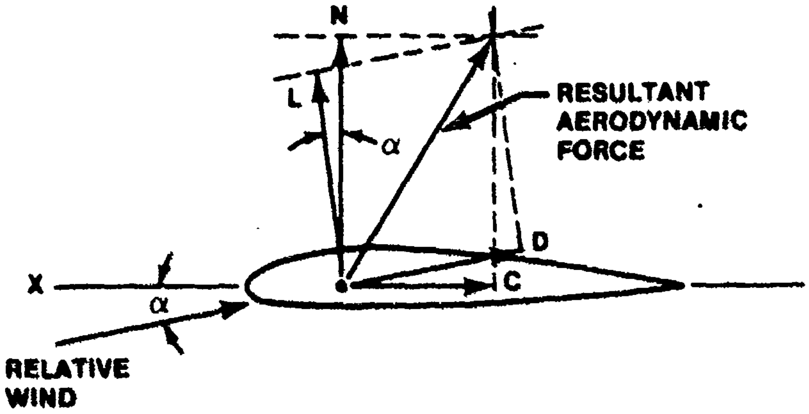


FIGURE 5.6. WING CONTRIBUTION TO STABILITY

Assuming the wing lift to be the airplane lift and the wing's angle of attack to be the airplane's angle of attack, the following relationship exists between the normal and lift forces (Figure 5.6)

$$N = L \cos \alpha + D \sin \alpha \qquad (5.9)$$

$$C = D \cos \alpha - L \sin \alpha \qquad (5.10)$$

Therefore, the coefficients are similarly related

$$C_N = C_L \cos \alpha + C_D \sin \alpha \qquad (5.11)$$

$$C_C = C_D \cos \alpha - C_L \sin \alpha \qquad (5.12)$$

The stability contributions, dC_N/dC_L and dC_C/dC_L, are obtained

$$\frac{dC_N}{dC_L} = \frac{dC_L}{dC_L}\cos\alpha - C_L\frac{d\alpha}{dC_L}\sin\alpha + \frac{dC_D}{dC_L}\sin\alpha - C_D\frac{d\alpha}{dC_L}\cos\alpha \tag{5.13}$$

$$\frac{dC_C}{dC_L} = \frac{dC_D}{dC_L}\cos\alpha - C_D\frac{d\alpha}{dC_L}\sin\alpha - \frac{dC_L}{dC_L}\sin\alpha - C_L\frac{d\alpha}{dC_L}\cos\alpha \tag{5.14}$$

Making an additional assumption that

$C_D = C_{D_p} + \dfrac{C_L^{\,2}}{\pi\, AR\, e}$ and that C_{D_p} is constant with changes in C_L

Then $\dfrac{dC_D}{dC_L} = \dfrac{2C_L}{\pi\, AR\, e}$

If the angles of attack are small such that $\cos\alpha \cong 1.0$ and $\sin\alpha \cong \alpha$, Equations 5.13 and 5.14 become

$$\frac{dC_N}{dC_L} = 1 + C_L\alpha\left(\frac{2}{\pi\, AR\, e} - \frac{d\alpha}{dC_L}\right) + C_D\frac{d\alpha}{dC_L} \tag{5.15}$$

$$\frac{dC_C}{dC_L} = \frac{2}{\pi\, AR\, e}C_L - C_D\frac{d\alpha}{dC_L}\alpha - \alpha - C_L\frac{d\alpha}{dC_L} \tag{5.16}$$

Examining the above equation for relative magnitude,

C_D is on the order of 0.02 to 0.30

C_L	usually ranges from 0.2 to 2.0
α	is small, $\leq$ 0.2 radians
$\frac{d\alpha}{dC_L}$	is nearly constant at 0.2 radians
$\frac{2}{\pi AR e}$	is on the order of 0.1

Making these substitutions, Equations 5.15 and 5.16 have magnitudes of

$$\frac{dC_N}{dC_L} = 1 - 0.04 + 0.06 = 1.02 \cong 1.0 \qquad (5.17)$$

$$\frac{dC_C}{dC_L} = 0.1\, C_L - 0.012 - 0.2 - 0.2C_L = -0.41 \qquad \left(\text{at } C_{L_{max}} = 2.0\right) \qquad (5.18)$$

The moment coefficient about the aerodynamic center is invariant with respect to angle of attack (see definition of aerodynamic center). Therefore

$$\frac{dC_{m_{ac}}}{dC_L} = 0$$

Rewriting the wing contribution of the Stability Equation, Equation 5.7,

$$\left.\frac{dC_m}{dC_L}\right)_{WING} = \frac{X_w}{c} - 0.41\,\frac{Z_w}{c} \qquad \left(\text{at } C_{L_{max}} = 2.0\right) \qquad (5.19)$$

From Figure 5.6 when α increases, the normal force increases and the chordwise force decreases. Equation 5.19 shows the relative magnitude of these changes. The position of the cg above or below the aerodynamic center (ac) has a much smaller effect on stability than does the position of the cg ahead of or behind the ac. With cg ahead of the ac, the normal force is stabilizing. From Equation 5.19, the more forward the cg location, the more stable the aircraft. With the cg below the ac, the chordwise force is stabilizing since this force decreases as the angle of attack increases. The further the cg is located below the ac, the more stable the aircraft or the more negative the value of dC_m/dC_L. The wing contribution to stability depends on the cg and the ac relationship shown in Figure 5.7.

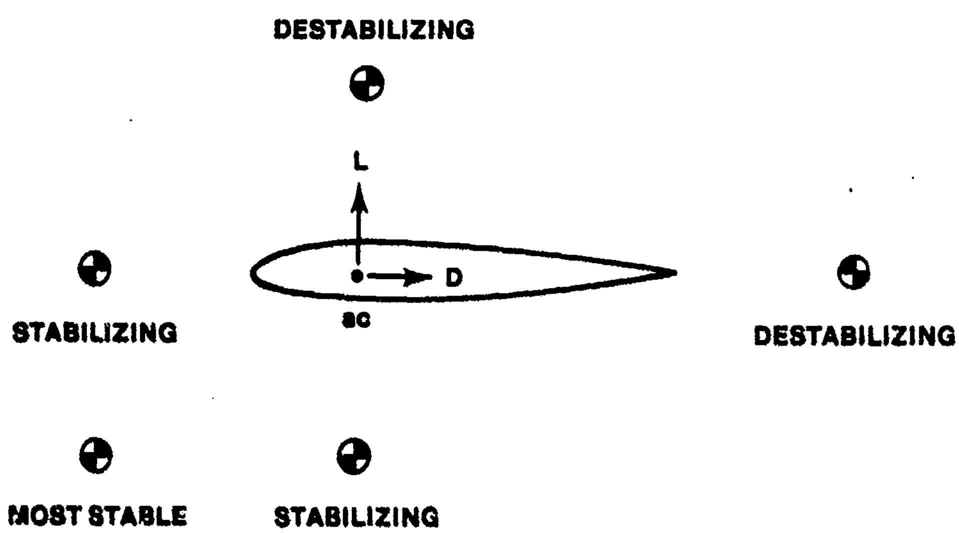


FIGURE 5.7. CG EFFECT ON WING CONTRIBUTION TO STABILITY

For a stable wing contribution to stability, the aircraft would be designed with a high wing aft of the center of gravity.

Fighter type aircraft and most low wing, large aircraft have cg's very close to the top of the mean aerodynamic chord. Z_w/c is on the order of 0.03. For these aircraft the chordwise force contribution to stability can be neglected. The wing contribution then becomes

$$\left.\frac{dC_m}{dC_L}\right)_{\text{WING}} = \frac{X_w}{c} \tag{5.20}$$

5.6.2 The Fuselage Contribution to Stability

The fuselage contribution is difficult to separate from the wing terms because it is strongly influenced by interference from the wing flow field. Wind tunnel tests of the wing-body combination are used by airplane designers to obtain information about the fuselage influence on stability.

A fuselage by itself is almost always destabilizing because the center of pressure is usually ahead of the center of gravity. The magnitude of the destabilization effects of the fuselage requires their consideration in the equilibrium and stability equations. In general, the effect of combining the wing and fuselage results in the combination aerodynamic center being forward of quarter-chord and the $C_{m_{ac}}$ of the combination being more negative than the wing value alone.

$$\left.\frac{dC_m}{dC_L}\right)_{\text{FUSELAGE}} = \text{Positive quantity}$$

5.6.3 The Tail Contribution to Stability

From equation 5.7, the tail contribution to stability was found to be

$$\left.\frac{dC_m}{dC_L}\right)_{TAIL} = -\frac{dC_{N_t}}{dC_L} V_H \eta_t$$

For small angles of attack, the lift curve slope of the tail is very nearly the same as the slope of the normal force curve.

$$a_t = \frac{dC_L}{d\alpha_L} \approx \frac{dC_N}{d\alpha_t} \qquad (5.22)$$

Therefore

$$C_N = a_t \alpha_t \qquad (5.23)$$

An expression for α_t in terms of C_L is required before solving for dC_{N_t}/dC_L

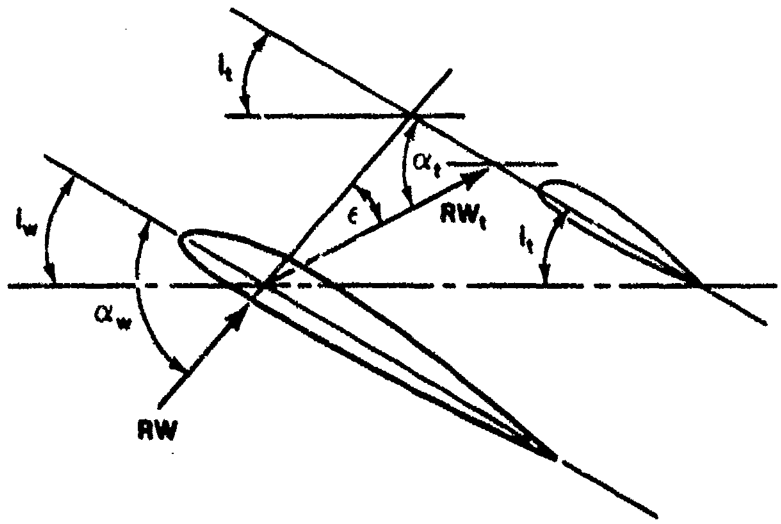


FIGURE 5.8. TAIL ANGLE OF ATTACK

From Figure 5.8

$$\alpha_t = \alpha_w - i_w + i_t - \varepsilon \tag{5.24}$$

Substituting Equation 5.24 into 5.23 and taking the derivative with respect to C_L, where $a_w = dC_L/d\alpha$

$$\frac{dC_{N_t}}{dC_L} = a_t \left(\frac{d\alpha_w}{dC_L} - \frac{d\varepsilon}{dC_L} \right) = a_t \left(\frac{1}{a_w} - \frac{d\varepsilon}{d\alpha} \frac{1}{a_w} \right) \tag{5.25}$$

upon factoring out $1/a_w$

$$\frac{dC_{N_t}}{dC_L} = \frac{a_t}{a_w} \left(1 - \frac{d\varepsilon}{d\alpha} \right) \tag{5.26}$$

Substituting Equation 5.26 into 5.21, the expression for the tail contribution becomes

$$\left.\frac{dC_m}{dC_L}\right)_{TAIL} = -\frac{a_t}{a_w} \left(1 - \frac{d\varepsilon}{d\alpha} \right) V_H \; \eta_t \tag{5.27}$$

The value of a_t/a_w is very nearly constant. These values are usually obtained from experimental data.

The tail volume coefficient, V_H, is a term determined by the geometry of the aircraft. To vary this term is to redesign the aircraft.

$$V_H = \frac{l_t S_t}{cS} \tag{5.28}$$

The further the tail is located aft of the cg (increase l_t) or the greater the tail surface area (S_t), the greater the tail volume coefficient (V_H), which increases the tail contribution to stability.

The expression, η_t, is the ratio of the tail dynamic pressure to the wing dynamic pressure and η_t varies with the location of the tail with respect to wing wake, prop slipstream, etc. For power-off considerations, η_t varies from 0.65 to 0.95 due to boundary layer losses.

The term $(1 - d\varepsilon/d\alpha)$ is an important factor in the stability contribution of the tail. Large positive values of $d\varepsilon/d\alpha$ produce destabilizing effects by reversing the sign of the term $(1 - d\varepsilon/d\alpha)$ and consequently, the sign of $dC_m/dC_{L_{Tail}}$.

For example, at high angles of attack the F-104 experiences a sudden increase in $d\varepsilon/d\alpha$. The term $(1 - d\varepsilon/d\alpha)$ goes negative causing the entire tail contribution to be positive or destabilizing, resulting in aircraft pitchup. The stability of an aircraft is definitely influenced by the wing vortex system. For this reason, the downwash variation with angle of attack should be evaluated in the wind tunnel.

The horizontal stabilizer provides the necessary positive stability contribution (negative dC_m/dC_L) to offset the negative stability of the wing-fuselage combination and to make the entire aircraft stable and balanced (Figure 5.9).

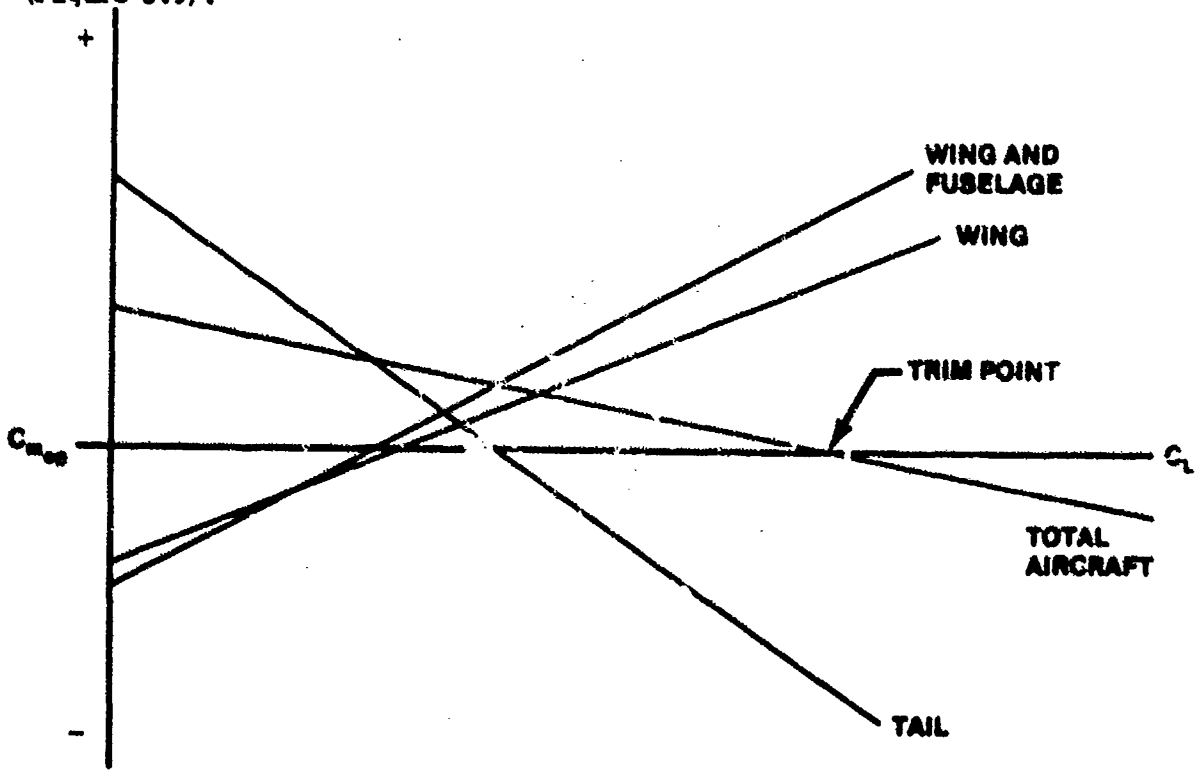


FIGURE 5.9. AIRCRAFT COMPONENT CONTRIBUTIONS TO STABILITY

The stability may be written as,

$$\frac{dC_m}{dC_L} = \frac{X_w}{c} + \frac{dC_m}{dC_{L_{Fus}}} - \frac{a_t}{a_w} V_H \eta_t \left(1 - \frac{d\varepsilon}{d\alpha}\right) \tag{5.29}$$

5.6.4 The Power Contribution to Stability

The addition of a power plant to the aircraft may have a decided effect on the equilibrium as well as the stability equations. The overall effect may be quite complicated. This section will be a qualitative discussion of power effects. The actual end result of the power effects on trim and stability should come from large scale wind tunnel models or actual flight tests.

5.6.4.1 Power Effects of Propeller Driven Aircraft - The power effects of a propeller driven aircraft which influence the static longitudinal stability of the aircraft are:

1. Thrust effect - effect on stability from the thrust acting along the propeller axis.
2. Normal force effect - effect on stability from a force normal to the thrust line and in the plane of the propeller.
3. Indirect effects - power plant effects on the stability contribution of other parts of the aircraft.

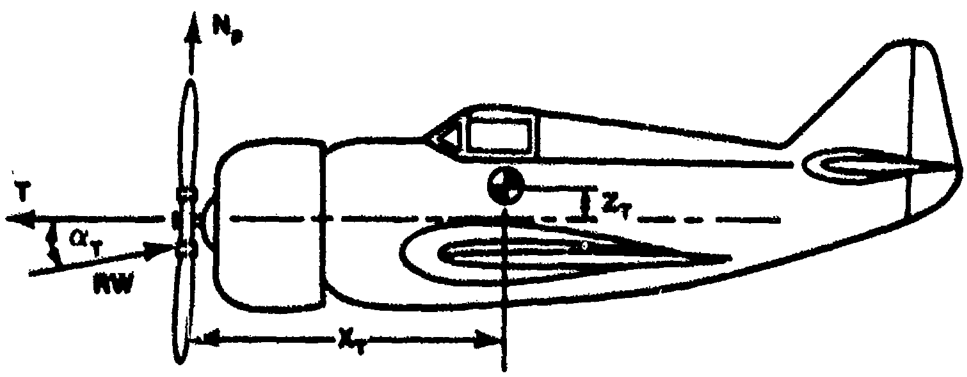


FIGURE 5.10. PROPELLER THRUST AND NORMAL FORCE

Writing the moment equation for the power terms as

$$M_{cg} = T Z_T + N_p X_T \tag{5.30}$$

In coefficient form

$$C_{m_{cg}} = C_T \frac{Z_T}{c} + C_{N_p} \frac{X_T}{c} \tag{5.31}$$

The direct power effect on the aircraft's stability equation is then

$$\left.\frac{dC_m}{dC_L}\right)_{POWER} = \frac{dC_T}{dC_L} \frac{Z_T}{c} + \frac{dC_{N_p}}{dC_L} \frac{X_T}{c} \tag{5.32}$$

The sign of $dC_m/dC_{L_{POWER}}$ then depends on the sign of the derivatives dC_{N_p}/dC_L and dC_T/dC_L.

First consider the dC_T/dC_L derivative. If the speed varies at different flight conditions with throttle position held constant, then C_T varies in a manner that can be represented by dC_T/dC_L. The coefficient of thrust for a reciprocating power plant varies with C_L and propeller efficiency. Propeller efficiency, which is available from propeller performance estimates in the manufacturer's data, decreases rapidly at high C_L. Coefficient of thrust variation with C_L is nonlinear $\left(\text{it varies with } C_L^{3/2}\right)$ with the derivative large at low speeds. The combination of these two variations approximately linearize C_T versus C_L (Figure 5.11). The sign of dC_T/dC_L is positive.

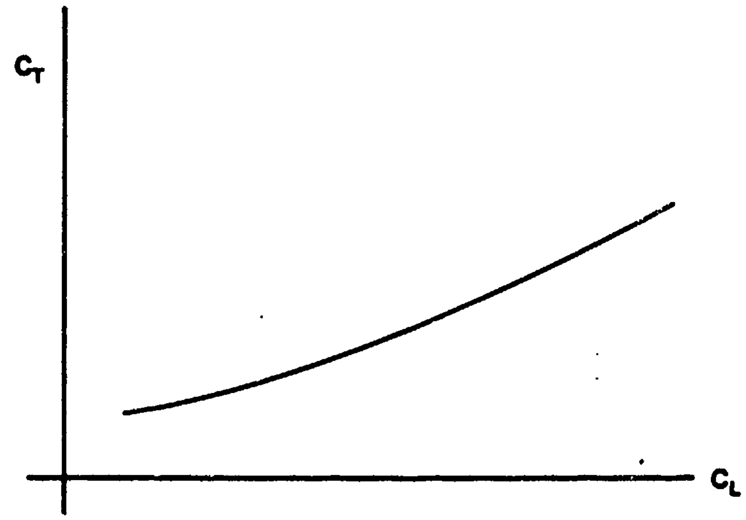


FIGURE 5.11. COEFFICIENT OF THRUST CURVE FOR A RECIPROCATING POWER PLANT WITH PROPELLER

The derivative dC_{N_p}/dC_L is positive since the normal propeller force increases linearly with the local angle of attack of the propeller axis, α_T. The direct power effects are then destabilizing if the cg is as shown in Figure 5.10 where the power plant is ahead and below the cg. The indirect power effects must also be considered in evaluating the overall stability contribution of the propeller power plant. No attempt will be made to determine their quantitative magnitudes. However, their general influence on the aircraft's stability and trim condition can be great.

5.6.4.1.1 Increase in angle of downwash, ε. Since the normal force on the propeller increases with angle of attack under powered flight, the slipstream is deflected downward, netting an increase in downwash on the tail. The downwash in the slipstream will increase more rapidly with the angle of attack than the downwash outside the slipstream. The derivative $d\varepsilon/d\alpha$ has a positive increase with power. The term $(1 - d\varepsilon/d\alpha)$ in Equation 5.27 is reduced causing the tail trim contribution to be less negative or less stable than the power-off situation.

5.6.4.1.2 Increase of $\eta_t = (q_t/q_w)$. The dynamic pressure, q, of the tail is increased by the slipstream and η_t is greater than unity. From equation 5.27, the increase of η_t with an application of power increases the tail contribution to stability.

Both slipstream effects mentioned above may be reduced by locating the

horizontal stabilizer high on the tail and out of the slipstream at operating angles of attack.

5.6.4.2 Power effects of the turbojet/turbofan/ramjet. The magnitude of the power effects on jet powered aircraft are generally smaller than on propeller driven aircraft. By assuming that jet engine thrust does not change with velocity or angle of attack, and by assuming constant power settings, smaller power effects would be expected than with a similar reciprocating engine aircraft.

There are three major contributions of a jet engine to the equilibrium static longitudinal stability of the aircraft. These are:

1. Direct thrust effects.
2. Normal force effects at the air duct inlet and at angular changes in the duct.
3. Indirect effects of induced flow at the tail.

The thrust and normal force contribution may be determined from Figure 5.12.

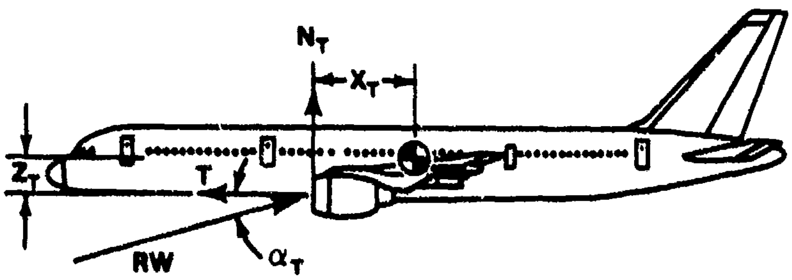


FIGURE 5.12. JET THRUST AND NORMAL FORCE

Writing the equation

$$M_{cg} = TZ_T + N_T X_T \tag{5.33}$$

or

$$C_{m_{cg}} = \frac{T}{qSc} Z_T + C_{N_T} \frac{X_T}{c} \tag{5.34}$$

With the aircraft in unaccelerated flight, the dynamic pressure is a function of lift coefficient.

$$q = \frac{W}{C_L S} \tag{5.35}$$

Therefore,

$$C_{m_{cg}} = \frac{T}{W} \frac{Z_T}{c} C_L + C_{N_T} \frac{X_T}{c} \tag{5.36}$$

If thrust is considered independent of speed, then

$$\left.\frac{dC_m}{dC_L}\right|_{POWER} = \frac{T}{W} \frac{Z_T}{c} + \frac{dC_{N_T}}{dC_L} \frac{X_T}{c} \tag{5.37}$$

The thrust contribution to stability then depends on whether the thrust line is above or below the cg. Locating the engine below the cg causes a destabilizing influence.

The normal force contribution depends on the sign of the derivative dC_{N_T}/dC_L. The normal force N_T is created at the air duct inlet to the turbojet engine. This force is created as a result of the momentum change of the free stream which bends to flow along the duct axis. The magnitude of the

force is a function of the engine's airflow rate, W_a, and the angle α_T between the local flow at the duct entrance and the duct axis.

$$N_T = \frac{W_a}{g} U_0 \alpha_T \tag{5.38}$$

With an increase in α_T, N_T will increase, causing dC_N/dC_L to be positive. The normal force contribution will be destabilizing if the inlet duct is ahead of the center of gravity. The magnitude of the destabilizing moment will depend on the distance the inlet duct is ahead of the center of gravity.

For a jet engine to definitely contribute to positive longitudinal stability (dC_m/dC_L negative), the jet engine would be located above and behind the center of gravity.

The indirect contribution of the jet unit to longitudinal stability is the effect of the jet induced downwash at the horizontal tail. This applies to the situation where the jet exhaust passes under or over the horizontal tail surface. The jet exhaust as it discharges from the tailpipe spreads outward. Turbulent mixing causes outer air to be drawn in towards the exhaust area. Downwash at the tail may be affected. The F-4 is a good example where entrained air from the jet exhaust causes downwash angle at the horizontal tail.

5.6.4.3 Power Effects of Rocket Aircraft. Rocket powered aircraft such as the Space Shuttle, and rocket augmented aircraft such as the C-130 with JATO installed, can be significantly affected longitudinally depending on the magnitude of the rocket thrust involved. Since the rocket system carries its oxidizer internally, there is no mass flow and no normal force contribution. The thrust contribution may be determined from Figure 5.13.

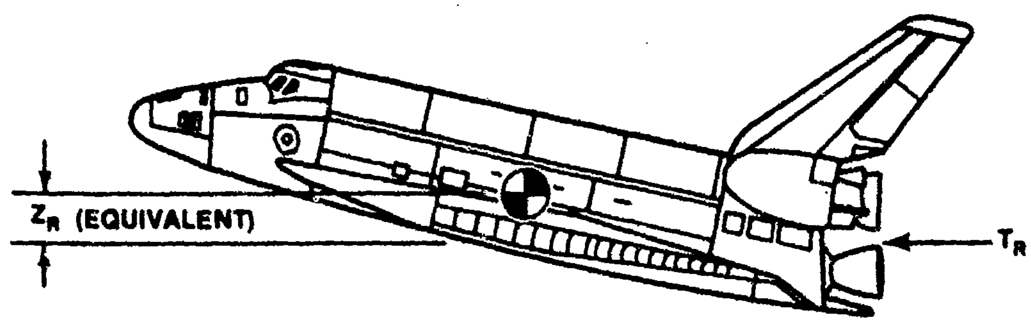


FIGURE 5.13. ROCKET THRUST EFFECTS

Writing the equation

$$M_{cg} = T_R Z_R \quad \text{or} \quad C_{m_{cg}} = \frac{T_R Z_R}{qSc} \tag{5.39}$$

Assuming that the rocket thrust is constant with changes in airspeed, the dynamic pressure is a function of lift coefficient.

$$q = \frac{W}{C_L S} \quad \text{therefore} \quad C_{m_{cg}} = \frac{T_R Z_R C_L}{W c}$$

and

$$\left.\frac{dC_m}{dC_L}\right|_{POWER} = \frac{T_R Z_R}{W c} \tag{5.40}$$

From the above discussion, it can be seen that several factors are important in deciding the power effect on stability. Each aircraft must be examined individually. This is the reason that aircraft are tested for stability in several configurations and at different power settings.

5.7 THE NEUTRAL POINT

The stick fixed neutral point is defined as the center of gravity position at which the aircraft displays neutral stability or where $dC_m/dC_L = 0$

The symbol h is used for the center of gravity position where

$$h = \frac{X_{cg}}{c} \tag{5.41}$$

The stability equation for the powerless aircraft is

$$\frac{dC_m}{dC_L} = \frac{X_w}{c} + \frac{dC_m}{dC_{L_{Fus}}} - \frac{a_t}{a_w} V_H \eta_t \left(1 - \frac{d\varepsilon}{d\alpha}\right) \tag{5.29}$$

Looking at the relationship between cg and ac in Figure 5.14

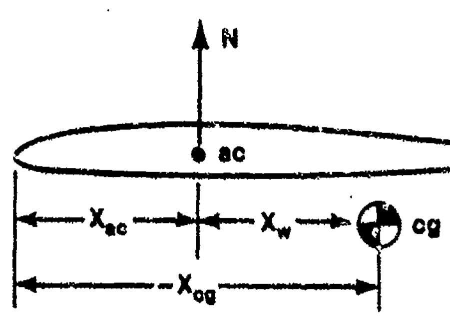


FIGURE 5.14. CG AND AC RELATIONSHIP

$$\frac{X_w}{c} = h - \frac{X_{ac}}{c} \tag{5.42}$$

Substituting Equation 5.42 into Equation 5.29,

$$\frac{dC_m}{dC_L} = h - \frac{X_{ac}}{c} + \frac{dC_m}{dC_{L_{Fus}}} - \frac{a_t}{a_w} V_H \eta_t \left(1 - \frac{d\varepsilon}{d\alpha}\right) \tag{5.43}$$

If we set $dC_m/dC_L = 0$, then $h \approx h_n$ and Equation 5.43 gives

$$h_n = \frac{X_{ac}}{c} - \frac{dC_m}{dC_{L_{Fus}}} + \frac{a_t}{a_w} V_H \eta_t \left(1 - \frac{d\varepsilon}{d\alpha}\right) \tag{5.44}$$

This is the cg location where the aircraft exhibits neutral static stability: the neutral point.

Substituting Equation 5.44 back into Equation 5.43, the stick-fixed stability derivative in terms of cg position becomes

$$\frac{dC_m}{dC_L} = h - h_n \tag{5.45}$$

The stick-fixed static stability is equal to the distance between the cg position and the neutral point in percent of the mean aerodynamic chord. "Static Margin" refers to the same distance, but is positive in sign for a stable aircraft.

$$\text{Static Margin} = h_n - h \tag{5.46}$$

It is the test pilot's responsibility to evaluate the aircraft's handling qualities and to determine the acceptable static margin for the aircraft.

5.8 ELEVATOR POWER

For an aircraft to be a usable flying machine, it must be stable and

balanced thorughout the useful C_L range. For trimmed, or equilibrium flight, $C_{m_{cg}}$ must be zero. Some means must be available for balancing the various terms in Equation 5.47

$$C_{m_{cg}} = C_N \frac{X_w}{c} + C_C \frac{Z_w}{c} - C_{m_{ac}} + C_{m_f} - (a_t \alpha_t \ V_H \eta_t) \qquad (5.47)$$

(Equation 5.47 is obtained by substituting Equation 5.23 into Equation 5.6.)

Several possibilities are available. The center of gravity could be moved fore and aft, or up and down, thus changing X_w/c or Z_w/c. However, this would not only affect the equilibrium lift coefficient, but would also change dC_m/dC_L in Equation 5.48. This is undesirable.

$$\frac{dC_m}{dC_L} = \frac{dC_N}{dC_L} \frac{X_w}{c} + \frac{dC_C}{dC_L} \frac{Z_w}{c} + \frac{dC_m}{dC_{L_{Fus}}} - \frac{a_t}{a_w} V_H \eta_t \left(1 - \frac{d\varepsilon}{d\alpha}\right) \qquad (5.48)$$

Equation 5.48 is obtained by substituting Equation 5.26 into Equation 5.7. The pitching moment coefficient about the aerodynamic center could be changed by effectively changing the camber of the wing by using trailing edge flaps as is done in flying wing vehicles. On the conventional tail-to-the rear aircraft, trailing edge wing flaps are ineffective in trimming the pitching moment coefficient to zero. The combined use of trailing edge flaps and trim from the tail may serve to reduce drag, as used on some sailplanes and the F-5E.

The remaining solution is to change the angle of attack of the horizontal tail to achieve a $\left(C_{m_{cg}} = 0\right)$ without a change to the basic aircraft stability.

The control means is either an elevator on the stabilizer or an all moving stabilizer (slab or stabilator). The slab is used on most high speed aircraft and is the most powerful means of longitudinal control.

Movement of the slab or elevator changes the effective angle of attack of the horizontal stabilizer and, consequently, the lift on the horizontal tail. This in turn changes the moment about the center of gravity due to the horizontal tail. It is of interest to know the amount of pitching moment change associated with an increment of elevator deflection. This may be determined by differentiating Equation 5.47 with respect to δ_e.

$$\frac{dC_m}{d\delta_e} = -a_t V_H \eta_t \frac{d\alpha_t}{d\delta_e} \qquad (5.49)$$

$$C_{m_{\delta_e}} = -a_t V_H \eta_t \tau \qquad (5.50)$$

This change in pitching moment coefficient with respect to elevator deflection $C_{m_{\delta_e}}$ is referred to as "elevator power". It indicates the capability of the elevator to produce moments about the center of gravity. The term $d\alpha_t/d\delta_e$ in Equation 5.49 is termed "elevator effectiveness" and is given the shorthand notation τ. The elevator effectiveness may be considered as the equivalent change in effective tail plane angle of attack per unit change in elevator deflection. The relationship between elevator effectiveness τ and the effective angle of attack of the stabilizer is seen in Figure 5.15.

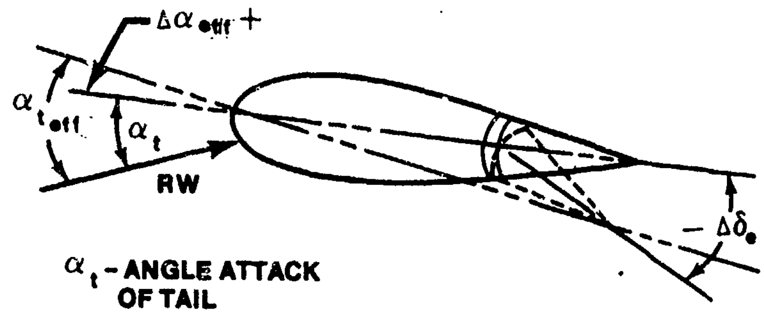


FIGURE 5.15. CHANGE WITH EFFECTIVE ANGLE OF ATTACK WITH ELEVATOR DEFLECTION

Elevator effectiveness is a design parameter and is determined from wind tunnel tests. Elevator effectiveness is a negative number for all tail-to-the-rear aircraft. The values range from zero to the limiting case of the all moving stabilizer (slab) where τ equals -1. The tail angle of attack would change plus one degree for every minus degree the slab moves. For the elevator-stabilizer combination, the elevator effectiveness is a function of the ratio of overall elevator area to the entire horizontal tail area.

5.9 ALTERNATE CONFIGURATIONS

Although tail-to-the-rear is the configuration normally perceived as standard, two other configurations merit some discussion. The tailless aircraft, or flying wing, has been used in the past, and some modern designs contemplate the use of this concept. The canard configuration has become increasingly more popular in modern designs over the past several years as evidenced by aircraft like the B-1B and the X-29.

5.9.1 Flying Wing Theory

In order for a flying wing to be a usable aircraft, it must be balanced (fly in equilibrium at a useful positive C_L) and be stable. The problem may be analyzed as follows:

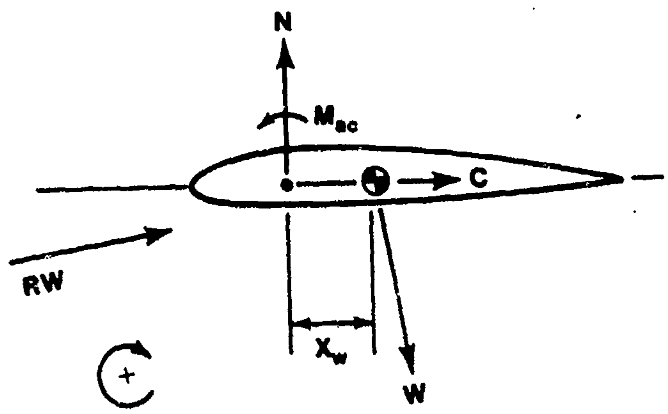


FIGURE 5.16. AFT CG FLYING WING

For the wing in Figure 5.16, assuming that the chordwise force acts through the cg, the equilibrium in pitch may be written

$$M_{cg} = NX_w - M_{ac} \tag{5.51}$$

or in coefficient form

$$C_{m_{cg}} = C_N \frac{X_w}{c} - C_{m_{ac}} \tag{5.52}$$

For controls fixed, the stability equation becomes

$$\frac{dC_{m_{cg}}}{dC_L} = \frac{dC_N}{dC_L} \frac{X_w}{c} \tag{5.53}$$

Equations 5.52 and 5.53 show that the wing in Figure 5.16 is balanced and unstable. To make the wing stable, or dC_m/dC_L negative, the center of gravity must be ahead of the wing aerodynamic center. Making this cg change, however, now changes the signs in Equation 5.51 . The equilibrium and stability equations become

$$C_{m_{cg}} = - C_N \frac{X_w}{c} - C_{m_{ac}} \qquad (5.54)$$

$$\frac{dC_{m_{cg}}}{dC_L} = - \frac{dC_N}{dC_L} \frac{X_w}{c} \qquad (5.55)$$

The wing is now stable but unbalanced. The balanced condition is possible with a positive $C_{m_{ac}}$

Three methods of obtaining a positive $C_{m_{ac}}$ are:

1. Use a negative camber airfoil section. The positive $C_{m_{ac}}$ will give a flying wing that is stable and balanced (Figure 5.17).

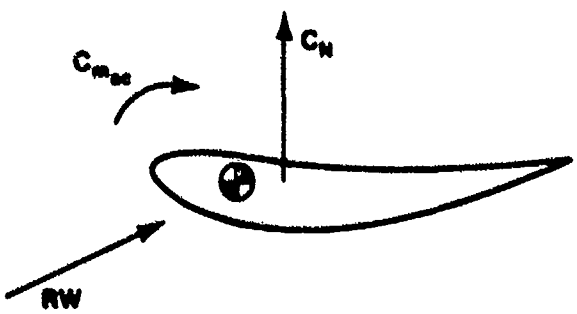


FIGURE 5.17. NEGATIVE CAMBERED FLYING WING

This type of wing is not realistic because of unsatisfactory dynamic characteristics, small cg range, and extremely low C_L capability.

2. A reflexed airfoil section reduces the effect of camber by creating a download near the trailing edge. Similar results are possible with an upward deflected flap on a symmetrical airfoil.

3. A symmetrical airfoil section in combination with sweep and wingtip washout (reduction in angle of incidence at the tip) will produce a positive $C_{m_{ac}}$ by virtue of the aerodynamic couple produced between the downloaded tips and the normal lifting force. This is shown in Figure 5.18.

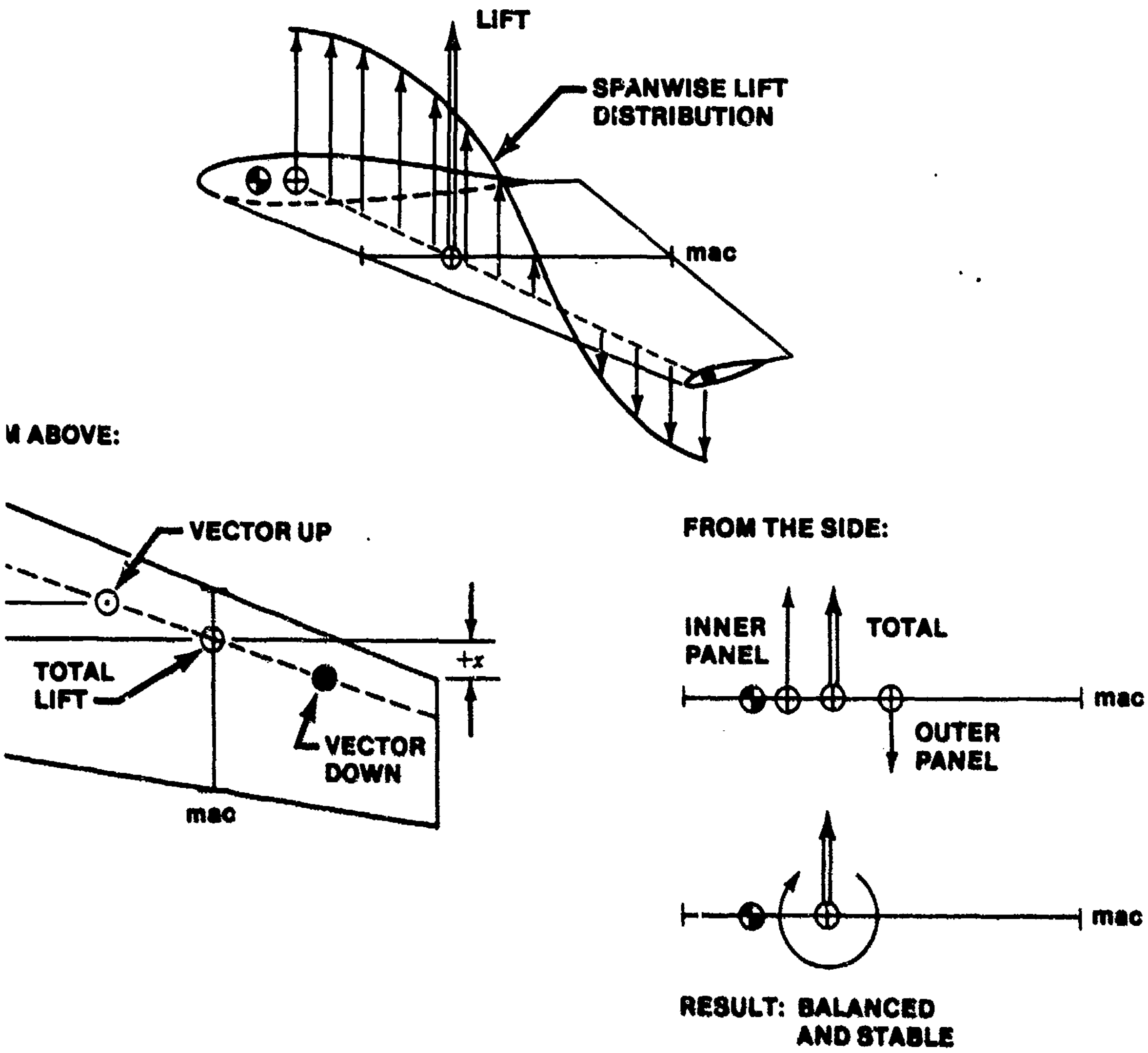


FIGURE 5.18. THE SWEPT AND TWISTED FLYING WING

Figure 5.19 shows idealized $C_{m_{ac}}$ versus C_L for various wings in a control fixed position. Only two of the wings are capable of sustained flight.

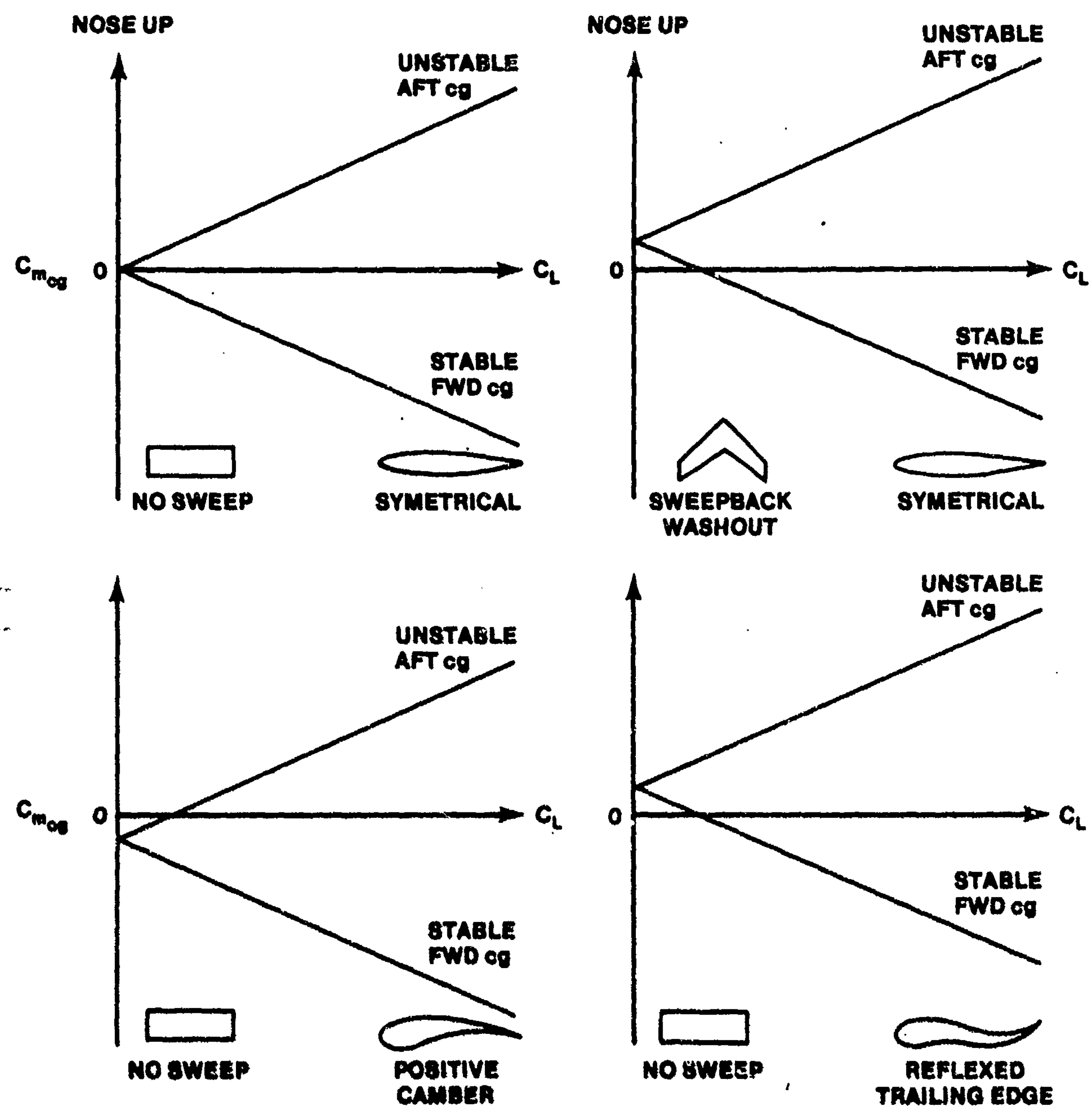


FIGURE 5.19. VARIOUS FLYING WINGS

5.9.2 The Canard Configuration

Serious work on aircraft with the canard configuration has been sporadic from the time the Wright brothers' design evolved into the tail to the rear airplanes of World War I, until the early 1970's. One of the first successful canard airplanes in quantity production was the Swedish JA-37 "Viggen" fighter. Other projects of significance were the XB-70, the Mirage Milan, the TU-144, and the prolific designs of Burt Rutan (Vari-Viggen, Vari-Eze and Long-Eze, Defiant, Grizzly, and Solitaire). The future seems to indicate that we will see more of the canard configuration, as evidenced by the X-29 Forward Swept Wing project and the OMAC airplane. A test pilot, knowing what the future may hold, should have more than a passing interest in how a canard affects longitudinal stability.

There are many reasons for a designer to select the canard configuration. A few of them are listed below:

a. Both the wing and canard surface contribute to the production of lift.

b. Since the cg is between the centers of pressure of the wing and canard, a larger cg range is possible. (The canard and conventional aircraft are shown balanced in Figure 5.20.)

c. The aircraft structure may be built more efficiently and simpler control arrangements are possible.

d. Better pitch control is available at high angle of attack because the canard is not in the wake of the wing, and the stall characteristics may be made benign by having the canard lose lift before the wing stalls.

e. The stick-free canard (reversible control system) provides load alleviation in gusts.

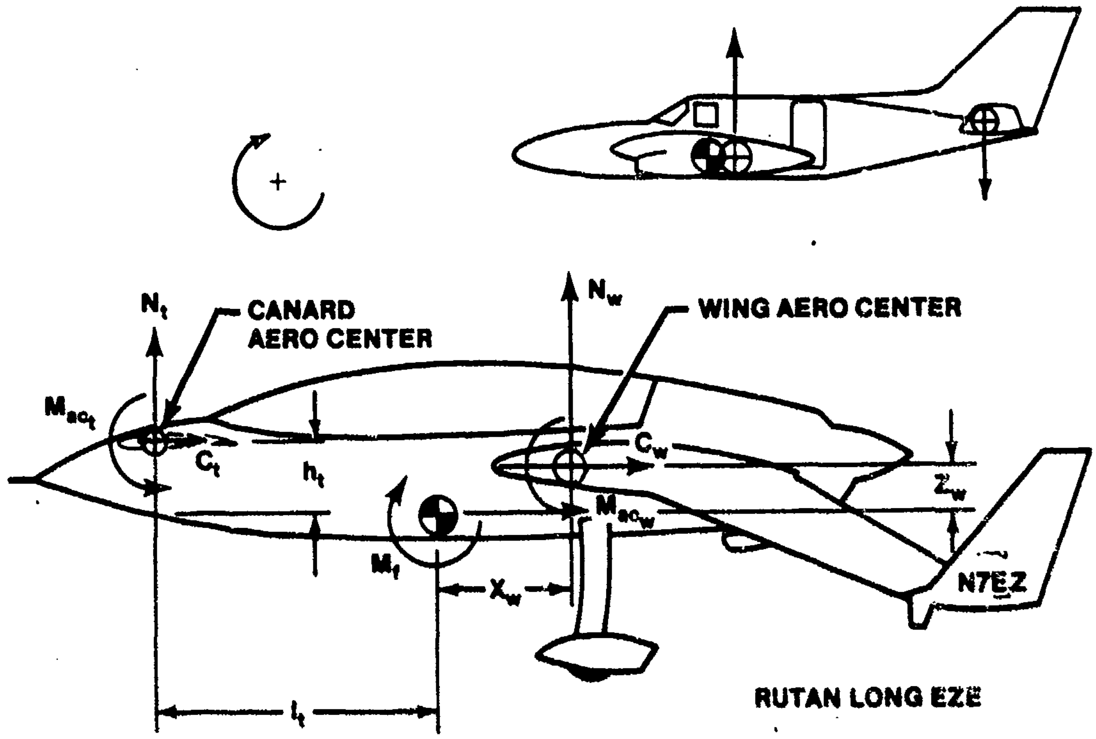


FIGURE 5.20. BALANCE COMPARISON

5.9.2.1 The Balance Equation. From Figure 5.20 the balance equation can be written as follows:

$$M_{cg} = -N_w X_w + C_w Z_w - M_{ac_w} + M_f + N_t l_t + C_t h_t - M_{ac_t} \quad (5.56)$$

Simplifying assumptions are

$C_t h_t$ and M_{ac_t} are small and may be neglected

Then

$$C_{m_{cg}} = \frac{M_{cg}}{q_w S_w c_w} = -C_N \frac{X_w}{c} + C_c \frac{Z_w}{c} - C_{m_{ac_w}} + C_{m_f} + \frac{N_t l_t}{q_w S_w c_w} \frac{q_t S_t}{q_t S_t} \quad (5.57)$$

Combining Terms:

$$C_{m_{cg}} = -C_N \frac{X_w}{c} + C_c \frac{Z_w}{c} - C_{m_{ac_w}} + C_{m_f} + C_{N_t} V_H \eta_t \qquad \boxed{\text{BALANCE EQUATION}} \qquad (5.58)$$

5.9.2.2 <u>The Stability Equation</u>. Although the canard can be a balanced configuration, it remains to be seen if it demonstrates static stability or "gust stability". By taking the derivative with respect to C_L, Equation 5.58 becomes

$$\frac{dC_{m_{cg}}}{dC_L} = -\frac{dC_N}{dC_L}\frac{X_w}{c} + \frac{dC_c}{dC_L}\frac{Z_w}{c} + \frac{dC_{m_f}}{dC_L} + \frac{dC_{N_t}}{dC_L} V_H \eta_t \qquad \boxed{\text{STABILITY EQUATION}} \qquad (5.59)$$

The small angle assumption allows us to say that C_N equals C_L, and by saying that the cg is close to the wing chord vertically. Equations 5.58 and 5.59 reduce to

$$C_{m_{cg}} = -C_L \frac{X_w}{c} - C_{m_{ac}} + C_{m_f} + C_{L_t} V_H \eta_t \qquad (5.60)$$

$$\frac{dC_m}{dC_L} = -\frac{X_w}{c} + \frac{dC_{m_f}}{dC_L} + \frac{dC_{L_t}}{dC_L} V_H \eta_t \qquad (5.61)$$

Equation 5.59 indicates that the normal (or lift) force of the wing now has a stabilizing influence (negative in sign), and the canard term is destabilizing due to its positive sign. It is obviously a misnomer to call the canard a horizontal stabilizer, because in reality it is a "destabilizer"! The degree of instability must be overcome by the wing-fuselage combination in order for the airplane to exhibit positive static stability dC_m/dC_L (negative in sign). This is shown graphically in Figure 5.21.

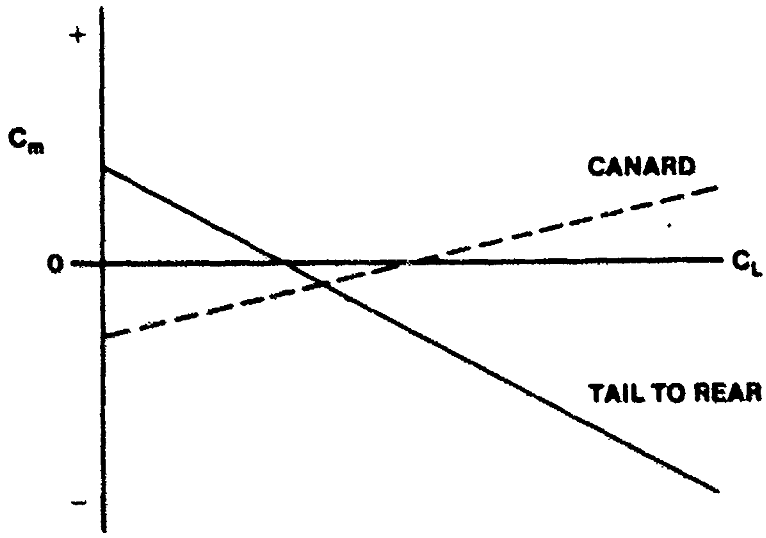


FIGURE 5.21. CANARD EFFECTS ON dC_m/dC_L

5.9.2.3 Upwash Contribution to Stability. In a manner similar to the way a rear mounted horizontal tail experiences a downwash field from the wake of the wing, the canard will see upwash ahead of the wing. This upwash field has a destabilizing effect on longitudinal stability because it makes the tail term in the stability equation more positive.

The tail contribution from Equation 5.61 can be examined for the effects of upwash, ϵ'

$$\frac{dC_{L_t}}{dC_L} V_H \eta_t = \frac{d\,(a_t \alpha_T)}{dC_L} V_H \eta_t = a_t V_H \eta_t \frac{d\alpha_t}{dC_L}$$

The tail angle of attack, α_t, can be expressed in terms of incidence and upwash, as described in Figure 5.22.

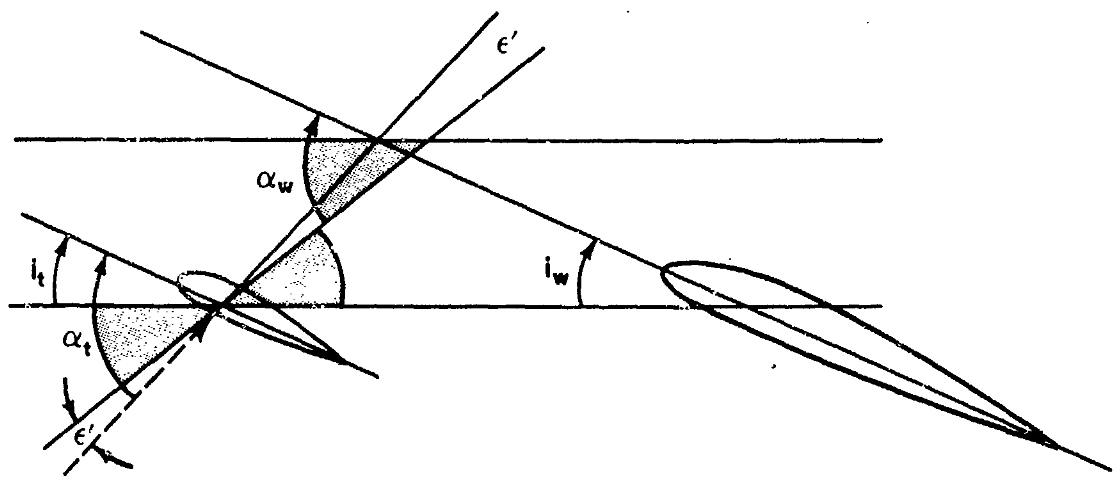


FIGURE 5.22. CANARD ANGLE OF ATTACK

$$\alpha_t - i_t - \varepsilon' = \alpha_w - i_w \tag{5.62}$$

Therefore

$$\alpha_t = \alpha_w - i_w + i_t + \varepsilon' \tag{5.63}$$

The tail contribution now becomes

$$= a_t V_H \eta_t \frac{d(\alpha_w - i_w + i_t + \varepsilon')}{dC_L}$$

$$= a_t V_H \eta_t \left(\frac{1}{a_w} + \frac{d\varepsilon'}{d\alpha_w} \frac{d\alpha_w}{dC_L} \right)$$

$$= a_t V_H \eta_t \left(\frac{1}{a_w} + \frac{d\varepsilon'}{d\alpha_w} \frac{1}{a_w} \right)$$

Therefore,

$$\frac{dC_{L_t}}{dC_L} V_H \eta_t = \frac{a_t}{a_w} V_H \eta_t \left(1 + \frac{d\varepsilon'}{d\alpha_w}\right) \tag{5.64}$$

It is extremely important to note that the upwash and downwash interaction between the canard and the wing are critical to the success of the design. The wing will see a downwash field from the canard over a portion of the leading edge. Aerodynamic tailoring and careful selection of the airfoil is required for the airplane to meet its design objectives at all canard deflections and flap settings on the wing. Designs which tend to be tandem-wing become even more sensitive to upwash and downwash.

5.10 STABILITY CURVES

Figure 5.23 is a wind tunnel plot of C_m versus C_L for an aircraft tested under two cg positions and two elevator positions.

Assuming the elevator effectiveness and the elevator power to be constant, equal elevator deflections will produce equal moments about the cg. Points A and B represent the same elevator deflection corresponding to the $C_{m_{cg}}$ needed to maintain equilibrium. For an elevator deflection of 10°, in the aft cg condition, the aircraft will fly in equilibrium or trim at point B. If the cg is moved forward with no change to the elevator deflection the equilibrium is now at A and at a new C_L. Note the increase in the stability of the aircraft (greater negative slope of dC_m/dC_L).

For equilibrium at a lower C_L or at A without changing the cg, the elevator is deflected to 5°. The stability level of the aircraft has not changed (same slope).

A cross plot of Figure 5.23 is elevator deflection versus C_L for $C_m = 0$. This is shown in Figure 5.24. The slopes of the cg curves are indicative of the aircraft's stability.

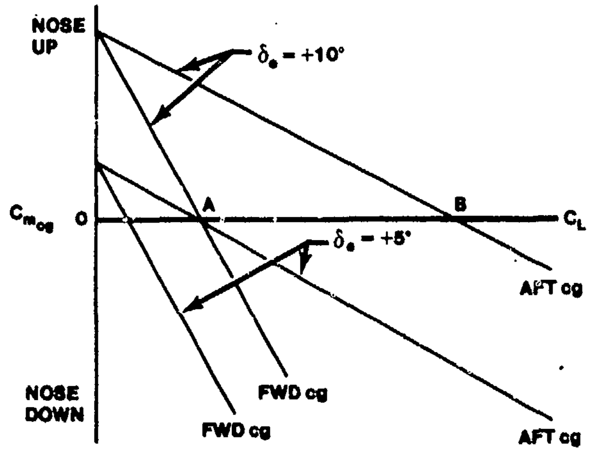


FIGURE 5.23. c_g AND δ_e VARIATION OF STABILITY

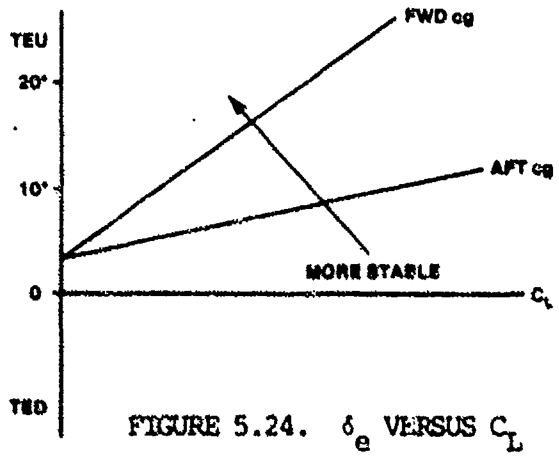


FIGURE 5.24. δ_e VERSUS C_L

5.11 FLIGHT TEST RELATIONSHIP

The stability equation previously derived cannot be directly used in flight testing. There is no aircraft instrumentation which will measure the change in pitching moment coefficient with change in lift coefficient or angle of attack. Therefore, an expression involving parameters easily measurable in flight is required. This expression should relate directly to the stick-fixed longitudinal static stability, dC_m/dC_L, of the aircraft.

The external moment acting longitudinally on an aircraft is

$$\mathcal{M} = f\,(U,\ \alpha,\ \dot{\alpha},\ Q,\ \delta_e)$$

Assuming that the aircraft is in equilibrium and in unaccelerated flight, then

$$\mathcal{M} = f\,(\alpha,\ \delta_e) \tag{5.65}$$

Therefore, using a Taylor series expansion,

$$\Delta\,\mathcal{M} = \frac{\partial \mathcal{M}}{\partial \alpha}\Delta\alpha + \frac{\partial \mathcal{M}}{\partial \delta_e}\,\Delta\delta_e \tag{5.66}$$

and

$$C_m = C_{m_\alpha}\,\Delta\alpha + C_{m_{\delta_e}}\,\Delta\delta_e = 0 \tag{5.67}$$

where

$$\Delta\alpha = \alpha - \alpha_o = \alpha$$

$$\Delta\delta_e = \delta_e - \delta_{e_o} = \delta_e$$

assuming

$$\alpha_o = 0$$

$$\delta_{e_o} = 0$$

The elevator deflection required to maintain equilibrium is,

$$\delta_e = -\frac{C_{m_\alpha}\alpha}{C_{m_{\delta_e}}} \tag{5.68}$$

Taking the derivative of δ_e with respect to C_L,

$$\frac{d\delta_e}{dC_L} = -\frac{\frac{dC_m}{d\alpha}\frac{d\alpha}{dC_L}}{C_{m_{\delta_e}}} = -\frac{\frac{dC_m}{dC_L}}{C_{m_{\delta_e}}} \tag{5.69}$$

In terms of the static margin, the flight test relationship is,

$$\frac{d\delta_e}{dC_L} = \frac{h_n - h}{C_{m_{\delta_e}}} = \frac{\text{static margin}}{\text{elevator power}} \tag{5.70}$$

The amount of elevator required to fly at equilibrium varies directly as the amount of static stick-fixed stability and inversely as the amount of elevator power.

5.12 LIMITATION TO DEGREE OF STABILITY

The degree of stability tolerable in an aircraft is determined by the physical limits of the longitudinal control. The elevator power and amount of elevator deflection is fixed once the aircraft has been designed. If the relationship between δ_e required to maintain the aircraft in equilibrium flight and C_L is linear, then the elevator deflection required to reach any C_L is,

$$\delta_e = \delta_{e_{\substack{\text{Zero}\\\text{Lift}}}} + \frac{d\delta_e}{dC_L} C_L \tag{5.71}$$

The elevator stop determines the absolute limit of the elevator deflection available. Similarly, the elevator must be capable of bringing the airplane into equilibrium at $C_{L_{Max}}$.

Recalling Equation 5.69

$$\frac{d\delta_e}{dC_L} = \frac{-\dfrac{dC_m}{dC_L}}{C_{m_{\delta_e}}} \tag{5.69}$$

Substituting Equation 5.69 into 5.71 and solving for $dC_m/dC_{L_{(Max)}}$ corresponding to $C_{L_{Max}}$

$$\frac{dC_m}{dC_{L_{Max}}} = \left(\frac{\delta_{e_{\substack{Zero\\Lift}}} - \delta_{e_{Limit}}}{C_{L_{Max}}} \right) C_{m_{\delta_e}} \tag{5.72}$$

Given a maximum C_L required for landing approach, Equation 5.72 represents the maximum stability possible, or defines the most forward cg position. A cg forward of this point prevents obtaining maximum C_L with limit elevator.

If a pilot were to maintain $C_{L_{Max}}$ for the approach, the value of dC_m/dC_L corresponding to this $C_{L_{Max}}$ would be satisfactory. However, the pilot usually desires additional C_L to compensate for gusts and to flare the aircraft. Additional elevator deflection is thus required. This requirement then dictates a $dC_m/dC_{L_{Max}}$ less than the value required for $C_{L_{Max}}$ only.

In addition to maneuvering the aircraft in the landing flare, the pilot must adjust for ground effect. The ground imposes a boundary condition which affects the downwash associated with the lifting action of the wing. This ground interference places the horizontal stabilizer at a reduced negative angle of attack. The equilibrium condition at the desired C_L is disturbed.

To maintain the desired C_L, the pilot must increase δ_e to obtain the original tail angle of attack. The maximum stability dC_m/dC_L must be further reduced to obtain additional δ_e to counteract the reduction in downwash.

The three conditions that limit the amount of static longitudinal stability or most forward cg position for landing are:

1. The ability to land at high C_L in ground effect.
2. The ability to maneuver at landing C_L (flare capability).
3. The total elevator deflection available.

Figure 5.25 illustrates the limitation in $dC_m/dC_{L_{Max}}$

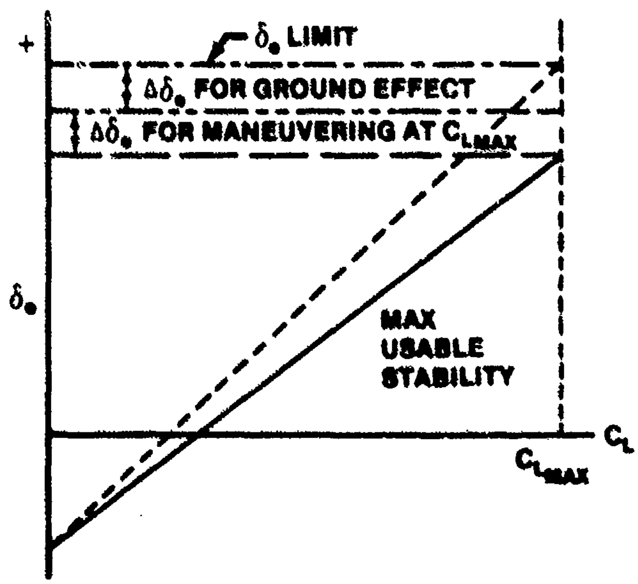


FIGURE 5.25. LIMITATIONS ON $dC_m/dC_{L_{Max}}$

5.13 STICK-FREE STABILITY

The name stick-free stability comes from the era of reversible control systems and is that variation related to the longitudinal stability which an aircraft could possess if the longitudinal control surface were left free to float in the slipstream. The control force variation with a change in airspeed is a flight test measure of this stability.

If an airplane had an elevator that would float in the slipstream when the controls were free, then the change in the pressure pattern on the stabilizer would cause a change in the stability level of the airplane. The change in the tail contribution would be a function of the floating characteristics of the elevator. Stick-free stability depends on the elevator hinge moments caused by aerodynamic forces which affect the total moment on the elevator.

An airplane with an irreversible control system has very little tendency for its elevator to float. Yet the control forces presented to the pilot during flight, even though artificially produced, appear to be the effects of having a free elevator. If the control feel system can be altered artificially, then the pilot will see only good handling qualities and be able to fly what would normally be an unsatisfactory flying machine.

Stick-free stability can be analyzed by considering the effect of freeing the elevator of a tail-to-the-rear aircraft with a reversible control system. In this case, the feel of stick-free stability would be indicated by the stick forces required to maintain the airplane in equilibrium at some speed other than trim.

The change in stability due to freeing the elevator is a function of the floating characteristics of the elevator. The floating characteristics depend upon the elevator hinge moments. These moments are created by the change in pressure distribution over the elevator associated with changes in elevator deflection and tail angle of attack.

The following analysis looks at the effect that pressure distribution has on the elevator hinge moments, the floating characteristics of the elevator, and the effects of freeing the elevator.

Previously, an expression was developed to measure the longitudinal static stability using elevator surface deflection, δ_e. This expression

represented a controls locked or stick-fixed flight test relationship where the aircraft was stabilized at various lift coefficients and the elevator deflections were then measured at these equilibrium values of C_L. The stick-free flight test relationship will be developed in terms of stick force, F_s, the most important longitudinal control parameter sensed by the pilot. In a reversible control system, the motion of the cockpit longitudinal control creates elevator control surface deflections which in turn create aerodynamic hinge moments, felt by the pilot as control forces. There is a direct feedback from the control surfaces to the cockpit control. The following analysis assumes a simple reversible flight control system as shown in Figure 5.26.

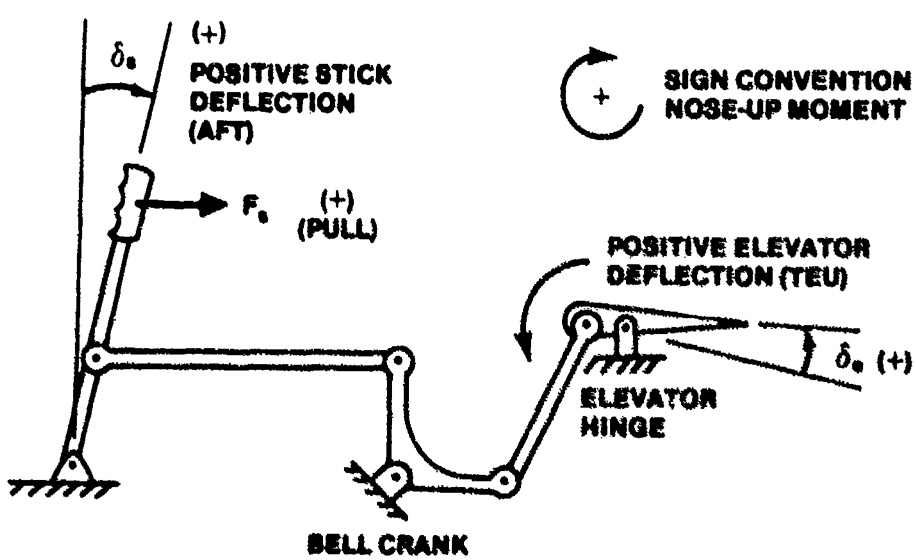


FIGURE 5.26. TAIL-TO-THE-REAR AIRCRAFT WITH A REVERSIBLE CONTROL SYSTEM

A discussion of hinge moments and their effect on the pitching moment and stability equations must necessarily precede analysis of the stick-free flight test relationship.

5.13.1 Aerodynamic Hinge Moment

An aerodynamic hinge moment is a moment generated about the control surface as a consequence of surface deflection and angle of attack. Figure 5.27 depicts the moment at the elevator hinge due to tail angle of attack ($\delta_e = 0$). Note the direction that the hinge moment would tend to rotate the elevator if the stick were released.

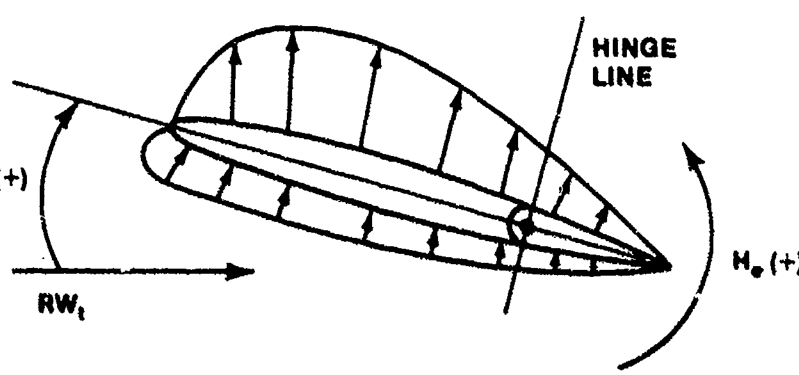


FIGURE 5.27. HINGE MOMENT DUE TO TAIL ANGLE OF ATTACK

If the elevator control were released in this case, the hinge moment, H_e would cause the elevator to rotate trailing edge up (TEU). Since the elevator TEU was previously determined to be positive, a positive hinge moment is that which, if the elevator control were released would cause the elevator to

deflect TEU. The general hinge moment equation may be expressed as

$$H_e = C_h q_t S_e c_e \tag{5.73}$$

Where S_e is elevator surface area aft of the hinge line and c_e is the root mean square chord of the elevator aft of the hinge line. The hinge moments due to elevator deflection, δ_e, and tail angle of attack, α_t, will be analyzed separately and each expressed in coefficient form.

5.13.2 Hinge Moment Due to Elevator Deflection

Figure 5.28 depicts the pressure distribution due to elevator deflection. This condition assumes $\alpha_t = 0$. The elevator is then deflected, δ_e. The resultant force aft of the hinge line produces a hinge moment, H_e, which is due to elevator deflection.

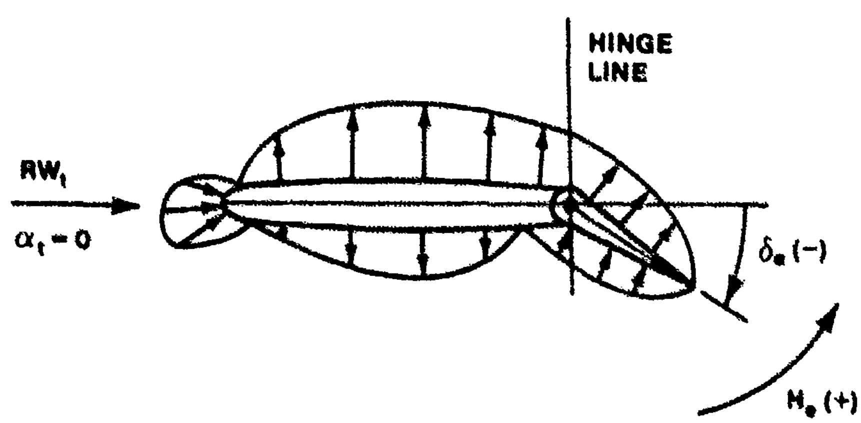


FIGURE 5.28. HINGE MOMENT DUE TO ELEVATOR DEFLECTION

Given the sign convention specified earlier, Figure 5.28 depicts the relationship of hinge moment coefficient to elevator deflection, where

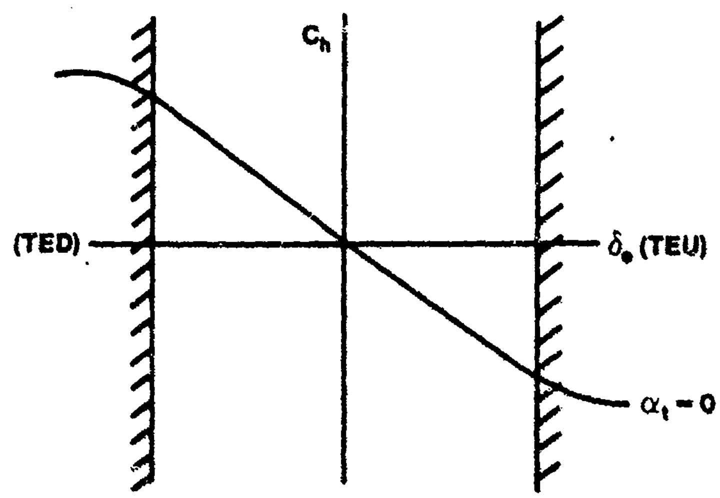


FIGURE 5.29. HINGE MOMENT COEFFICIENT DUE TO ELEVATOR DEFLECTION

most hinge moment curves are nonlinear at the extremes of elevator deflection or tail angle of attack. The boundaries shown on Figure 5.29 signify that only the linear portion of the curves is considered. The usefulness of this assumption will be apparent when the effect of elevator deflection and tail angle of attack are combined.

The slope of the curve in Figure 5.29 is $C_{h_{\delta_e}}$, the hinge moment derivative due to elevator deflection. It is negative in sign and constant in the linear region. The term $C_{h_{\delta_e}}$ is generally called the "restoring" moment coefficient.

5.13.3 Hinge Moment Due to Tail Angle of Attack

Figure 5.30 depicts the pressure distribution due to tail angle of attack. This condition assumes $\delta_e = 0$. The tail is placed at some angle of

attack. As in the previous case, the lift distribution produces a resultant force aft of the hinge line, which in turn generates a hinge moment. The term, $C_{h_{\alpha_t}}$, is generally referred to as the "floating" moment coefficient.

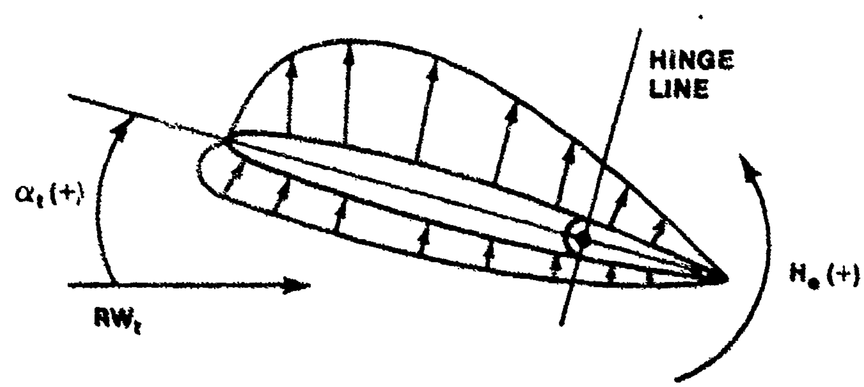


FIGURE 5.30. HINGE MOMENT DUE TO TAIL ANGLE OF ATTACK

Figure 5.31 depicts the relationship of hinge moment coefficient to tail angle of attack, where

$$C_h = \frac{H_e}{q_t S_e c_e}$$

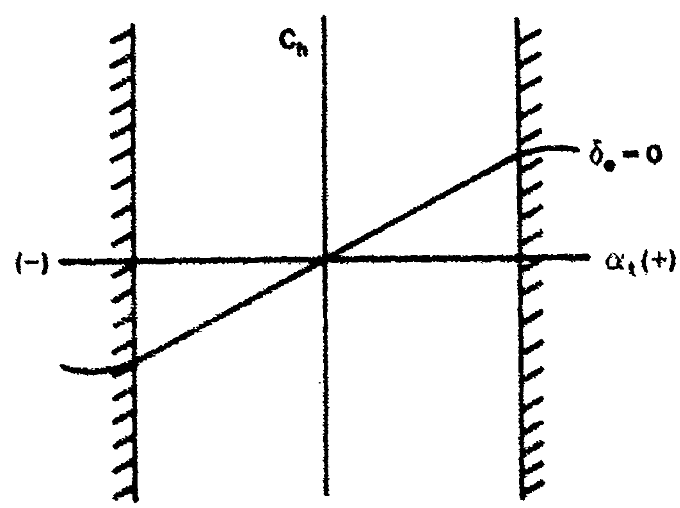


FIGURE 5.31. HINGE MOMENT COEFFICIENT DUE TO TAIL ANGLE OF ATTACK

5.13.4 Combined Effects of Hinge Moments

Given the previous assumption of linearity, the total aerodynamic hinge moment coefficient for a given elevator deflection and tail angle of attack may be expressed as

$$C_h = C_{h_0} + C_{h_{\delta_e}} \delta_e + C_{h_{\alpha_t}} \alpha_t \qquad (5.74)$$

Figure 5.32 is a graphical depiction of the above relationship, assuming a symmetrical tail so that $C_{h_0} = 0$.

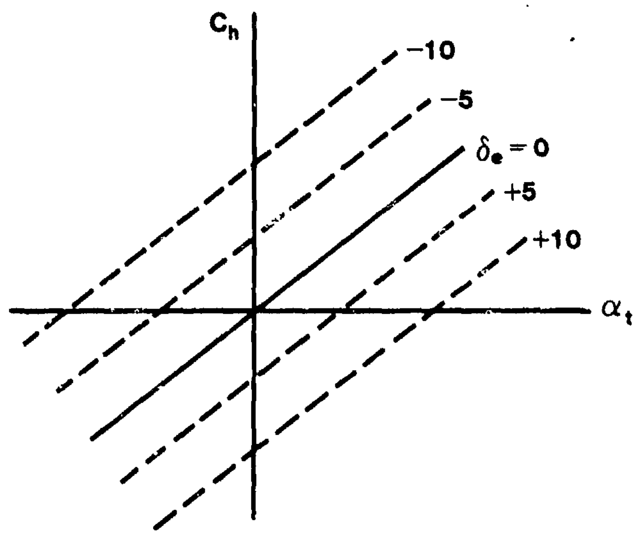


FIGURE 5.32. COMBINED HINGE MOMENT COEFFICIENTS

The "e" and "t" subscripts on the restoring and floating hinge moment coefficients are often dropped in the literature. For the remainder of this chapter:

Restoring Coefficient

$$\frac{\partial C_h}{\partial \delta_e} = C_{h_{\delta_e}} = C_{h_\delta} \tag{5.75}$$

Floating Coefficient

$$\frac{\partial C_h}{\partial \alpha_t} = C_{h_{\alpha_t}} = C_{h_\alpha} \tag{5.76}$$

Examining a floating elevator, it is seen that the total hinge moment coefficient is a function of elevator deflection, tail angle of attack, and mass distribution.

$$H_e = f(\delta_e, \alpha_t, W) \tag{5.77}$$

If the elevator is held at zero elevator deflection and zero angle of attack, there may be some residual aerodynamic hinge moment, C_{h_0}. If W is the weight of the elevator and x is the moment arm between the elevator cg and elevator hinge line, then the total hinge moment is,

$$C_h = C_{h_o} + C_{h_\alpha} \alpha_t + C_{h_\delta} \delta_e - \frac{W}{qS} \frac{x}{c} \tag{5.78}$$

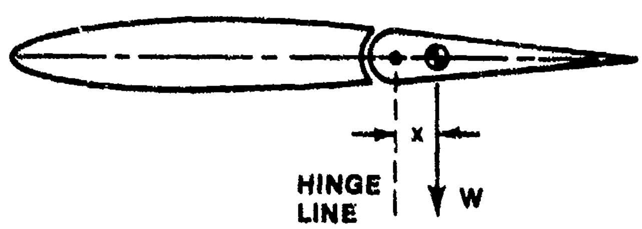


FIGURE 5.33. ELEVATOR MASS BALANCING REQUIREMENT

The weight effect is usually eliminated by mass balancing the elevator (Figure 5.33). Proper design of a symmetrical airfoil will cause C_{h_0} to be negligible.

When the elevator assumes its equilibrium position, the total hinge moments will be zero and solving for the elevator deflection at this floating position, which is shown in Figure 5.34

$$\delta_{e_{Float}} = -\frac{C_{h_\alpha}}{C_{h_\delta}} \alpha_t \tag{5.79}$$

The suitability of the aircraft with the elevator free is going to be affected by this floating position.

If the pilot desires to hold a new angle of attack from trim, he will have to deflect the elevator from this floating position to the position desired.

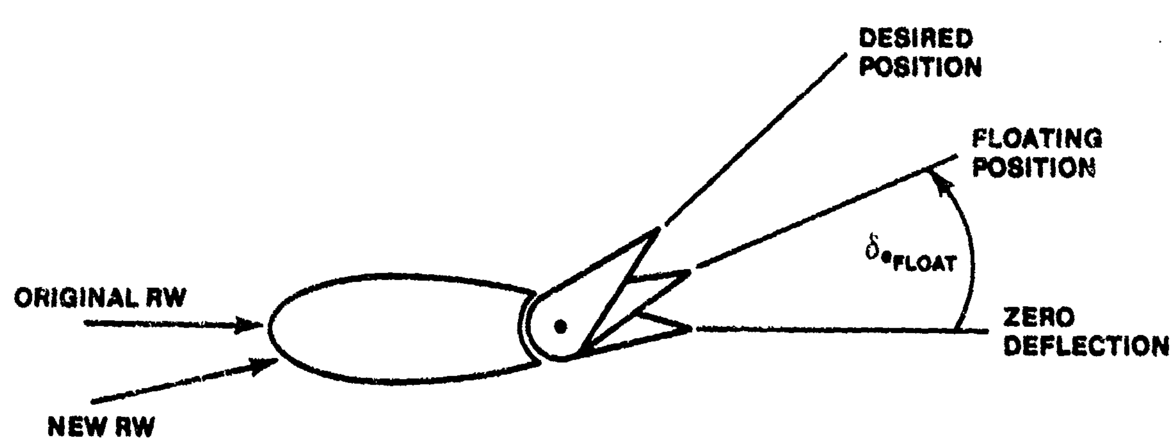


FIGURE 5.34. ELEVATOR FLOAT POSITION

The floating position will greatly affect the forces the pilot is required to use. If the ratio $C_{h_\alpha}/C_{h_\delta}$ can be adjusted, then the forces required of the pilot can be controlled.

If $C_{h_\alpha}/C_{h_\delta}$ is small, then the elevator will not float very far and the stick-free stability characteristics will be much the same as those with the stick fixed. But C_{h_δ} must be small or the stick forces required to hold deflection will be unreasonable. The values of C_{h_α} and C_{h_δ} can be controlled by aerodynamic balance. Types of aerodynamic balancing will be covered in a later section.

One additional method for altering hinge moments is through the use of a trim tab. There are numerous tab types that will be discussed in a later section. A typical tab installation is presented in Figure 5.35.

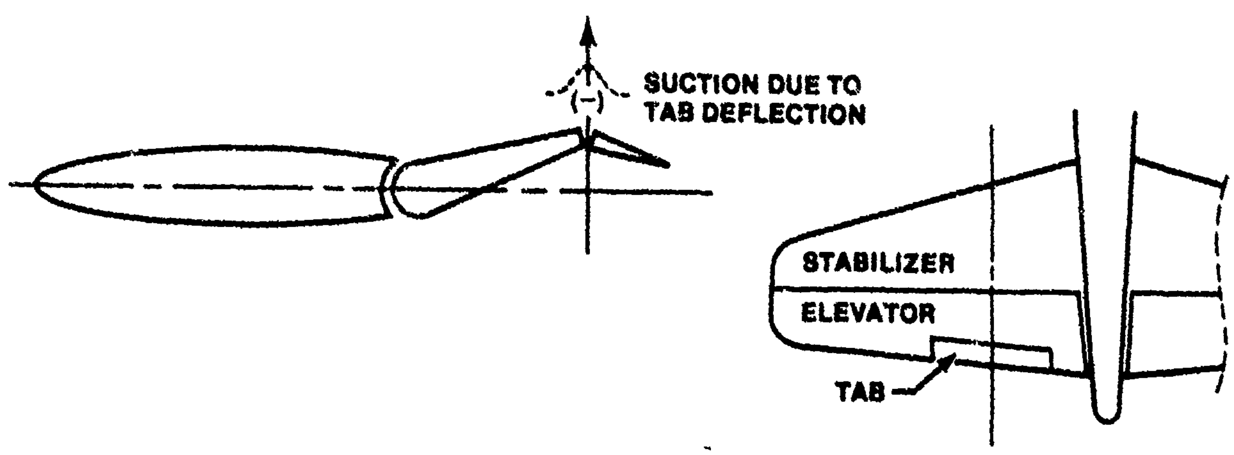


FIGURE 5.35. ELEVATOR TRIM TAB

Deflecting the tab down will result in an upward force on the trailing edge of the elevator. This tends to make the control surface float up. Thus a down tab deflection (tail-to-the-rear) results in a nose up pitching moment and is positive. This results in a positive hinge moment, and the slope of control hinge moment versus tab deflection must be positive.

The hinge moment contribution from the trim tab is thus,

$$\frac{\partial C_h}{\partial \delta_T} \delta_T \quad \text{or} \quad C_{h_{\delta_T}} \delta_T$$

and continuing with our assumption of linearity, the control hinge moment coefficient equation becomes,

$$C_h = C_{h_0} + C_{h_\alpha} \alpha_t + C_{h_\delta} \delta_e + C_{h_{\delta_T}} \delta_T \tag{5.80}$$

for a mass balanced elevator.

The elevator deflection for a floating symmetrical elevator (with tab) becomes,

$$\delta_{e_{Float}} = -\frac{C_{h_\alpha}}{C_{h_\delta}} \alpha_t - \frac{C_{h_{\delta_T}}}{C_{h_\delta}} \delta_T \tag{5.81}$$

5.14 THE STICK-FREE STABILITY EQUATION

The stick-free stability may be considered the summation of the stick-fixed stability and the contribution to stability of freeing the elevator.

$$\left.\frac{dC_m}{dC_L}\right|_{\substack{Stick-\\Free}} = \left.\frac{dC_m}{dC_L}\right|_{\substack{Stick-\\Fixed}} + \left.\frac{dC_m}{dC_L}\right|_{\substack{Free-\\Elev}} \tag{5.82}$$

The stability contribution of the free elevator depends upon the elevator floating position. Equation 5.83 relates to this position

$$\delta_{e_{Float}} = -\frac{C_{h_\alpha}}{C_{h_\delta}}\alpha_t \tag{5.83}$$

Substituting for α_t from Equation 5.24

$$\delta_{e_{Float}} = -\frac{C_{h_\alpha}}{C_{h_\delta}}(\alpha_w - i_w + i_t - \varepsilon) \tag{5.84}$$

Taking the derivative of Equation 5.84 with respect to C_L,

$$\frac{d\delta_e}{dC_L} = -\frac{C_{h_\alpha}}{C_{h_\delta}}\frac{1-\frac{d\varepsilon}{d\alpha}}{a_w} \tag{5.85}$$

Substituting the expression for elevator power, (Equation 5.50) into Equation 5.69 and combining with Equation 5.85.

$$C_{m_{\delta_e}} = -a_t V_H \eta_t \tau \tag{5.50}$$

$$\left.\frac{dC_m}{dC_L}\right|_{\substack{Free\\Elev}} = \frac{a_t}{a_w} V_H \eta_t \left(1 - \frac{d\varepsilon}{d\alpha}\right)\left(\tau\frac{C_{h_\alpha}}{C_{h_\delta}}\right) \tag{5.86}$$

Substituting Equation 5.86 and Equation 5.29 into Equation 5.82, the stick-free stability becomes

$$\frac{dC_m}{dC_L}_{\substack{\text{Stick-}\\\text{Free}}} = \frac{X_w}{c} + \frac{dC_m}{dC_L}_{\text{Fus}} - \frac{a_t}{a_w} V_H \eta_t \left(1 - \frac{d\varepsilon}{d\alpha}\right)\left(1 - \tau \frac{C_{h_\alpha}}{C_{h_\delta}}\right) \qquad (5.87)$$

The difference between stick-fixed and stick-free stability is the multiplier in Equation 5.87 $\left(1 - \tau\, C_{h_\alpha} \Big/ C_{h_\delta}\right)$, called the "free elevator factor" which is designated F. The magnitude and sign of F depends on the relative magnitudes of τ and the ratio of $C_{h_\alpha} \Big/ C_{h_\delta}$. An elevator with only slight floating tendency has a small $C_{h_\alpha} \Big/ C_{h_\delta}$ giving a value of F around unity. Stick-fixed and stick-free stability are practically the same. If the elevator has a large floating tendency $\left(\text{ratio of } C_{h_\alpha} \Big/ C_{h_\delta} \text{ large,}\right)$ the stability contribution of the horizontal tail is reduced ($dC_m/dC_{L_{\text{Stick-Free}}}$ is less negative). For instance, a ratio of $C_{h_\alpha} \Big/ C_{h_\delta} = -2$ and a τ of $-.5$, the floating elevator can eliminate the whole tail contribution to stability. Generally, freeing the elevator causes a destabilizing effect. With elevator free to float, the aircraft is less stable.

The stick-free neutral point, h_n', is that cg position at which $dC_m/dC_{L_{\text{Stick-Free}}}$ is zero. Continuing as in the stick-fixed case, the stick-free neutral point is,

$$h_n' = \frac{X_{ac}}{c} - \frac{dC_m}{dC_L}_{\text{Fus}} + \frac{a_t}{a_w} V_H \eta_t \left(1 - \frac{d\varepsilon}{d\alpha}\right) F \qquad (5.88)$$

and

$$\left.\frac{dC_m}{dC_L}\right|_{\substack{\text{Stick-}\\\text{Free}}} = h - h_n' \tag{5.89}$$

The stick-free static margin is defined as

$$\text{Static Margin} \equiv h_n' - h \tag{5.90}$$

5.15 FREE CANARD STABILITY

While it is not the intent of this paragraph to go into stick-free stability aspects of the canard, it is useful to present a summary of the effects of freeing the elevator. Remember that the tail term will be multiplied by the free elevator factor F

$$F = 1 - \tau \frac{C_{h_\alpha}}{C_{h_\delta}}$$

As F becomes less than unity, the tail (canard) contribution to stability becomes less positive, making the airplane more stable. In turbulence, stick free, the nose tends to fall slightly from an up gust, resulting in a sort of load alleviation or ride smoothing characteristic (reversible control system). The opposite is true for a tail to the rear airplane.

Table 5.1 compares the differences in stability derivatives and control terms between the canard and tail to the rear aircraft.

TABLE 5.1
STABILITY AND CONTROL DERIVATIVE COMPARISON

	Tail-to-Rear	Canard
C_{m_α}	(-)	(-)
C_{m_q}	(-)	(-)
$C_{m_{\delta_e}}$	(+)	(+)
C_{h_δ}	(-)	(-)
C_{h_α}	(+)	(-)

5.16 STICK-FREE FLIGHT TEST RELATIONSHIP

As was done for stick-fixed stability, a flight test relationship is required that will relate measurable flight test parameters with the stick-free stability of the aircraft, $dC_m/dC_{L_{\text{Stick-Free}}}$.

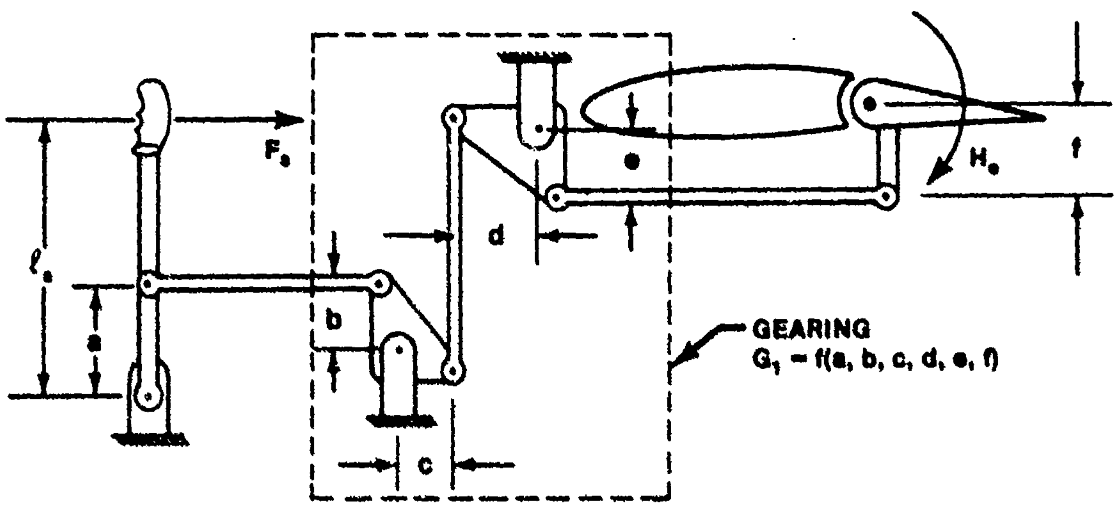


FIGURE 5.36. ELEVATOR-STICK GEARING

The pilot holds a stick deflected with a stick force F_s. The control system transmits the moment from the pilot through the gearing to the elevator Figure 5.36. The elevator deflects and the aerodynamic pressure produces a hinge moment at the elevator that exactly balances the moment produced by the pilot
with force F_s.

$$F_s l_s = G_i H_e$$

If the length l_s is included with the gearing, the stick force becomes

$$F_s = -GH_e \tag{5.91}$$

The hinge moment H_e may be written

$$H_e = C_h q S_e c_e \tag{5.92}$$

Equation 5.91 then becomes

$$F_s = - G C_h q S_e c_e \tag{5.93}$$

Substituting

$$C_h = C_{h_0} + C_{h_\alpha} \alpha_t + C_{h_\delta} \delta_e + C_{h_{\delta_T}} \delta_T \tag{5.80}$$

and using

$$\delta_e = \delta_{e_{\substack{\text{Zero}\\\text{Lift}}}} + \frac{d\delta_e}{dC_L} C_L \tag{5.71}$$

and

$$\alpha_t = \alpha_w - i_w + i_t - \varepsilon \tag{5.24}$$

With no small amount of algebraic manipulation, Equation 5.93 may be written

$$F_s = Aq \left[B + C_{h_{\delta_T}} \delta_T - \frac{C_L C_{h_\delta}}{C_{m_{\delta_e}}} \frac{dC_m}{dC_L}_{\substack{\text{Stick-}\\\text{Free}}} \right] \tag{5.94}$$

where

$$A = - G S_e c_e$$

$$B = C_{h_0} + C_{h_\alpha} (\alpha_{OL} - i_w + i_t) + C_{h_\delta} \delta_{e_{\text{Zero Lift}}}$$

Writing Equation 5.94 as a function of airspeed and substituting for unaccelerated flight, $C_L q = W/S$ and using equivalent airspeed, V_e,

$$F_s = 1/2\, \rho_0\, V_e^2\, A \left(B + C_{h_{\delta_T}} \delta_T\right) - A \frac{W}{S} \frac{C_{h_\delta}}{C_{m_{\delta_e}}} \frac{dC_m}{dC_L}_{\substack{\text{Stick-}\\\text{Free}}} \tag{5.95}$$

Simplifying Equation 5.95 by combining constant terms,

$$F_s = K_1 V_e^2 + K_2 \tag{5.96}$$

K_1 contains δ_T which determines trim speed. K_2 contains $dC_m/dC_{L_{\text{Stick-Free}}}$.

Equation 5.96 gives a relationship between an in-flight measurement of stick force gradient and stick-free stability. The equation is plotted in Figure 5.37.

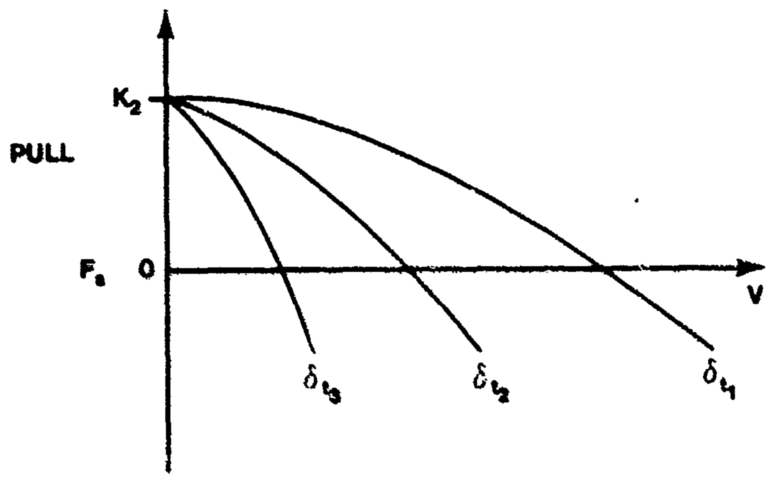


FIGURE 5.37. STICK FORCE VERSUS AIRSPEED

The plot is made up of a constant force springing from the stability term plus a variable force proportional to the velocity squared, introduced through constants and the tab term $C_{h_{\delta_T}}\ \delta_T$. Equation 5.96 introduces the interesting fact that the stick-force variation with airspeed is apparently dependent on the first term only and independent in general of the stability level. That is, the slope of the curve F_s versus V is not a direct function of $dC_m/dC_{L_{\text{Stick-Free}}}$. If the derivative of Equation 5.96 is taken with respect to V, the second term containing the stability drops out. For constant stability level and trim tab setting, stick force gradient is a function of trim airspeed.

$$\frac{dF_s}{dV} = \rho_0\, V_e\, A\left(B + C_{h_{\delta_T}}\ \delta_T\right) \tag{5.97}$$

However, dF_s/dV is a function of the stability term if the trim tab setting, δ_T, is adjusted to trim at the original trim airspeed after a change in stability level, e.g., movement of the aircraft cg. The tab setting, δ_T, in Equation 5.95 should be adjusted to obtain $F_s = 0$ at the original trim velocity $\left(C_{h_\delta} \text{ is affected by moving the tab}\right)$

$$\delta_T = f\left(V_{Trim}, \frac{dC_m}{dC_L}_{\substack{Stick-\\Free}}\right) \tag{5.98}$$

This new value of δ_T for $F_s = 0$ is then substituted into Equation 5.97 so

$$\frac{dF_s}{dV_{Trim}} = f\left(V_{Trim}, \frac{dC_m}{dC_L}_{\substack{Stick-\\Free}}\right) \tag{5.99}$$

Thus, it appears that if an aircraft is flown at two cg locations and dF_s/dV_{Trim} is determined at the same trim speed each time, then one could extrapolate or interpolate to determine the stick-free neutral point h_n. Unfortunately, if there is a significant amount of friction in the control system, it is impossible to precisely determine this trim speed. In order to investigate briefly the effects of friction on the longitudinal control system, suppose that the aircraft represented in Figure 5.38 is perfectly trimmed at V_1 i.e., $\left(\delta_e = \delta_{e_1} \text{ and } \delta_T = \delta_{T_1}\right)$. If the elevator is used to decrease or increase airspeed with no change to the trim setting, the friction in the control system will prevent the elevator from returning all the way back to δ_{e_1} when the controls are released. The aircraft will return

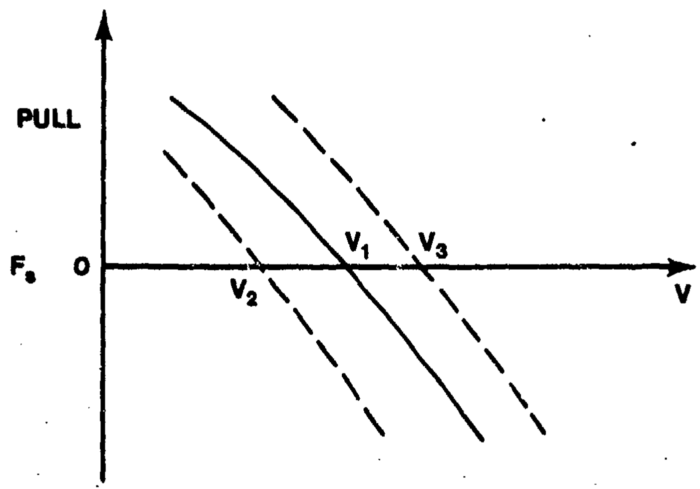


FIGURE 5.38. CONTROL SYSTEM FRICTION

only to V_2 or V_3. With the trim tab at δ_{T_1} , the aircraft is content to fly at any speed between V_2 and V_3. The more friction that exists in the system, the wider this speed range becomes. If you refer to the flight test methods section of this chapter, you will find that the F_s versus V_e plot shown in Figure 5.38 matches the data plotted in Figure 5.65. In theory, dF_s/dV_e is the slope of the parabola formed by Equation 5.96. With that portion of the parabola from $V = 0$ to V_{Stall} removed, Figures 5.37 and 5.38 predict flight test data quite accurately.

Therefore, if there is a significant amount of friction in the control system, it becomes impossible to say that there is one exact speed for which

the aircraft is trimmed. Equation 5.99 is something less than perfect for predicting the stick-free neutral point of an aircraft. To reduce the undesirable effect of friction in the control system, a different approach is made to Equation 5.94.

If Equation 5.94 is divided by the dynamic pressure, q, then,

$$F_s/q = A\left(B + C_{h_{\delta_T}}\delta_T\right) - \frac{AC_L C_{h_\delta}}{C_{m_{\delta_e}}} \frac{dC_m}{dC_L}_{\substack{Stick-\\Free}} \tag{5.100}$$

Differentiating with respect to C_L,

$$\frac{dF_s/q}{dC_L} = -\frac{AC_{h_\delta}}{C_{m_{\delta_e}}} \frac{dC_m}{dC_L}_{\substack{Stick-\\Free}} \tag{5.101}$$

or

$$\frac{dF_s/q}{dC_L} = f\left(\frac{dC_m}{dC_L}_{\substack{Stick-\\Free}}\right) \tag{5.102}$$

Trim velocity is now eliminated from consideration and the prediction of stick-free neutral point h_n' is exact. A plot of $(dF_s/q)/dC_L$ versus cg position may be extrapolated to obtain h_n'.

5.17 APPARENT STICK-FREE STABILITY

Speed stability or stick force gradient dF_s/dV, in most cases does not reflect the actual stick-free stability $dC_m/dC_{L_{\substack{Stick-\\Free}}}$ of an aircraft. In

fact, this apparent stability dF_s/dV, may be quite different from the actual stability of the aircraft. Where the actual stability of the aircraft may be marginal $\left(dC_m/dC_{L_{\text{Stick-Free}}} \text{ small}\right)$, or even unstable $\left(dC_m/dC_{L_{\text{Stick-Free}}} \text{ positive}\right)$, apparent stability of dF_s/dV may be such as to make the aircraft quite acceptable. In flight, the test pilot feels and evaluates the apparent stability of the aircraft and not the actual stability, $dC_m/dC_{L_{\text{Stick-Free}}}$. The apparent stability dF_s/dV is affected by:

1. Changes in $dC_m/dC_{L_{\text{Stick-Free}}}$
2. Aerodynamic balancing
3. Downsprings, bob weights, etc.

The apparent stability of the stick force gradient through a given trim speed increases if $dC_m/dC_{L_{\text{Stick-Free}}}$ is made more negative. The constant K_2 of Equation 5.96 is made more positive and in order for the stick force to continue to pass through the desired trim speed, a more positive tab selection is required. An aircraft operating at a given cg with a tab setting δ_{T_1} is shown in Figure 5.39, Line 1.

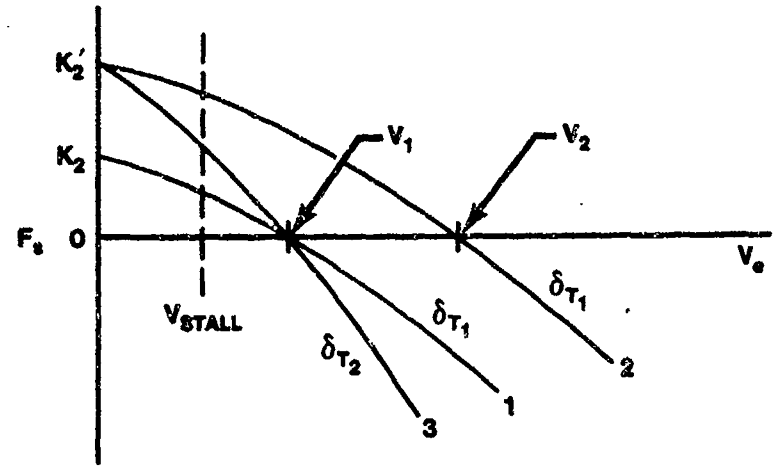


FIGURE 5.39. EFFECT ON APPARENT STABILITY

If $dC_m/dC_{L_{\text{Stick-Free}}}$ is increased by moving the cg forward, then K_2 (which is a function of $dC_m/dC_{L_{\text{Stick-Free}}}$ in Equation 5.95) becomes more positive or increases, and the equation becomes

$$F_s = K_1 V_e^2 + K_2' \qquad (5.103)$$

This equation plots as Line 2 in Figure 5.39. The aircraft with no change in tab setting δ_{T_1} operates on Line 2 and is trimmed to V_2. Stick forces at all airspeeds have increased. At this juncture, although the actual stability $dC_m/dC_{L_{\text{Stick-Free}}}$ has increased, there has been little effect on the stick force gradient or apparent stability. (The slopes of Line 1 and Line 2 being about the same.) So as to retrim to the original trim airspeed V_1, the pilot applies additional nose up tab to δ_{T_2}. The aircraft is now operating on line

3. The stick force gradient through V_1 has increased because of an increase in the K_1 term in Equation 5.96. The apparent stability dF_s/dV has increased.

Aerodynamic balancing of the control surface affects apparent stability in the same manner as cg movement. This is a design means of controlling the hinge moment coefficients, C_{h_α} and C_{h_δ}. The primary reason for aerodynamic balancing is to increase or reduce the hinge moments and, in turn, the control stick forces. Changing C_{h_δ} affects the stick forces as seen in Equation 5.100. In addition to the influence on hinge moments, aerodynamic balancing affects the real and apparent stability of the aircraft. Assuming that the restoring hinge moment coefficient C_{h_δ} is increased by an appropriate aerodynamically balanced control surface, the ratio of $C_{h_\alpha}/C_{h_\delta}$ in Equation 5.87 is decreased.

$$\frac{dC_m}{dC_{L_{\substack{\text{Stick}\\\text{Free}}}}} = \frac{X_w}{c} + \frac{dC_m}{dC_{L_{Fus}}} - \frac{a_t}{a_w} V_H \eta_t \left(1 - \frac{d\varepsilon}{d\alpha}\right)\left(1 - \tau \frac{C_{h_\alpha}}{C_{h_\delta}}\right) \quad (5.87)$$

The combined increase in $dC_m/dC_{L_{\substack{\text{Stick-}\\\text{Free}}}}$ and C_{h_δ}, increases the K_2 term in Equation 5.96 since

$$K_2 = -A\frac{W}{S}\frac{C_{h_\delta}}{C_{m_{\delta_e}}}\frac{dC_m}{dC_{L_{\substack{\text{Stick-}\\\text{Free}}}}} \quad (5.104)$$

Figure 5.39 shows the effect of increased K_2. The apparent stability is not affected by the increase in K_2 if the aircraft stabilizes at V_2. However, once the aircraft is retrimmed to the original airspeed V_1 by increasing the tab setting to δ_{T_2}, the apparent stability is increased.

5.17.1 Set-Back Hinge

Perhaps the simplest method of reducing the aerodynamic hinge moments is to move the hinge line rearward. The hinge moment is reduced because the moment arm between the elevator lift and the elevator hinge line is reduced. (One may arrive at the same conclusion by arguing this part of the elevator lift acting behind the hinge line has been reduced, while that in front of the hinge line has been increased.) The net result is a reduction in the absolute value of both C_{h_α} and C_{h_δ}. In fact, if the hinge line is set back far enough, the sign of both derivatives can be changed. A set-back hinge is shown in Figure 5.40.

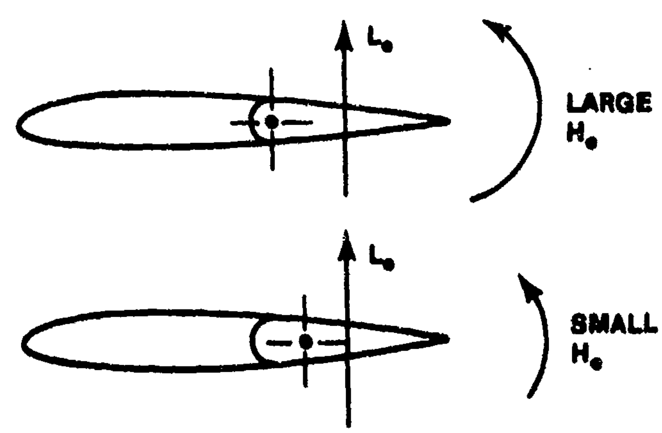


FIGURE 5.40. SET-BACK HINGE

5.17.2 Overhang Balance

This method is simply a special case of set-back hinge in which the

elevator is designed so that when the leading edge protrudes into the airstream, the local velocity is increased significantly, causing an increase in negative pressure at that point. This negative pressure peak creates a hinge moment which opposes the normal restoring hinge moment, reducing C_{h_δ}. Figure 5.41 shows an elevator with an overhang balance.

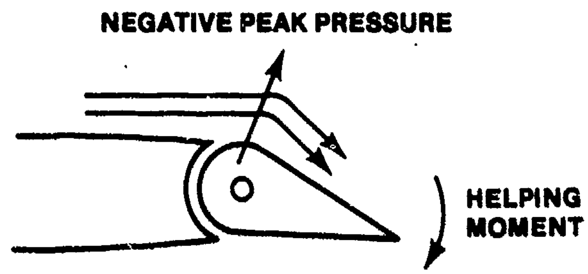


FIGURE 5.41. OVERHANG BALANCE

5.17.3 Horn Balance

The horn balance works on the same principle as the set-back hinge; i.e., to reduce hinge moments by increasing the area forward of the hinge line. The horn balance, especially the unshielded horn, is very effective in reducing C_{h_α} and C_{h_δ}. This arrangement shown in Figure 5.42 is also a handy way of improving the mass balance of the control surface.

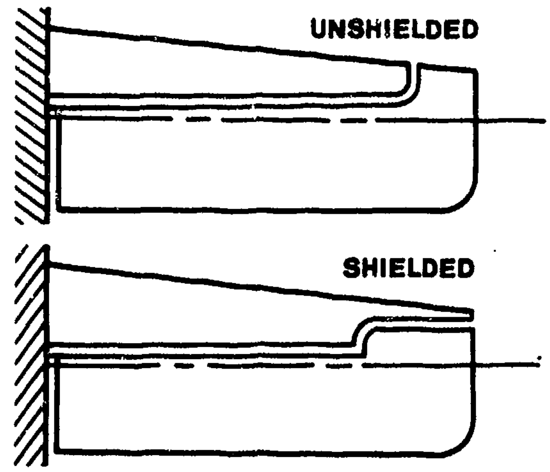


FIGURE 5.42. HORN BALANCE

5.17.4 Internal Balance or Internal Seal

The internal seal allows the negative pressure on the upper surface and the positive pressure on the lower surface to act on an internally sealed surface forward of the hinge line in such a way that a helping moment is created, again opposing the normal hinge moments. As a result, the absolute values of C_{h_α} and C_{h_δ} are both reduced. This method is good at high indicated airspeeds, but is sometimes troublesome at high Mach. Figure 5.43 shows an elevator with an internal seal.

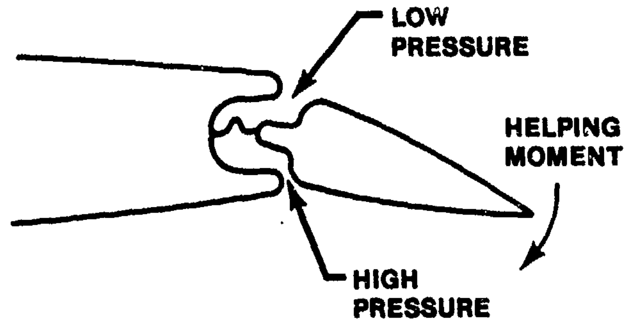


FIGURE 5.43. INTERNAL SEAL

5.17.5 Elevator Modifications

Bevel Angle on Top or on Bottom of the Stabilizer. This device which causes flow separation on one side, but not on the other, reduces the absolute values of C_{h_α} and C_{h_δ}.

Trailing Edge Strips. This device increases both C_{h_α} and C_{h_δ}. In combination with a balance tab, trailing edge strips produce a very high positive C_{h_α}, but still a low C_{h_δ}. This results in what is called a favorable "Response Effect," (i.e., it takes a lower control force to hold a deflection than was originally required to produce it).

Convex Trailing Edge. This type surface produces a more negative C_{h_δ}, but tends as well to produce a dangerous short period oscillation.

5.17.6 Tabs

A tab is simply a small flap which has been placed on the trailing edge of a larger one. The tab greatly modifies the flap hinge moments, but has only a small effect on the lift of the surface or the entire airfoil. Tabs in general are designed to modify stick forces, and therefore C_{h_δ}, but will not affect C_{h_α} very much.

By rewriting Equation 5.96, the expression for stick force as a function of airspeed, the hinge moment created by deflecting the trim tab can be determined by setting (i.e. trimming) stick force to zero.

$$C_{h_{\delta_T}} \delta_T = -B + \frac{1}{1/2\rho_0 V_e^2} \frac{W}{S} \frac{C_{h_\delta}}{C_{m_{\delta_e}}} \left.\frac{dC_m}{dC_L}\right|_{\substack{\text{Stick-}\\\text{Free}}} \tag{5.105}$$

Using a bungee (spring), an adjustable horizontal stabilizer, or a simple trim tab will have only a small effect on actual airplane stability. Using tabs to tailor stick forces, and hence the flight test relationship or "speed stability," may require that the trim and balance tabs be combined in a single tab.

5.17.6.1 <u>Trim Tab</u>. A trim tab is a tab which is controlled independently of the normal elevator control by means of a wheel or electric motor. The purpose of the trim tab is to alter the elevator hinge moment and in doing so drive the stick force to zero for a given flight condition. A properly designed trim tab should be allowed to do this throughout the flight envelope and across the allowable cg range. The trim tab reduces stick forces to zero primarily by changing the elevator hinge moment at the elevator deflection required for trim. This is illustrated in Figure 5.44.

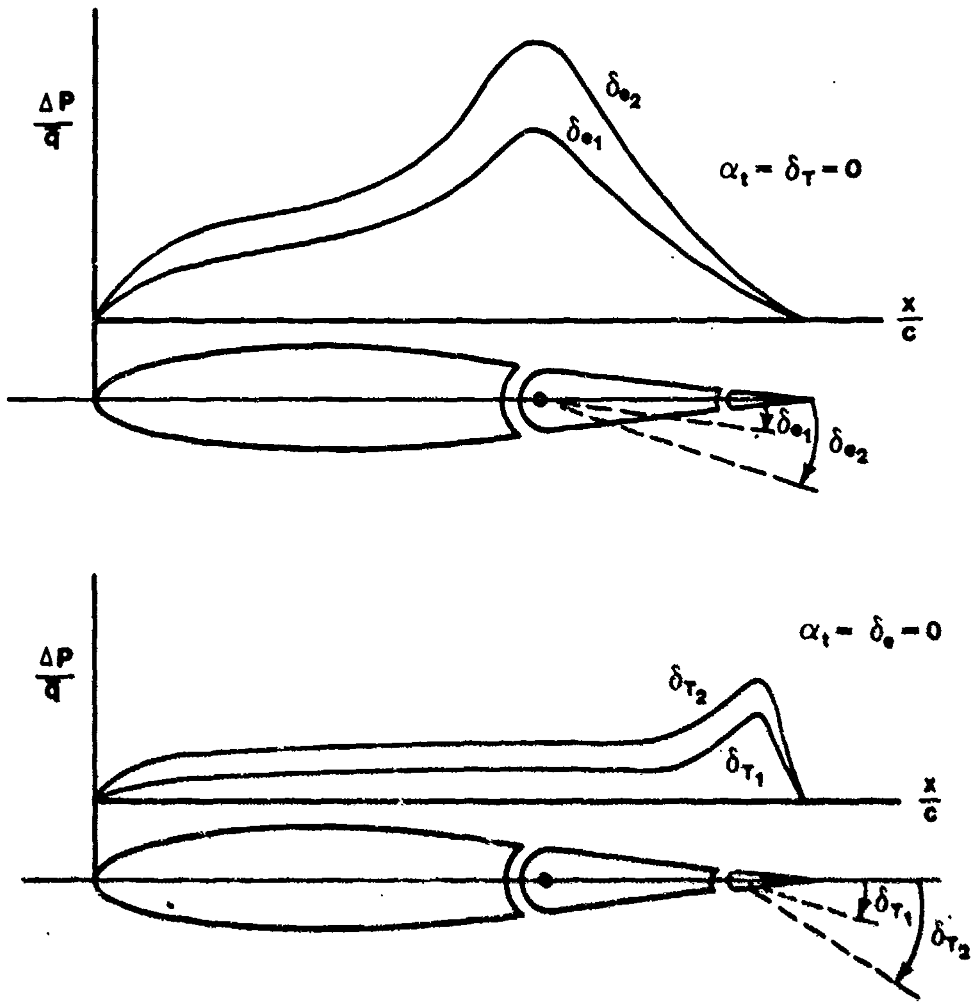


FIGURE 5.44. TRIM TAB

5.17.6.2 Balance Tab. A balance tab is a simple tab, not a part of the longitudinal control system, which is mechanically geared to the elevator so that a certain elevator deflection produces a given tab deflection. If the tab is geared to move in the same direction as the surface, it is called a leading tab. If it moves in the opposite direction, it is called a lagging tab. The purpose of the balance tab is usually to reduce the hinge moments and stick force (lagging tab) at the price of a certain loss in control effectiveness. Sometimes, however, a leading tab is used to increase control effectiveness at the price of increased stick forces. The leading tab may also be used for the express purpose of increasing control force. Thus C_{h_δ} may be increased or decreased, while C_{h_α} remains unaffected. If the linkage shown in Figure 5.45 is made so that the length may be varied by the pilot, then the tab may also serve as a trimming device.

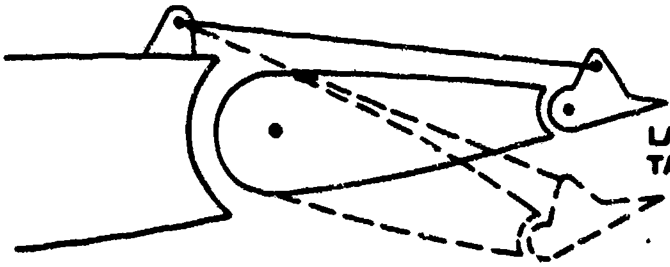


FIGURE 5.45. BALANCE TAB

5.17.6.3 Servo or Control Tab. The servo tab is linked directly to the aircraft longitudinal control system in such a manner that the pilot moves the tab and the tab moves the elevator, which is free to float. The summation of elevator hinge moment due to elevator deflection just balances out the hinge moments due to α_t and δ_T. The stick forces are now a function of the tab hinge moment or $C_{h_{\delta_T}}$. Again C_{h_α} is not affected.

5.17.6.4 Spring Tab. A spring tab is a lagging balance tab which is geared in such a way that the pilot receives the most help from the tab at high speeds where he needs it the most; i.e., the gearing is a function of dynamic pressure. The spring tab is shown in Figure 5.46.

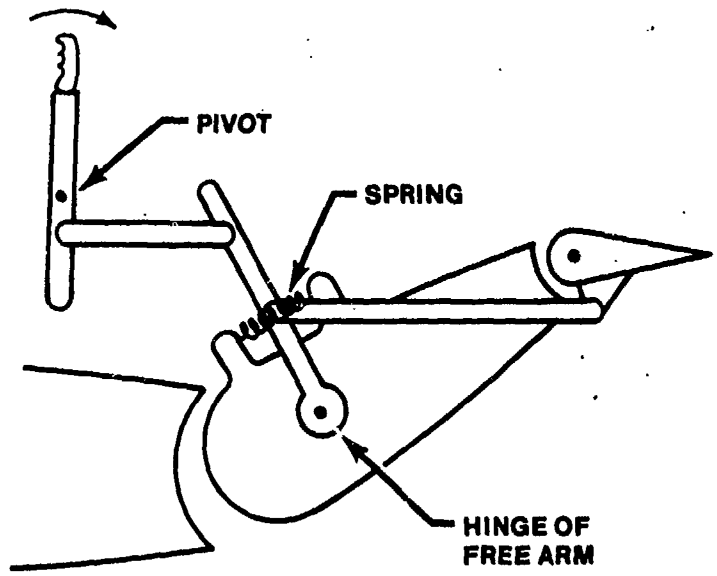


FIGURE 5.46. SPRING TAB

The basic principles of its operation are:

1. An increase in dynamic pressure causes an increase in hinge moment and stick force for a given control deflection.

2. The increased stick force causes an increased spring deflection and, therefore, an increased tab deflection.

3. The increased tab deflection causes a decrease in stick force. Thus an increased proportion of the hinge moment is taken by the tab.

4. Therefore, the spring tab is a geared balance tab where the gearing is a function of dynamic pressure.

5. Thus, the stick forces are more nearly constant over the speed range of the aircraft. That is, the stick force required to produce a given deflection at 300 knots is still greater than at 150 knots, but not by as much as before. Note that the pilot cannot tell what is causing the forces he feels at the stick. This appears a change in "speed stability," but in fact will change actual stability or dC_m/dC_L.

6. After full spring or tab deflection is reached the balancing feature is lost and the pilot must supply the full force necessary for further deflection. (This acts as a safety feature.)

A plot comparing the relative effects of the various balances on hinge moment parameters is given in Figure 5.47 below. The point indicated by the circle represents the values of a typical plain control surface. The various lines radiating from that point indicate the manner in which the hinge moment parameters are changed by addition of various kinds of balances. Figure 5.48 is also a summary of the effect of various types of balances on hinge moment coefficients.

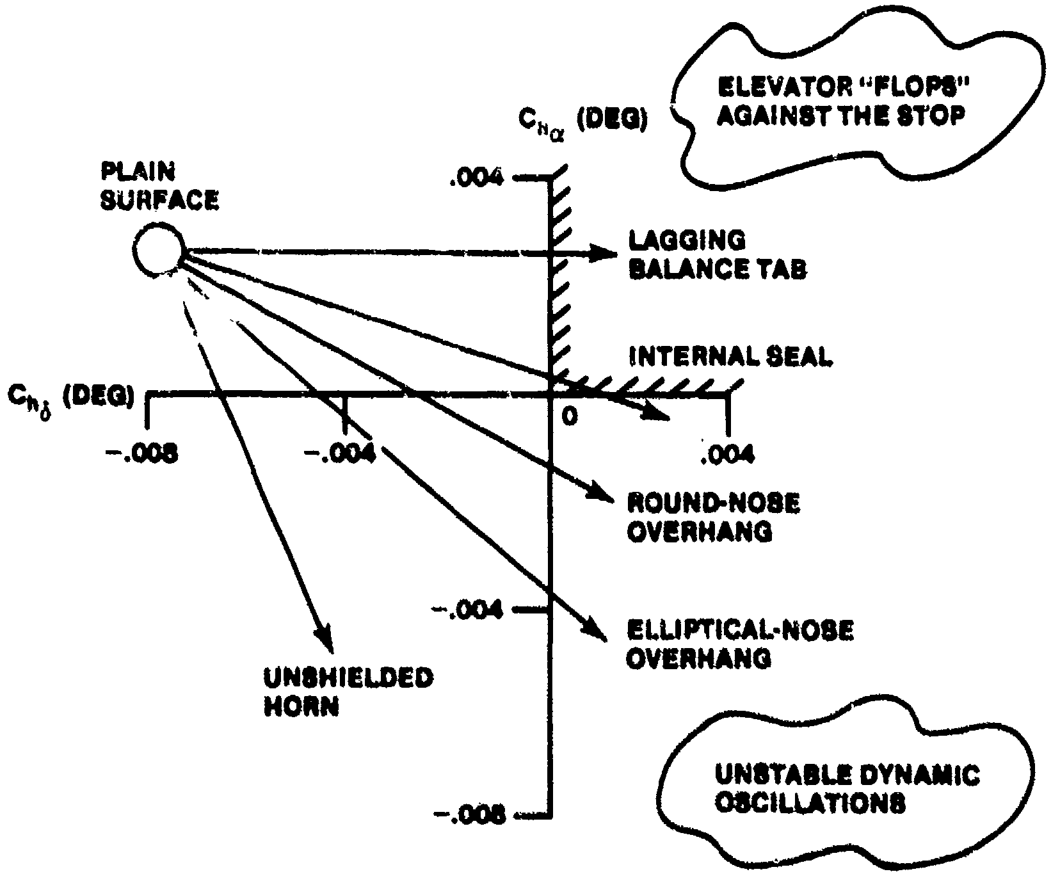


FIGURE 5.47. TYPICAL HINGE MOMENT COEFFICIENT VALUES

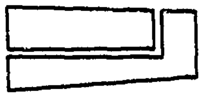
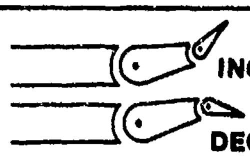
NOMENCLATURE	C_{h_α} SIGN NORMALLY (+)	C_{h_δ} SIGN ALWAYS (−)	
SET-BACK HINGE	REDUCED	REDUCED	
OVERHANG	REDUCED	REDUCED	
UNSHIELDED HORN	REDUCED	REDUCED	 TOP VIEW
INTERNAL SEAL	REDUCED	REDUCED	
BEVEL ANGLE STRIPS	REDUCED	REDUCED	
TRAILING EDGE STRIPS	INCREASED	INCREASED	
CONVEX TRAILING EDGE	NO CHANGE	INCREASED	
TRIM TABS	NO CHANGE	INCREASED OR DECREASED	 INCREASED DECREASED
LAGGING BALANCE TAB	NO CHANGE	DECREASED	
LEADING BALANCE TAB	NO CHANGE	INCREASED	
BLOW DOWN TAB OR SPRING TAB	NO CHANGE	INCREASED, DECREASES WITH "q"	

FIGURE 5.48. METHODS OF CHANGING AERODYNAMIC HINGE MOMENT COEFFICIENT MAGNITUDES (TAIL-TO-THE-REAR AIRCRAFT)

In summary, aerodynamic balancing is "tailoring" the values of values of C_{h_α} and C_{h_δ} during the aircraft design phase in order to increase or decrease hinge moments. It is a method of controlling stick forces and affects the real and apparent stability of the aircraft. In the literature, dynamic control balancing is often defined as making C_{h_α} small or just slightly positive. Note from Figure 5.47 that addition of an unshielded horn balance changes C_{h_α} without affecting C_{h_δ} very much. If C_{h_α} is made exactly zero, the aircraft's stick-fixed and stick-free stability are the same. Making C_{h_α} negative is defined as overbalancing. If C_{h_α} is made slightly negative, then the aircraft is more stable stick-free than stick-fixed. Early British flying quality specifications permitted an aircraft to be unstable stick-fixed as long as stick-free stability was maintained. Overbalancing increases stick forces.

Because of the very low force gradients in most modern aircraft at the aft center of gravity, improvements in stick-free longitudinal stability are obtained by devices which produce a constant pull force on the stick independent of airspeed which allows a more noseup tab setting and steeper stick force gradients. Two devices for increasing the stick force gradients are the downspring and bobweight. Both effectively increase the apparent stability of the aircraft.

5.17.7 Downspring

A virtually constant stick force may be incorporated into the control system by using a downspring or bungee which tends to pull the top of the stick forward. From Figure 5.49 the force required to counteract the spring is

$$F_{s_{Downspring}} = T \frac{l_1}{l_2} = K_3 \tag{5.106}$$

If the spring is a long one, the tension in it will be increased only slightly as the top moves rearward and can be considered to be constant.

The equation with the downspring in the control system becomes,

$$F_s = K_1 V_e^2 + K_2 + K_{3_{Downspring}} \quad (5.107)$$

As shown in Figure 5.37, the apparent stability will increase when the aircraft is once again retrimmed by increasing the tab setting. Note that the downspring increases apparent stability, but does not affect the actual stability of the aircraft ($dC_m/dC_{L_{Stick-Free}}$; no change to K_2).

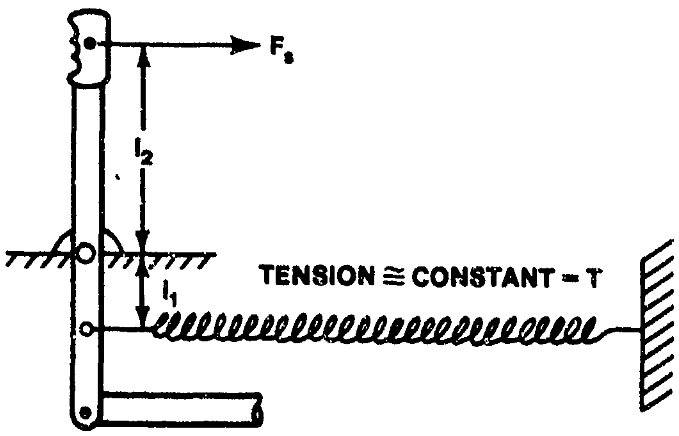


FIGURE 5.49. DOWNSPRING

5.17.8 Bobweight

Another method of introducing a nearly constant stick force is by placing a bobweight in the control system which causes a constant moment (Figure 5.50). The force to counteract the bobweight is,

$$F_{s_{Bobweight}} = nW \frac{l_1}{l_2} = K_3 \quad (5.108)$$

Like the downspring, the bobweight increases the stick force throughout the airspeed range and, at increased tab settings, the apparent stability or stick force gradient. The bobweight has no effect on the actual stability of the aircraft $\left(dC_m/dC_{L_{\text{Stick-Free}}}\right)$.

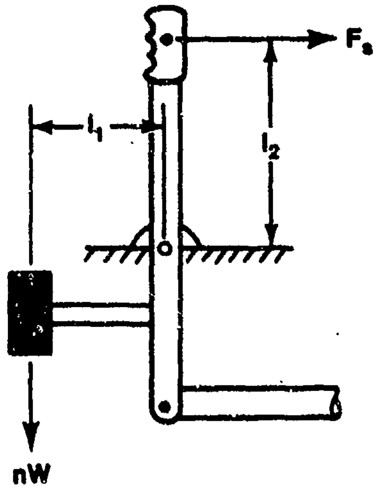


FIGURE 5.50. BOBWEIGHT

A spring may be used as an "unspring," and a bobweight may be placed on the opposite side of the stick in the control system. Those configurations are illustrated in Figure 5.51. In this configuration, the stick force would be decreased, and the apparent stability also decreased. It should again be emphasized that regardless of spring or bobweight configuration, there is no effect on the actual stability $\left(dC_m/dC_{L_{\text{Stick-Free}}}\right)$ of the aircraft.

Further use of these control system devices will be discussed in Chapter 6, Maneuvering Flight.

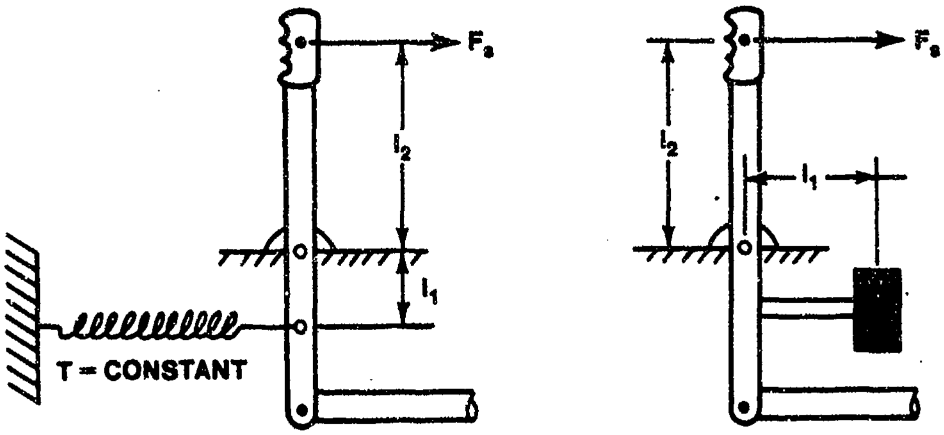


FIGURE 5.51. ALTERNATE SPRING & BOBWEIGHT CONFIGURATIONS

To examine the effect of the stick force gradient dF_s/dV on Equation 5.102 and to find h_n', Equation 5.94 is rewritten with a control system device

$$F_s = Aq(B + C_{h_{\delta_T}}\delta_T) - AC_L q \frac{C_{h_\delta}}{C_{m_{\delta_e}}} \left.\frac{dC_m}{dC_L}\right|_{\text{Stick-Free}} + K_{3_{\text{Device}}} \tag{5.109}$$

$$F_s/q = A(B + C_{h_{\delta_T}}\delta_T) - AC_L \frac{C_{h_\delta}}{C_{m_{\delta_e}}} \left.\frac{dC_m}{dC_L}\right|_{\text{Stick-Free}} + \frac{K_3 C_L}{W/S} \tag{5.110}$$

$$\frac{dF_s/q}{dC_L} = K1_2 \left.\frac{dC_m}{dC_L}\right|_{\text{Stick-Free}} + \frac{K_3}{W/S} \tag{5.111}$$

The cg location at which $(dF_s/q)/dC_L$ goes to zero will not be the true h_n' when a device such as a spring or a bobweight is included.

5.18 HIGH SPEED LONGITUDINAL STATIC STABILITY

The effects of high speed (transonic and supersonic) on longitudinal static stability can be analyzed in the same manner as that done for subsonic speeds. However, the assumptions that were made for incompressible flow are no longer valid.

Compressibility effects both the longitudinal static stability, dC_m/dC_L (gust stability) and speed stability, dF_s/dV. The gust stability depends mainly on the contributions to stability of the wing, fuselage, and tail in the stability equation below during transonic and supersonic flight.

$$\frac{dC_m}{dC_L} = \frac{X_w}{c} + \frac{dC_m}{dC_{L_{Fus}}} - \frac{a_t}{a_w} V_H \eta_t \left(1 - \frac{d\varepsilon}{d\alpha}\right) \tag{5.29}$$

5.18.1 The Wing Contribution

In subsonic flow the aerodynamic center is at the quarter chord. At transonic speed, flow separation occurs behind the shock formations causing the aerodynamic center to move forward of the quarter chord position. The immediate effect is a reduction in stability since X_w/c increases. As speed increases further the shock moves off the surface and the wing recovers lift. The aerodynamic center moves aft towards the 50% chord position. There is a sudden increase in the wing's contribution to stability since X_w/c is reduced (Figure 5.2).

The extent of the aerodynamic center shift depends greatly on the aspect ratio of the aircraft. The shift is least for low aspect ratio aircraft. Among the planforms, the rectangular wing has the largest shift for a given aspect ratio, whereas the triangular wing has the least (Figure 5.52).

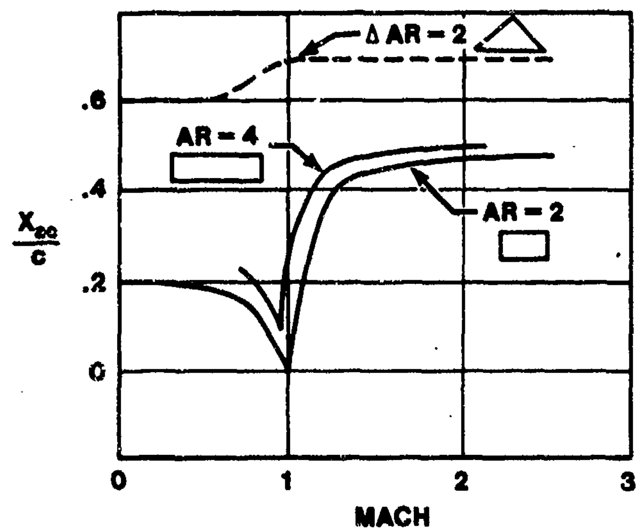


FIGURE 5.52. AC VARIATION WITH MACH

5.18.2 The Fuselage Contribution

In supersonic flow, the fuselage center of pressure moves forward causing a positive increase in the fuselage dC_m/dC_L or a destabilizing influence on the fuselage term. Variation with Mach is usually small and will be ignored.

5.18.3 The Tail Contribution

The tail contribution to stability depends on the variation of lift curve slopes, a_w and a_t, plus downwash ε with Mach during transonic and supersonic flight. It is expressed as:

$$(-a_t/a_w)\ V_H\ \eta_t\ \ (1 - d\varepsilon/d\alpha)$$

During subsonic flight a_t/a_w remains approximately constant. The slope of the lift curve, a_w varies as shown in Figure 5.53. This variation of a_w in the transonic speed range is a function of geometry (i.e., aspect ratio, thickness, camber, and sweep). a_t varies in a similar manner. Limiting further discussion to aircraft designed for transonic flight or aircraft which employ airfoil shapes with small thickness to chord ratios, then a_w and a_t increase slightly in the transonic regime. For all airfoil shapes, the values of a_w and a_t decrease as the airspeed increases supersonically.

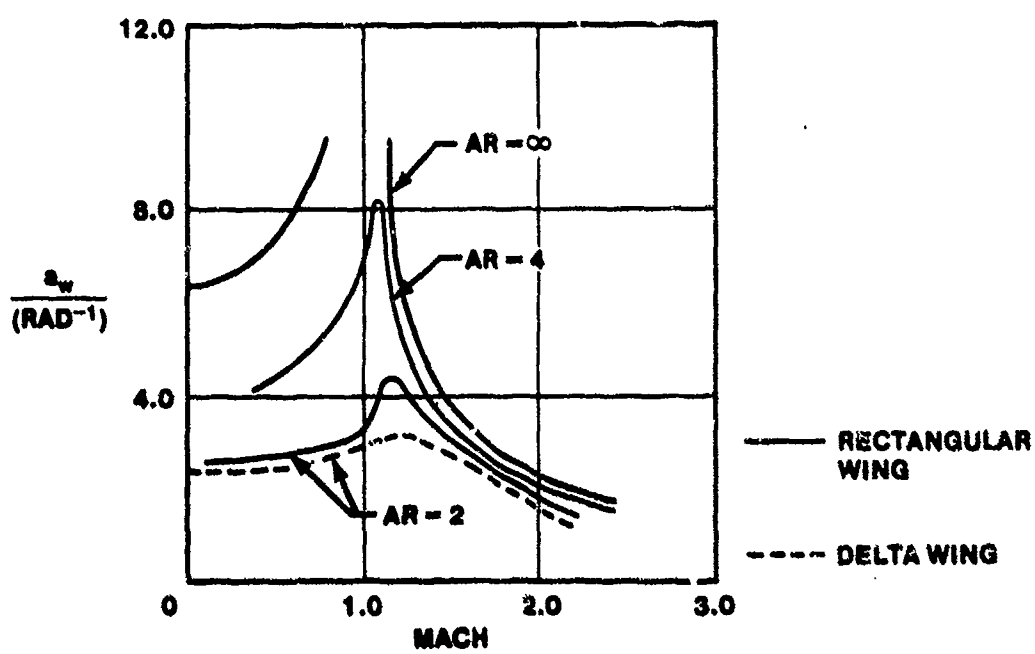


FIGURE 5.53. LIFT CURVE SLOPE VARIATION WITH MACH

The tail contribution is further affected by the variation in downwash, ε, with Mach increase. The downwash at the tail is a result of the vortex system associated with the lifting wing. A sudden loss of downwash occurs transonically with a resulting decrease in tail angle of attack. The effect is to require the pilot to apply additional up elevator with increasing airspeed to maintain altitude. This additional up elevator contributes to speed instability. (Speed stability will be covered more thoroughly later.) Typical downwash variation with Mach is seen in Figure 5.54.

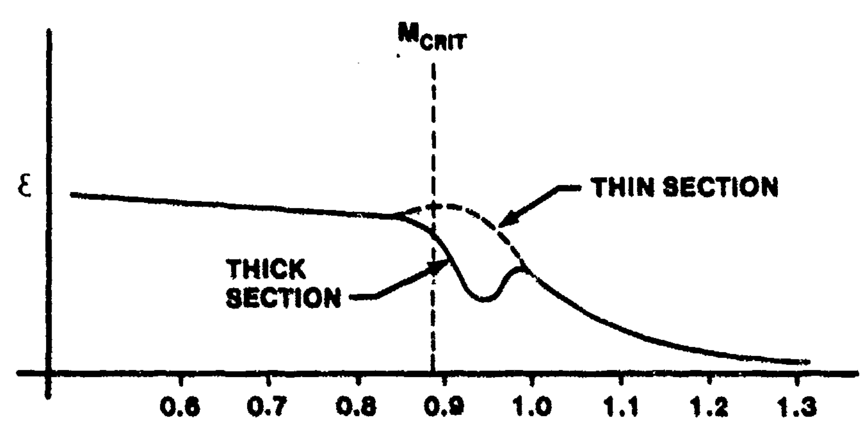


FIGURE 5.54. TYPICAL DOWNWASH VARIATION WITH MACH

The variation of $d\varepsilon/d\alpha$ with Mach greatly influences the aircraft's gust stability dC_m/dC_L. Recalling from subsonic aerodynamics,

$$\varepsilon = \frac{114.6C_L}{\pi\ AR} \quad \text{and} \quad \frac{d\varepsilon}{d\alpha} = \frac{114.6a_W}{\pi\ AR}$$

Since the downwash angle behind the wing is directly proportional to the lift coefficient of the wing, it is apparent that the value of the derivative $d\varepsilon/d\alpha$ is a function of a_W. The general trend of $d\varepsilon/d\alpha$ is an initial increase with

Mach starting at subsonic speeds. Above Mach 1.0, $d\varepsilon/d\alpha$ decreases and approaches zero. This variation depends on the particular wing geometry of the aircraft. As shown in Figure 5.55, $d\varepsilon/d\alpha$ may dip for thicker wing sections where considerable flow separation occurs. Again, $d\varepsilon/d\alpha$ is very much dependent upon a_w.

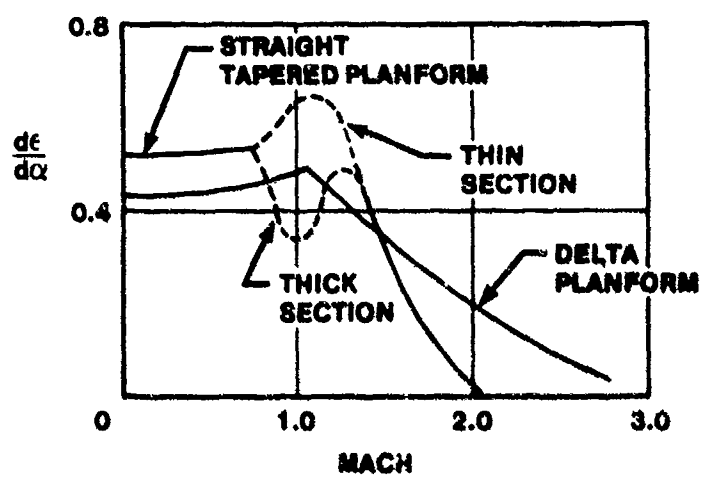


FIGURE 5.55. DOWNWASH DERIVATIVE VS MACH

For an aircraft designed for high speed flight, the variation of $d\varepsilon/d\alpha$ with increasing Mach results in a slight destabilizing effect in the transonic regime and contributes to increased stability in the supersonic speed regime; therefore, the overall tail contribution to stability is difficult to predict.

loss of stabilizer effectiveness is experienced in supersonic flight as ɔmes a less efficient lifting surface. The elevator power, $C_{m_{\delta_e}}$, ses as airspeed approaches Mach 1 . Beyond Mach 1, elevator iveness decreases. Typical variations in $C_{m_{\delta_e}}$ with Mach are shown ure 5.56.

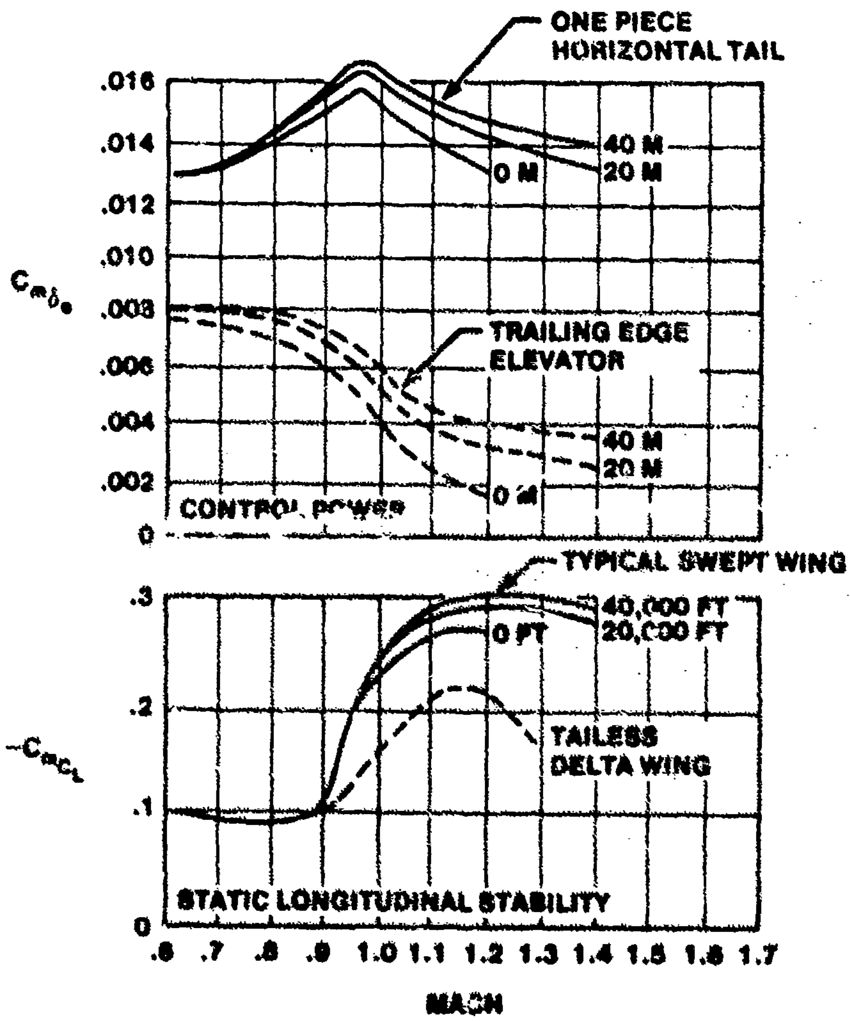


FIGURE 5.56. MACH VARIATIONS ON $C_{m_{\delta_e}}$ AND $C_{m_{C_L}}$

The overall effect of transonic and supersonic flight on gust stability or dC_m/dC_L is also shown in Figure 5.56. Static longitudinal stability increases supersonically. The speed stability of the aircraft is affected as well. The pitching moment coefficient equation developed in Chapter 4 can be written,

$$C_m = C_{m_0} + C_{m_\alpha} \Delta\alpha + C_{m_\alpha} \Delta\dot{\alpha} + C_{m_{\delta_e}} \Delta\delta_e + C_{m_u} \Delta U + C_{m_q} \Delta Q \qquad (5.112)$$

Assuming no pitch rates, Equation 5.112 can be written

$$C_m = C_{m_\alpha} \Delta\alpha + C_{m_{\delta_e}} \Delta\delta_e + C_{m_u} \Delta U \qquad (5.113)$$

where C_{m_α} is $dC_m/dC_L \cdot dC_L/d\alpha$. All three of the stability derivatives in Equation 5.113 are functions of Mach. The elevator deflection required to trim as an aircraft accelerates from subsonic to supersonic flight depends on how these derivatives vary with Mach. For supersonic aircraft, speed stability is provided entirely by the artificial feel system. However, it usually depends on how δ_e varies with Mach. A reversal of elevator deflection with increasing airspeed usually requires a relaxation of forward pressure or even a pull force to maintain altitude or prevent a nose down pitch tendency.

Elevator deflection versus Mach curves for several supersonic aircraft are shown in Figure 5.57. The important point from this figure is that supersonically $d\delta_e/dC_L$ is no longer a valid indication of gust stability. All of the aircraft shown in Figure 5.57 are more stable supersonically than subsonically, if you were to look purely at dF_s/dV.

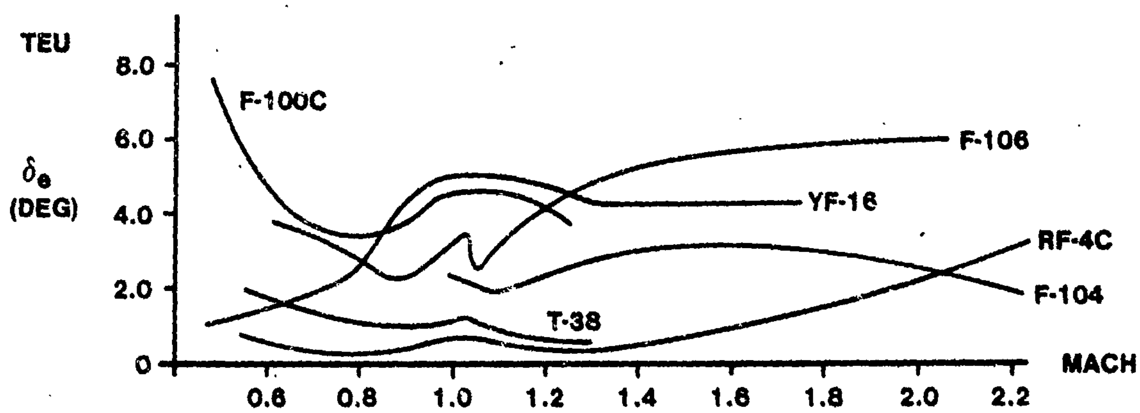


FIGURE 5.57. STABILIZER DEFLECTION VS MACH FOR SEVERAL SUPERSONIC AIRCRAFT

Whether the speed instability or reversal in elevator deflections and stick forces are objectionable depends on many factors such as magnitude of variation, length of time required to transverse the region of instability, control system characteristics, and conditions of flight.

In the F-100C, a pull of 14 pounds was required when accelerating from Mach 0.87 to 1.0. The test pilot described this trim change as disconcerting while attempting to maneuver the aircraft in this region and recommended that a "q" or Mach sensing device be installed to eliminate this characteristic. Consequently, a mechanism was incorporated to automatically change the artificial feel gradient as the aircraft accelerates through the transonic range. Also, the longitudinal trim is automatically changed in this region by the use of a "Mach Trimmer."

F-104 test pilots stated that F-104 transonic trim changes required an aft stick movement with increasing speed and a forward stick movement when decreasing speed but described this speed instability as acceptable.

F-106 pilots stated that the Mach 1.0 to 1.1 region is characterized by a moderate trim change to avoid large variations in altitude during accelerations. Minor speed instabilities were not unsatisfactory.

T-38 test pilots described the transonic trim change as being hardly perceptible.

Aircraft design considerations are influenced by the stability aspects of high speed flight. It is desirable to design an aircraft where trim changes through transonic speeds are small. A tapered wing without camber, twist, or incidence or a low aspect ratio wing and tail provide values of X_w/c, a_w, a_t, and $d\varepsilon/d\alpha$ which vary minimally with Mach. An all-moving tail (slab) gives negligible variation of $C_{m_{\delta_e}}$ with Mach and maximum control effectiveness. A full power, irreversible control system is necessary to counteract the erratic changes in pressure distribution which affect C_{h_α} and C_{h_δ}.

5.19 HYPERSONIC LONGITUDINAL STATIC STABILITY

The X-15 is an example of an airplane with a low aspect ratio wing and an all moving horizontal tail. Having achieved a maximum speed of Mach 6.7, it was definitely a hypersonic vehicle. For an airplane to overcome the thermal and aerodynamic problems of atmospheric entry, the delta or double delta configuration with a blunt aft end seems to be one answer.

Configurations such as the Space Shuttle experience low longitudinal stability in the high subsonic to transonic region. To increase stability in this region, these vehicles have used boat-tail flaps or extended rudder surfaces to move the center of pressure aft, creating "shuttlecock stability." As expected, stability improves significantly due to Mach effects in the 0.9 to 1.4 Mach region, as the center of pressure shifts aft. The transonic and supersonic longitudinal stability curves for the delta configuration are shown in Figure 5.58.

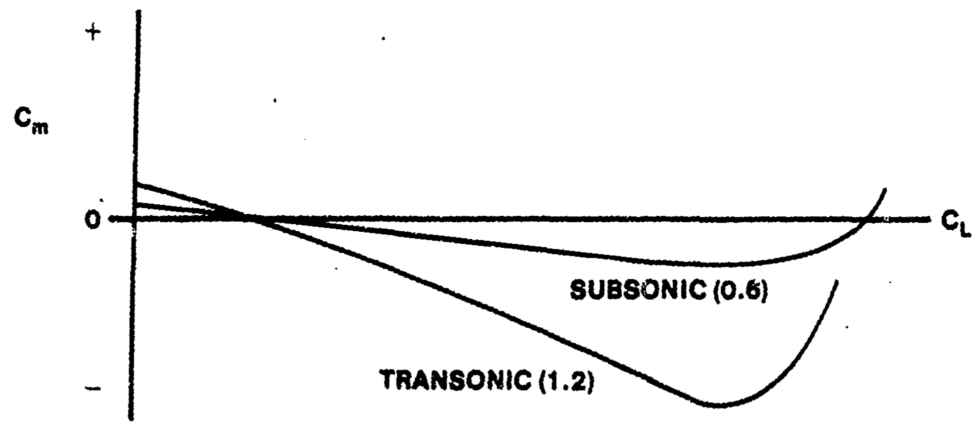


FIGURE 5.58. TRANSONIC AND SUPERSONIC LONGITUDINAL STABILITY - DELTA PLANFORM

At Mach 2.0 this configuration is stable at low α (low C_L) and neutral to slightly unstable at high α (high C_L). The opposite is true at Mach 4.0, where the vehicle is more stable at high α than at low α. This is shown in Figure 5.59. It should be noted that the region around Mach 3.0 is one of uncertainty in all axes. The shock wave does not lie close to the lower surface as it does at Mach 4.0 and above, and there is a large low pressure area at the blunt aft end that makes the elevons less effective.

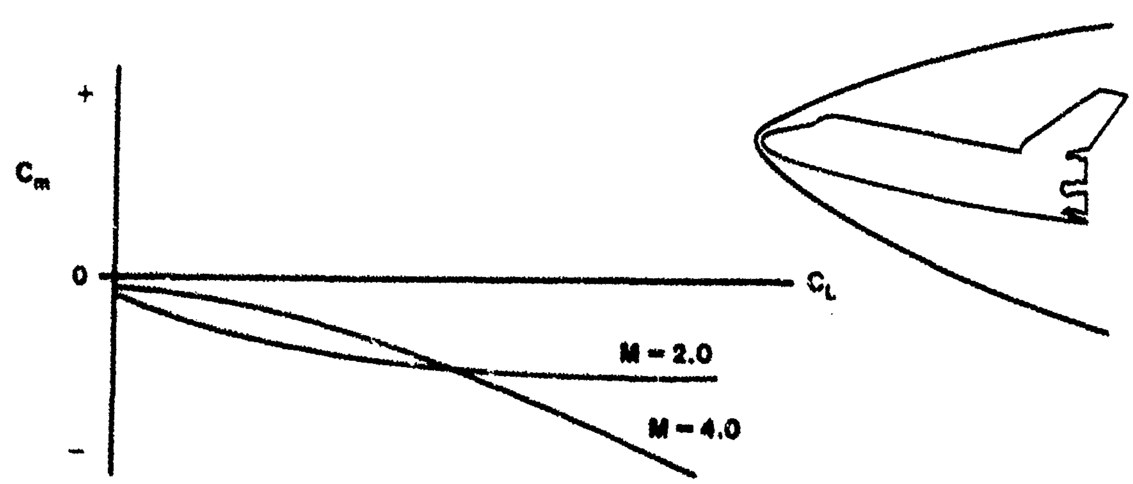


FIGURE 5.59. DELTA CONFIGURATION AT M = 2.0 AND M = 4.0

Above Mach 4.0, there will still be some problems at low α, but the vehicle will be stable in the 20° to 60° α range. At these speeds the shock is adjacent to the lower surface and adds to reasonable elevon control. As shown in Figure 5.60, a stable break occurs in the plot of dC_m/dC_L at C_L's corresponding to 15° to 20°.

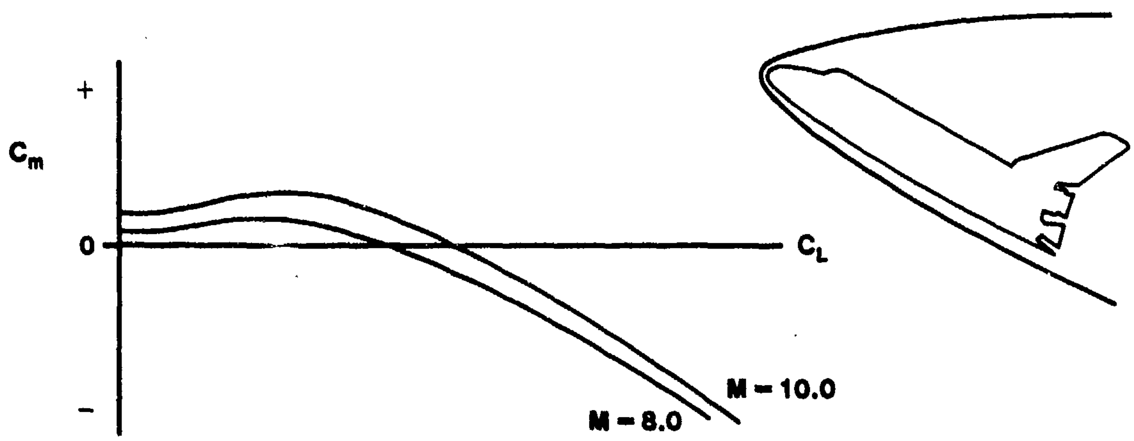


FIGURE 5.60. DELTA CONFIGURATION AT M = 8.0 AND M = 10.0

Control power, $C_{m_{\delta_e}}$, can be a problem during hypersonic flight, even in the 20° to 60° range. Figure 5.61 illustrates that as the elevon moves TEU, it moves into the low pressure area at the aft end of the vehicle and becomes less effective. At certain cg's you may not have enough elevon effectiveness to trim. The Space Shuttle uses its body flap as an additional trimming surface to keep the elevons close to zero degrees deflection throughout the allowable cg range.

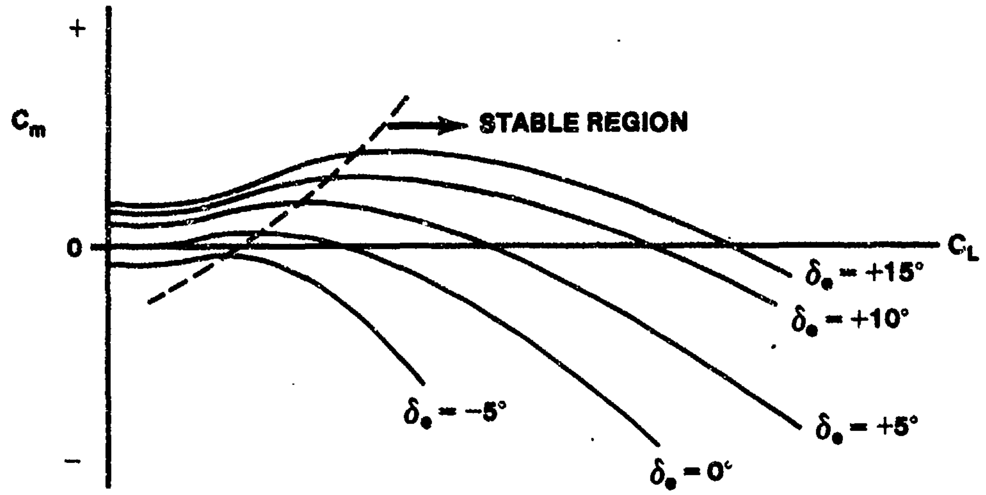


FIGURE 5.61. HYPERSONIC CONTROL POWER

From the knowledge of pitching moment characteristics at Hypersonic Mach numbers, a schedule of α to fly during an atmospheric entry emerges. From Mach 24 to Mach 12, the Shuttle flies at 40° α. As Mach decreases, α decreases to maintain stability. As shown in Figure 5.62, however, longitudinal stability is by no means the only limitation in determining the entry profile.

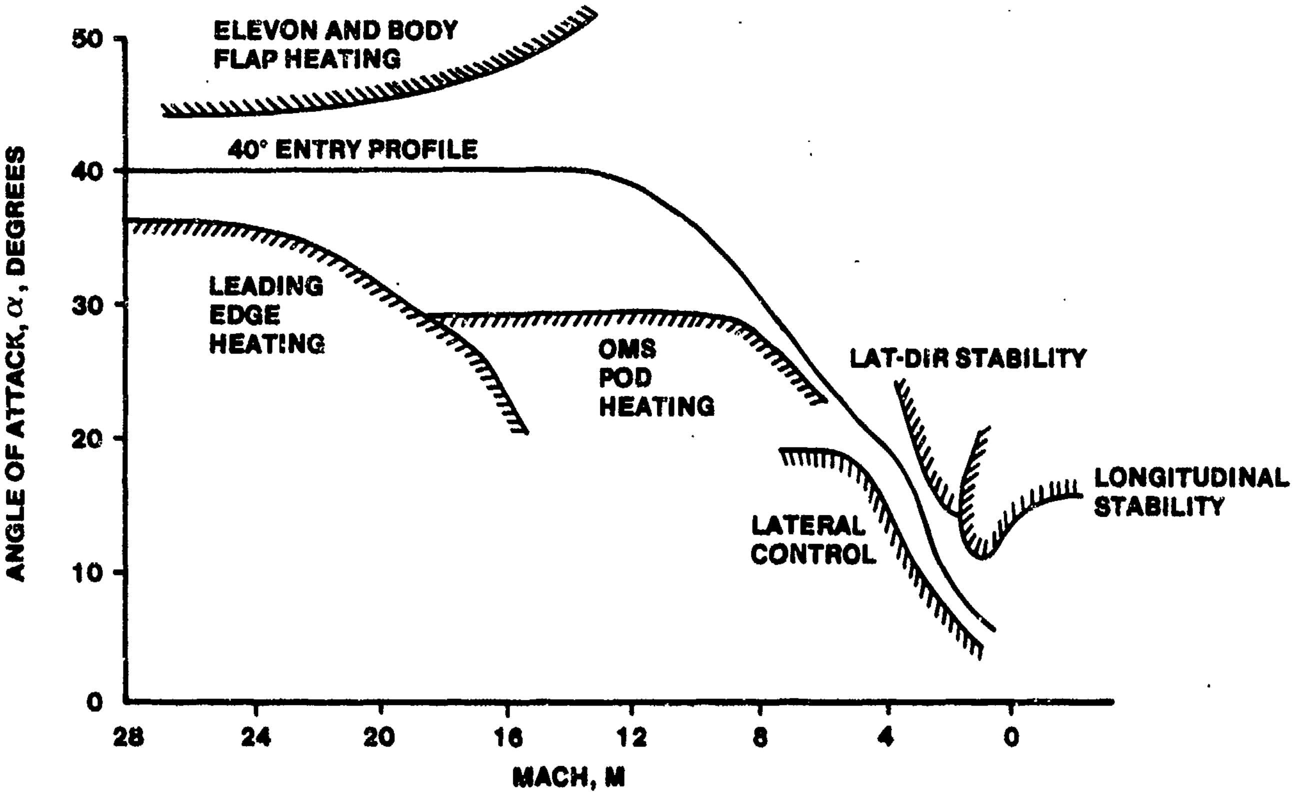


FIGURE 5.62. SPACE SHUTTLE ENTRY PROFILE AND LIMITATIONS

5.20 LONGITUDINAL STATIC STABILITY FLIGHT TESTS

The purpose of these flight tests is to determine the longitudinal static stability characteristics of an aircraft. These characteristics include gust stability, speed stability, and friction/breakout. Trim change tests will also be discussed.

An aircraft is said to be statically stable longitudinally (positive gust stability) if the moments created when the aircraft is disturbed from trimmed flight tend to return the aircraft to the condition from which it was disturbed. Longitudinal stability theory shows the flight test relationships for stick-fixed and stick-free gust stability, dC_m/dC_L, to be

stick-fixed:

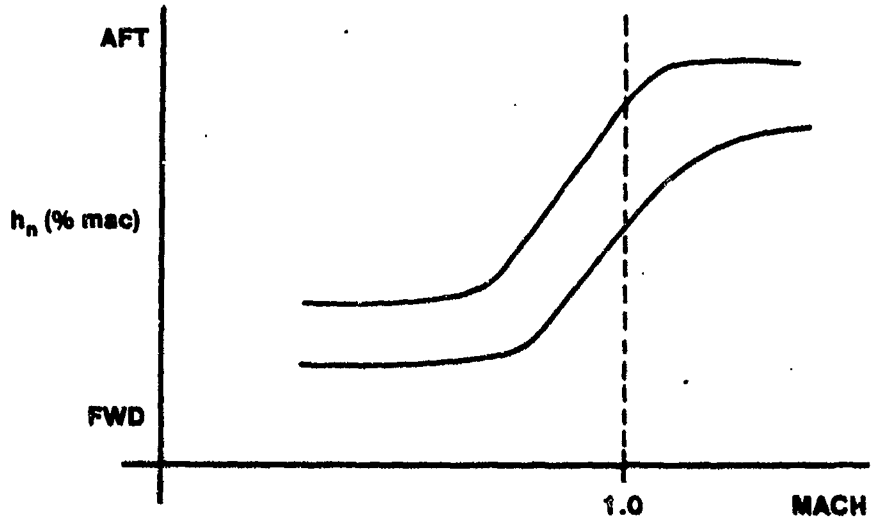
$$\frac{d\delta_e}{dC_L} = -\frac{1}{C_{m_{\delta_e}}} \frac{dC_m}{dC_L}_{\substack{\text{Stick}\\\text{Fixed}}} \tag{5.69}$$

stick-free:

$$\frac{d(F_s/q)}{dC_L} = -A \frac{C_{h_\delta}}{C_{m_{\delta_e}}} \frac{dC_m}{dC_L}_{\substack{\text{Stick}\\\text{Free}}} \tag{5.101}$$

Stick force (F_s), elevator deflection (δ_e), equivalent velocity (V_e) and gross weight (W) are the parameters measured to solve the above equations. When $d\delta_e/dC_L$ is zero, an aircraft has neutral stick-fixed longitudinal static stability. As $d\delta_e/dC_L$ increases, the stability of the aircraft increases. The same statements about stick-free longitudinal static stability can be made with respect to $d(F_s/q)/dC_L$. The neutral point is the cg location which gives neutral stability, stick-fixed or stick-free. These neutral points are determined by flight testing at two or more cg locations, and extrapolating the curves of $d\delta_e/dC_L$ and $d(F_s/q)/dC_L$ versus cg to zero.

The neutral point so determined is valid for the trim altitude and airspeed at which the data were taken and may vary considerably at other trim conditions. A typical variation of neutral point with Mach is shown in Figure 5.63.



5.63. STICK-FIXED NEUTRAL POINT VERSUS MACH

The use of the neutral point theory to define gust stability is therefore time consuming. This is especially true for aircraft that have a large airspeed envelope and aeroelastic effects.

Speed stability is the variation in control stick forces with airspeed changes. Positive stability requires that increased aft stick force be required with decreasing airspeed and vice versa. It is related to gust stability but may be considerably different due to artificial feel and stability augmentation systems. Speed stability is the longitudinal static stability characteristic most apparent to the pilot, and it therefore receives the greatest emphasis.

Flight-path stability is defined as the variation in flight-path angle when the airspeed is changed by use of the elevator alone. Flight-path stability applies only to the power approach flight phase and is basically determined by aircraft performance characteristics. Positive flight-path stability ensures that the aircraft will not develop large changes in rate of descent when corrections are made to the flight-path with the throttle fixed. The exact limits are prescribed in MIL-F-8785C, paragraph 3.2.1.3. An

aircraft likely to encounter difficulty in meeting these limits would be one whose power approach airspeed was far up on the "backside" of the power required curve. A corrective action might be to increase the power approach airspeed, thereby placing it on a flatter portion of the curve or by installing an automatic throttle.

5.20.1 Military Specification Requirements

The 1954 version of MIL-F-8785 established longitudinal stability requirements in terms of the neutral point. While the neutral point criteria is still valid for testing certain types of aircraft, this criteria was not optimum for aircraft operating in flight regimes where other factors were more important. MIL-F-8785C (5 Nov 80) does not mention neutral points. Instead, section 3.2.1 of MIL-F-8785C specifies longitudinal stability with respect to speed and flight-path. The requirements of this section are relaxed in the transonic speed range except for those aircraft which are designed for prolonged transonic operation. As technology progresses, highly augmented aircraft and aircraft with fly-by-wire control systems may be designed with neutral speed stability. The F-15, F-16, and F-20 are examples of aircraft with neutral speed stability. For these aircraft, the program manager may require a mil spec written specifically for the aircraft and control system involved.

5.20.2 Flight Test Methods

There are two general test methods (stabilized and acceleration/deceleration) used to determine either speed stability or neutral points.

5.20.2.1 Stabilized Method. This method is used for aircraft with a small airspeed range in the cruise flight phase and virtually all aircraft in the power approach, landing or takeoff flight phases. Propeller type aircraft are normally tested by this method because of the effects on the elevator control power caused by thrust changes. It involves data taken at stabilized airspeeds at the trim throttle setting with the airspeed maintained constant by a rate of descent or climb. As long as the altitude does not vary

excessively (typically +/- 1,000 ft) this method gives good results, but it is time consuming.

The aircraft is trimmed carefully at the desired altitude and airspeed, and a trim shot is recorded. Without moving the throttle or trim setting, the pilot changes aircraft pitch attitude to achieve a lower or higher airspeed (typically in increments of +/- 10 knots) and maintains that airspeed.

Aircraft with both reversible and irreversible hydro-mechanical control systems exhibit varying degrees of friction and breakout force about trim. The friction force is the force required to begin a tiny movement of the stick. This initial movement will not cause an aircraft motion as observed on the windscreen. The breakout force is that additional amount of pilot-applied force required to produce the first tiny movement of the elevator. These small forces, friction and breakout, combine to form what is generally termed the "friction band."

Since the pilot has usually moved the control stick fore and aft through the friction band, he must determine which side of the friction band he is on before recording the test point data. The elevator position for this airspeed will not vary, but stick force varies relative to the instantaneous position within the friction band at the time the data is taken. Therefore, the pilot should (assuming an initial reduction in airspeed from the trim condition) increase force carefully until the nose starts to rise. The stick should be frozen at this point (where <u>any</u> increase in stick force will result in elevator movement and thus nose rise) and data recorded. The same technique should be used for all other airspeed points below trim. For airspeed above trim airspeed, the same technique is used although now the stick is frozen at a point where any increase in push force will result in nose drop.

5.20.2.2 <u>Acceleration/Deceleration Method</u>. This method is commonly used for aircraft that have a large airspeed envelope. It is always used for transonic testing. It is less time consuming than the stabilized method but introduces thrust effects. The U.S. Navy uses the acceleration/deceleration method but maintains the throttle setting constant and varies altitude to change airspeed. The Navy method minimizes thrust effects but introduces altitude effects.

The same trim shot is taken as in the stabilized method to establish trim conditions. MIL-F-8785C requires that the aircraft exhibit positive speed stability only within +/-50 knots or +/-15 percent of the trim airspeed, whichever is the less. This requires very little power change to traverse this band and maintain level flight unless the trim airspeed is near the back side of the thrust required curve. Before the 1968 revision to MIL-F-8785, the flight test technique commonly used to get acceleration/deceleration data was full military power or idle, covering the entire airspeed envelope. Unfortunately, this technique cannot be used to determine the requirements under the current specification with the non-linearities that usually exist in the control system. Therefore a series of trim points must be selected to cover the envelope with a typical plot (friction and breakout excluded) shown in Figure 5.64.

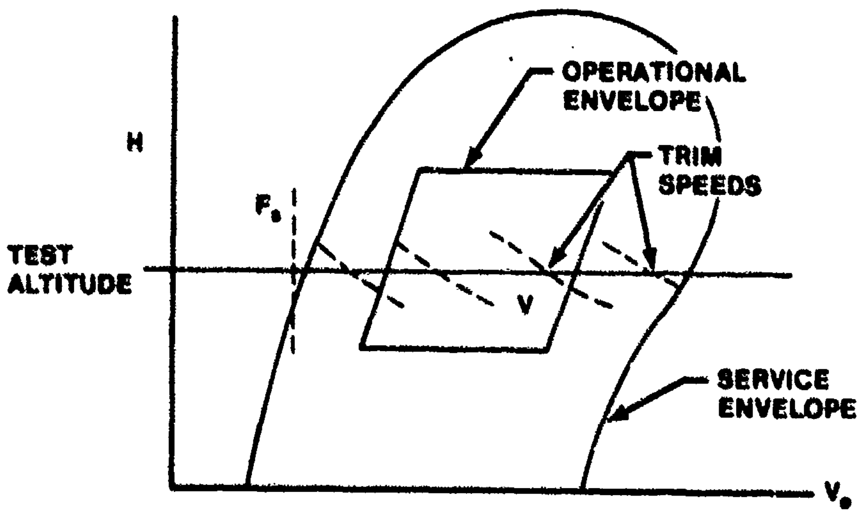


FIGURE 5.64. SPEED STABILITY DATA

The most practical method of taking data is to note the power setting required for trim and then either decrease or increase power to overshoot the data band limits slightly. Then turn on the instrumentation and reset trim power, and a slow acceleration or deceleration will occur back towards the trim point. The data will be valid only during the acceleration or deceleration with trim power set. A small percent change in the trim power setting may be required to obtain a reasonable acceleration or deceleration without introducing gross power effects. The points near the trim airspeed point will be difficult to obtain but they are not of great importance since they will probably be obscured by the control system breakout and friction (Figure 5.65).

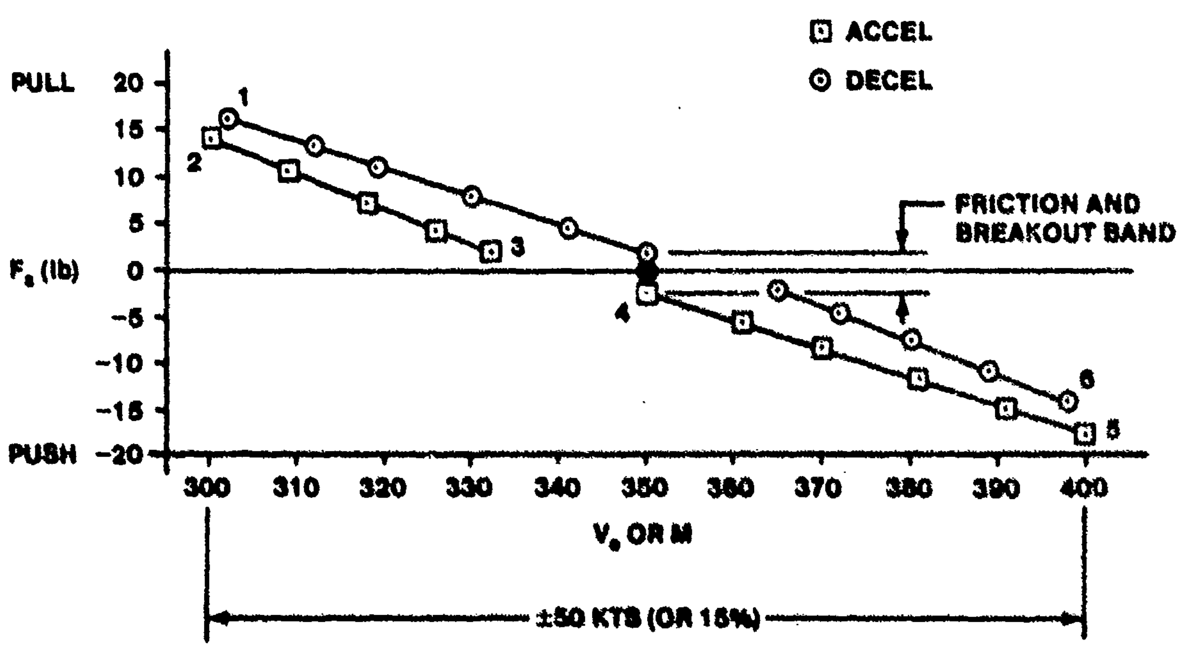


FIGURE 5.65. ACCELERATION/DECELERATION DATA
ONE TRIM SPEED, cg, ALTITUDE

Throughout the acceleration or deceleration, the primary parameter to control is stick force. It is important that the friction band not be reversed during the test run. A slight change in altitude is preferable (i.e. to let the aircraft climb slightly throughout an acceleration) to avoid the tendency to reverse the stick force by over-rotating the nose. The opposite is advisable during the deceleration.

There is a relaxation in the requirement for speed stability in the transonic area unless the aircraft is designed for continued transonic operation. The best way to define where the transonic range occurs is to determine the point where the F_s versus V goes unstable. In this area, MIL-F-8785C allows a specified maximum instability. The purpose of the transonic longitudinal static stability flight test in the transonic area is to determine the degree of instability.

The transonic area flight test begins with a trim shot at some high subsonic airspeed. The power is increased to maximum thrust and an acceleration is begun.

It is important that a stable gradient be established before entering the transonic area. Once the first sensation of instability is felt by the pilot, his primary control parameter changes from stick force to attitude. From this point until the aircraft is supersonic, the true altitude should be held as closely as possible. This is because the unstable stick force being measured will be in error if a climb or descent occurs. A radar altimeter output on an over-water flight or keeping a flight path on the horizon are precise ways to hold constant altitude, but if these are not available, the pilot will have to use the outside references to maintain level flight.

Once the aircraft goes supersonic, the test pilot should again concern himself with not reversing the friction band and with establishing a stable gradient. The acceleration should be continued to the limit of the service envelope to test for supersonic speed stability. The supersonic data will also have to be shown at +/-15% of the trim airspeed, so several trim shots may be required. A deceleration from V_{max} to subsonic speed should be made with a careful reduction in power to decelerate supersonically and transonically. The criteria for decelerating through the transonic region are the same as for the acceleration. Power reductions during this deceleration

will have to be done carefully to minimize thrust effects and still decelerate past the Mach drag rise point to a stable subsonic gradient.

5.20.3 Flight-Path Stability

Flight-path stability is a criterion applied to power approach and landing qualities. It is primarily determined by the performance characteristics of the aircraft and related to stability and control only because it places another requirement on handling qualities. The following is one way to look at flight-path stability. Thrust required curves are shown for two aircraft with the recommended final approach speed marked in Figure 5.66.

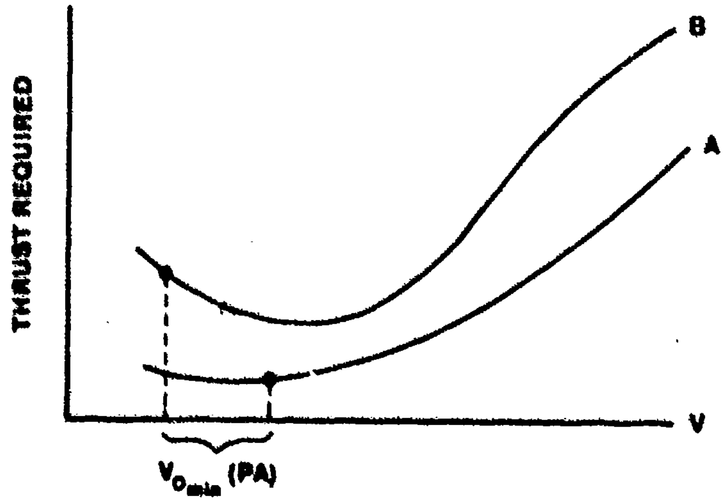


FIGURE 5.66. THRUST REQUIRED VS VELOCITY (TWO AIRCRAFT).

If both aircraft A and B are located on the glidepath shown in Figure 5.67, their relative flight-path stability can be shown.

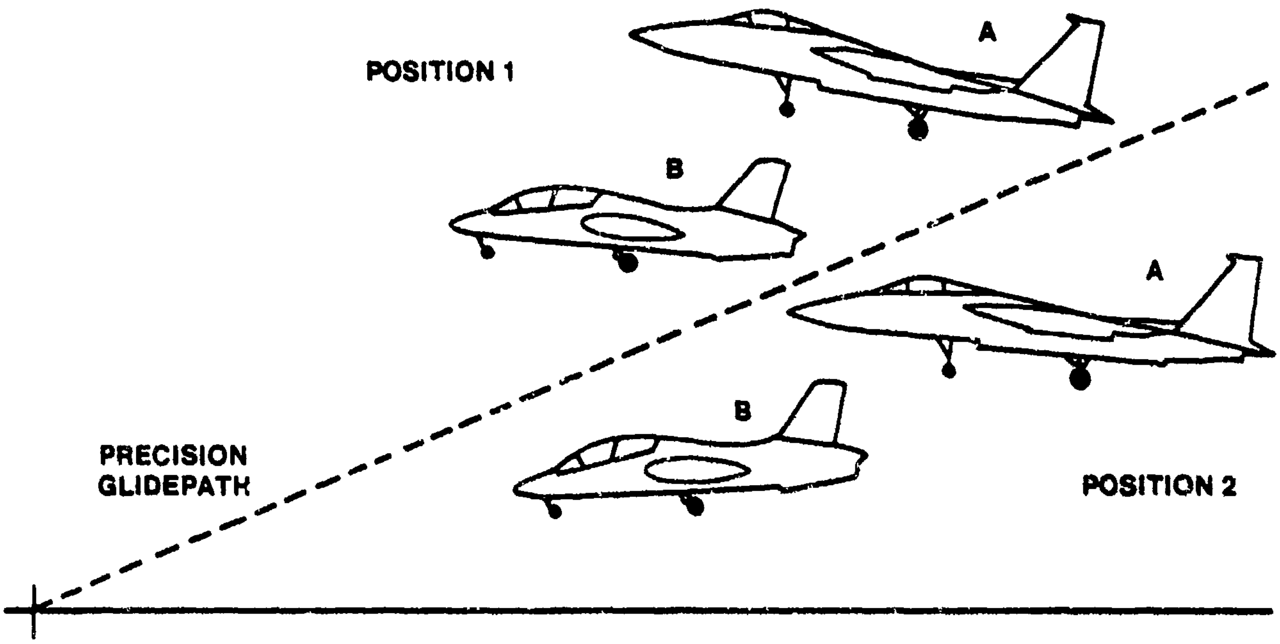


FIGURE 5.67. AIRCRAFT ON PRECISION APPROACH

At Position 1 the aircraft are in stable flight above the glidepath, but below the recommended final approach speed. If Aircraft A is in this position, the pilot can nose the aircraft over and descend to glidepath while the airspeed increases. Because the thrust required curve is flat at this point, the rate of descent at this higher airspeed is about the same as before the correction, so he does not need to change throttle setting to maintain the glidepath. Aircraft B, under the same conditions, will have to be flown differently. If the pilot noses the aircraft over, the airspeed will increase to the recommended airspeed as the glidepath is reached. The rate of descent

at this power setting is less than it was before so the pilot will go above glidepath if he maintains this airspeed.

At Position 2 the aircraft are in stable flight below the glidepath but above the recommended airspeed. Aircraft A can be pulled up to the glidepath and maintained on the glidepath with little or no throttle change. Aircraft B will develop a greater rate of descent once the airspeed decreases while coming up to glidepath and will fall below the glidepath again.

If the aircraft are in Position 1 with the airspeed higher than recommended instead of lower, the same situation will develop when correcting back to flight-path, but the required pilot compensation is increased. In all cases Aircraft A has better flight-path stability than Aircraft B. As mentioned earlier in this chapter, aircraft which have unsatisfactory flight-path stability can be improved by increasing the recommended final approach airspeed or by adding an automatic throttle.

Another way of looking at flight-path stability is by investigating the difficulty that a pilot has in maintaining glidepath even when using the throttles. This problem is seen in large aircraft for which the time lag in pitching the aircraft to a new pitch attitude is quite long. In these instances, incorporation of direct lift allows the pilot to correct the glidepath without pitching the aircraft. Direct lift control will also affect the influence of performance on flight-path stability.

5.20.4 Trim Change Tests

The purpose of this test is to determine the control force changes associated with normal configuration changes, trim system failure, or transfer to alternate control systems in relation to specified limits. It must also be determined that no undesirable flight characteristics accompany these configuration changes.Pitching moments on aircraft are normally associated with changes in the condition of any of the following: landing gear, flaps, speed brakes, power, bomb bay doors, rocket and missile doors, or any jettisonable device. The magnitude of the change in control forces resulting from these pitching moments is limited by Military Specification F-8785C, and it is the responsibility of the testing organization to determine if the actual forces are within the specified limits.

The pitching moment resulting from a given configuration change will normally vary with airspeed, altitude, cg loading, and initial configuration of the aircraft. The control forces resulting from the pitching moment will further depend on the aircraft parameters being held constant during the configuration change. These factors should be kept in mind when determining the conditions under which the given configuration change should be tested. Even though the specification lists the altitude, airspeed, initial conditions, and parameter to be held constant for most configuration changes, some variations may be necessary on a specific aircraft to provide information on the most adverse conditions encountered in operational use of the aircraft. The altitude and airspeed should be selected as indicated in the specifications or for the most adverse conditions. In general, the trim change will be greatest at the highest airspeed and the lowest altitude. The effect of cg location is not so readily apparent and usually has a different effect for each configuration change. A forward loading may cause the greatest trim change for one configuration change, and an aft loading may be most severe for another. Using the build up approach, a mid cg loading is normally selected since rapid movement of the cg in flight will probably not be possible. If a specific trim change appears marginal at this loading, it may be necessary to test it at other cg loadings to determine its acceptability.

Selection of the initial aircraft configurations will depend on the anticipated normal operational use of the aircraft. The conditions given in the specifications will normally be sufficient and can always be used as a guide, but again variations may be necessary for specific aircraft. The same holds true for selection of the aircraft paramater to hold constant during the change. The parameter that the pilot would normally want to hold constant in operational use of the aircraft is the one that should be selected. Therefore, if the requirements of MIL-F-8785C do not appear logical or complete, then a more appropriate test should be added or substituted.

In addition to the conditions outlined above, it may be necessary to test for some configuration changes that could logically be accomplished

simultaneously. The force changes might be additive and could be objectionably large. For example, on a go-around, power may be applied and the landing gear retracted at the same time. If the trim changes associated with each configuration change are appreciable and in the same direction, the combined changes should definitely be investigated. The specifications require that no objectionable buffet or undesirable flight chracteristics be associated with normal trim changes. Some buffet is normal with some configuration changes, e.g., gear extension, however, it would be considered if this buffet tended to mask the buffet associated with stall warning. The input of the pilot is the best measure of what actually constitutes "objectionable," but anything that would interfere with normal use of the aircraft would be considered objectionable. The same is true for "undesirable flight characteristics." An example would be a strong nose-down pitching moment associated with gear or flap retraction after take-off.

The specification also sets limits on the trim changes resulting from transfer to an alternate control system. The limits vary with the type of alternate system and the configuration and speed at the time of transfer but in no case may a dangerous flight condition result. A good example of this is the transfer to manual reversion in the A-10. It will probably be necessary for the pilot to study the operation of the control system and methods of effecting transfer in order to determine the conditions most likely to cause an unacceptable trim change upon transferring from one system to the other. As in all flight testing, a thorough knowledge of the aircraft and the objectives of the test will improve the quality and increase the value of the test results.

PROBLEMS

5.1 In Subsonic Aerodynamics the following approximation was developed for the Balance Equation

$$C_{m_{cg}} = C_{m_{ac}} + C_L\,(cg - ac) + C_{m_{tail}}$$

where $C_{m_{tail}}$ is the total stability contribution of the tail.

(a) Sketch the location of the forces, moments, and cg required to balance an airplane using the above equation.

(b) Using the data shown below, what contribution is required from the tail to balance the airplane?

$C_{m_{ac}}$ = −0.12 $\qquad$ $C_{m_{tail}}$ = ?

ac = 0.25 (25%) $\qquad$ C_L = 0.5

cg = 0.188 (18.8%)

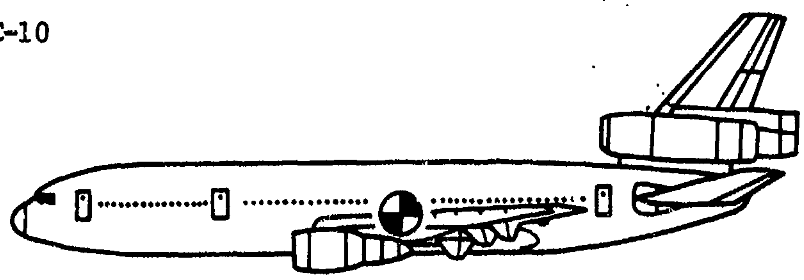
(c) If this airplane were a fixed hang-glider and a C_L = 1.3 were required to flare and land, how far aft must the cg be shifted to obtain the landing C_L without changing the tail contribution?

(d) If a 4% margin were desired between max aft cg and the aerodynamic center for safety considerations, how much will the tail contribution have to increase for the landing problem presented in (c)?

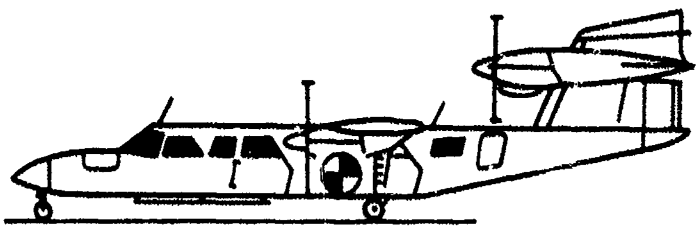
(e) List four ways of increasing the tail contribution to stability.

5.2 Given the aircraft configurations shown below, write the Balance and Stability Equations for the thrust contributions to stability. Sketch the forces involved and state which effects are stabilizing and which are destabilizing.

(a) DC-10



(b) Britten-Norman Trislander



5.3 (a) Are the two expressions below derived for the tail-to-the-rear aircraft valid for the canard aircraft configuration?

$$\frac{dC_{m_{cg}}}{dC_L} = h - h_n$$

$$\text{Static margin} \equiv h_n - h$$

(b) Derive an expression for elevator power for the canard aircraft configuration. Determine its sign.

(c) What is the expression for elevator effectiveness for the canard aircraft configuration? Determine its sign.

5.4 During wind tunnel testing, the following data were recorded:

C_L	δ_e (deg) $h = 0.20$	δ_e (deg) $h = 0.25$	δ_e (deg) $h = 0.30$
0.2	-2	-3	-4
0.6	4	[illegible]	-2
1.0	10	5	0

Elevator limits are $\pm 20^{o}$

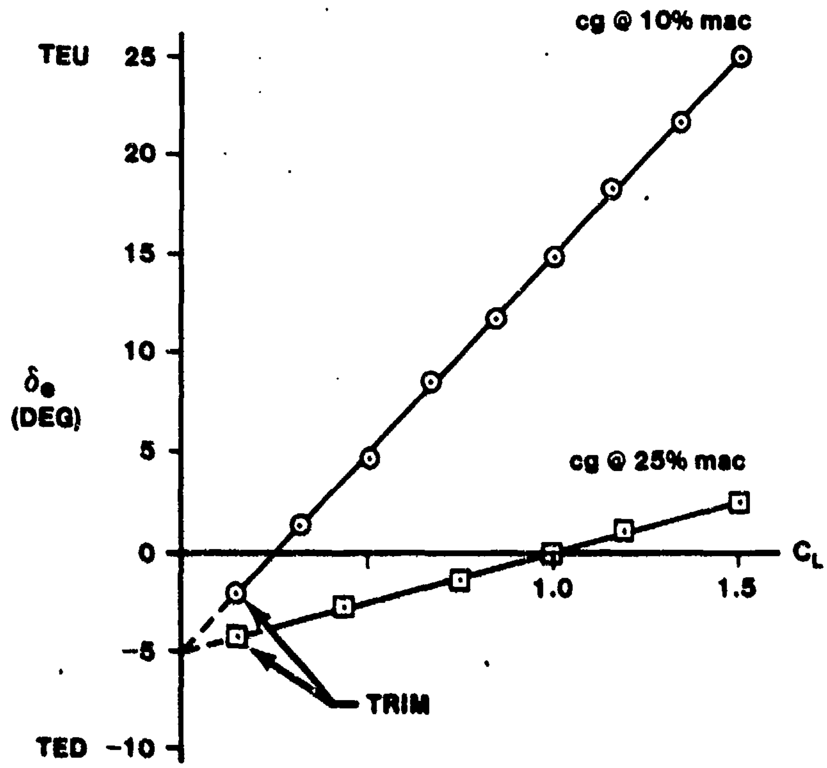
A. Find the stick-fixed static margin for $h = 0.20$

B. Find the numerical value for elevator power.

C. Find the most forward cg permissible if it is desired to be able to stabilize out of ground effect at a $C_{L_{max}} = 1.0$.

5.5 Given the flight test data below from the aircraft which was wind tunnel tested in Problem 5.4, answer the following questions:

A. Find the aircraft neutral point. Was the wind tunnel data conservative?



B. What is the flight test determined value of $C_{m_{\delta_e}}$

C. If the lift curve slope is determined to be 1.0, what is the flight test determined value of C_{m_α} (per deg) at a cg of 25%?

5.6 Given geometric data for the canard design in Problem 5.5, calculate an estimate for elevator power.

HINT:

$$C_{m_{\delta_e}} = a_T \; V_H \; \eta_T \; \frac{d\alpha_T}{d\delta_e}$$

Assume:

$a_T = a =$ wind tunnel value

$\eta_T = 1.0$

Given:

Slab canard

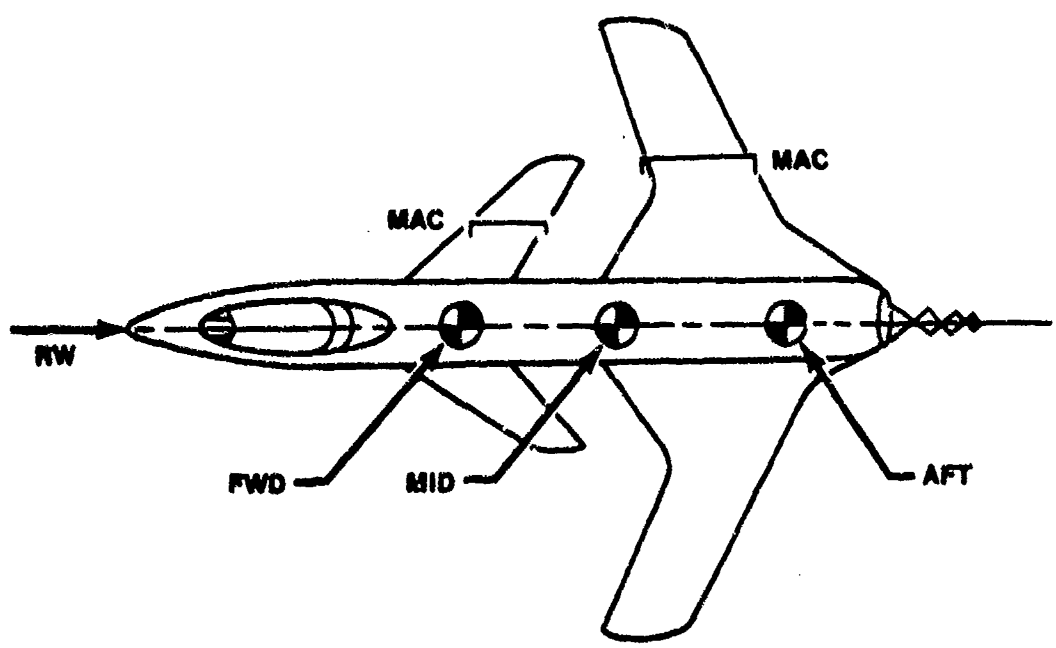
$\ell_T = 14\text{ft}$

$S_T = 10\text{ft}^2$

$c = 7\text{ft}$

$S = 200\text{ft}^2$

5.7 The Forward Swept Wing (FWS) technology aircraft designed by North American is shown below. The aerodynamic load is shared by the two "wings" with the forward wing designed to carry about 30% of the aircraft weight. For this design condition, answer the following multiple choice questions by circling the number of correct answer(s).



A. At the cg location marked FWD the aircraft:

(1) Can be balanced and is stable.

(2) Can be balanced and is stable only if the cg is ahead of the neutral point.

(3) Can be balanced and is unstable.

(4) Cannot be balanced.

B. At the cg location marked MID the aircraft:

(1) Can be balanced and is stable.

(2) Can be balanced and is stable only if the cg is ahead of the neutral point.

(3) Can be balanced and is unstable.

(4) Cannot be balanced.

C At the cg location marked AFT the aircraft:

(1) Can be balanced and is stable.

(2) Can be balanced and is stable only if the cg is ahead of the neutral point.

(3) Can be balanced and is unstable.

(4) Cannot be balanced.

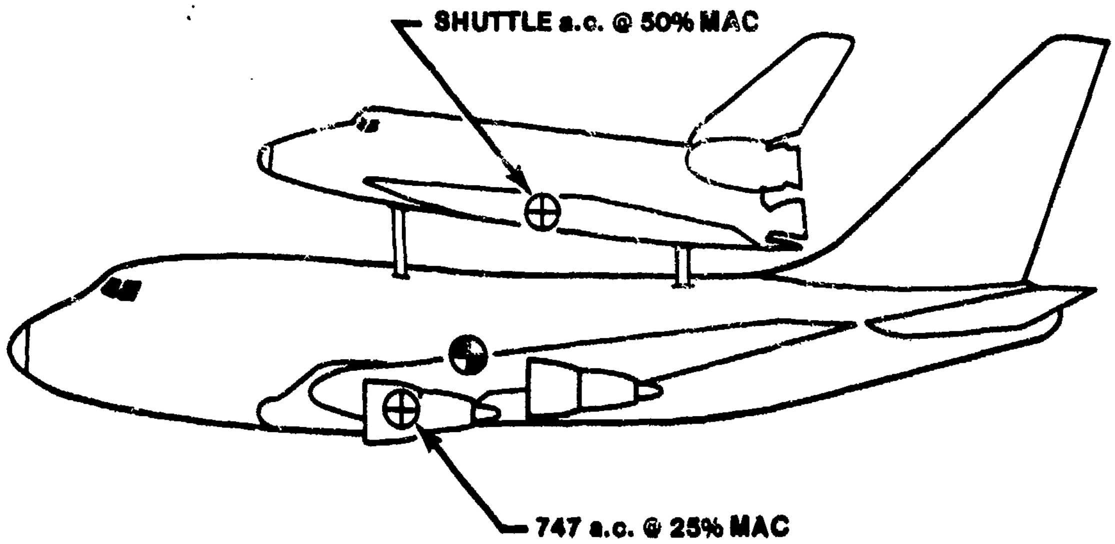
D. For this design the sign of control power is:

(1) Negative.

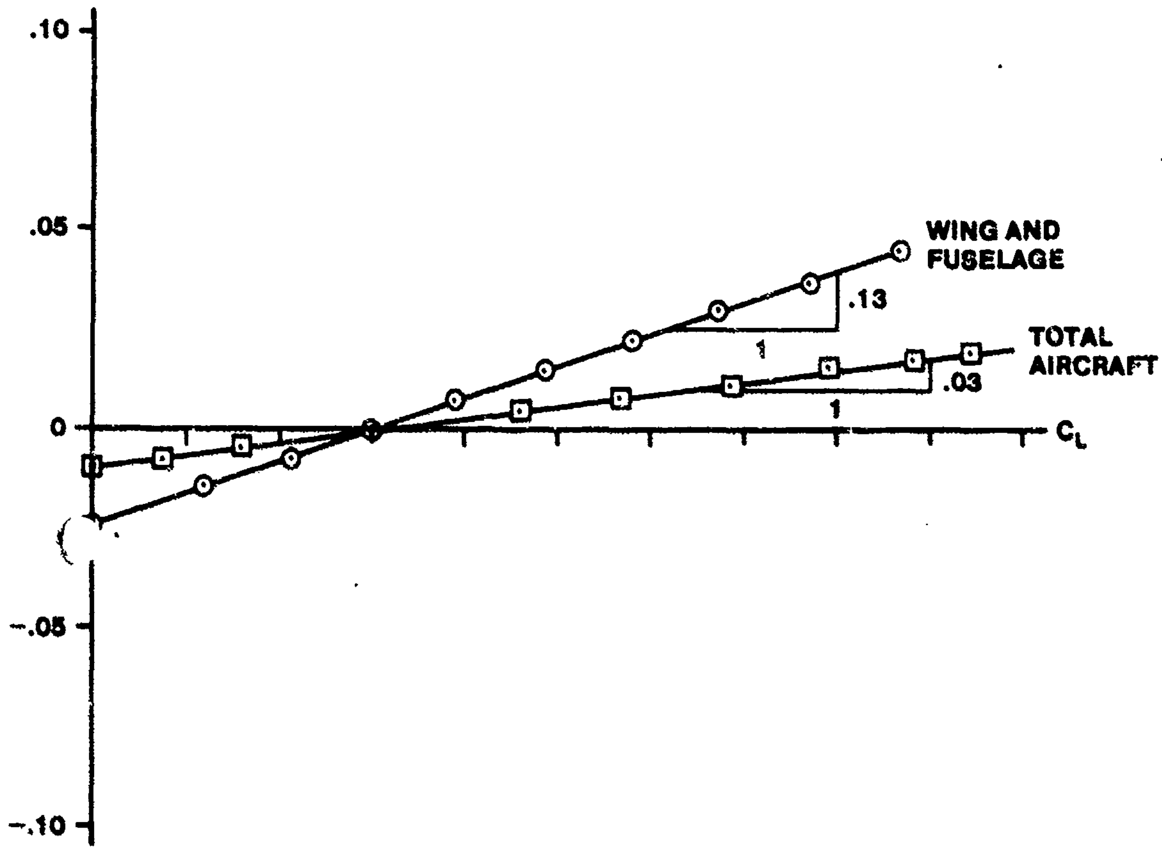
(2) Positive.

(3) Dependent on cg location.

5.8 Given the Boeing 747/Space Shuttle Columbia combination as shown below, is the total shuttle orbiter wing contribution to the combination stabilizing or destabilizing if the cg is located as shown? Briefly explain the reason for the answer given.



Given below is wind tunnel data for the YF-16.



Answer the following questions YES or NO:

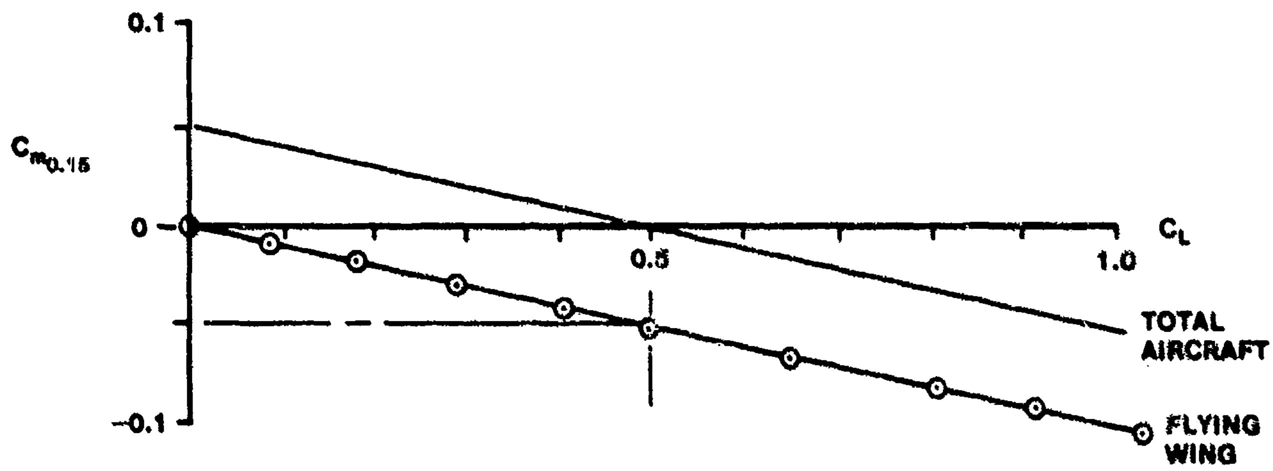
Is the total aircraft stable?

Is the wing-fuselage combination stable?

Is the tail contribution stabilizing?

B. How much larger (in percent) would the horizontal stabilizer have to be to give the F-16 a static margin of 2% at a cg of 35% MAC? Assume all other variables remain constant.

5.10 Given below is a $C_{m_{cg}}$ versus C_L curve for a rectangular flying wing from wind tunnel tests and a desired TOTAL AIRCRAFT trim curve.



A. Does the flying wing need a TAIL or CANARD added or can it attain the required TOTAL aircraft stability level by ELEVON deflection?

B. Is the TOTAL aircraft stable?

C. Is C_{m_α} positive or negative?

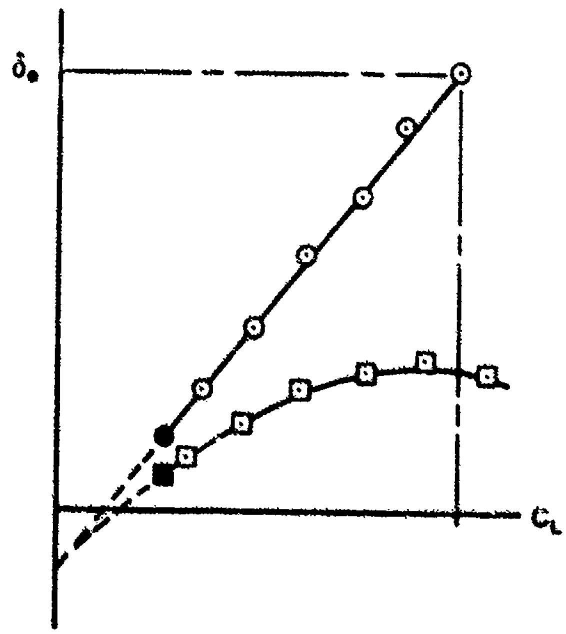
D. Does the flying wing have a symmetric wing section?

E. What is the flying wing's neutral point?

F. Is $C_{m_{\delta_e}}$ positive or negative?

G. What is the static margin for this trim condition?

5.11.2 Given the flight test data shown below, show how to obtain the neutral point(s). Label the two cg's tested as FWD and AFT.



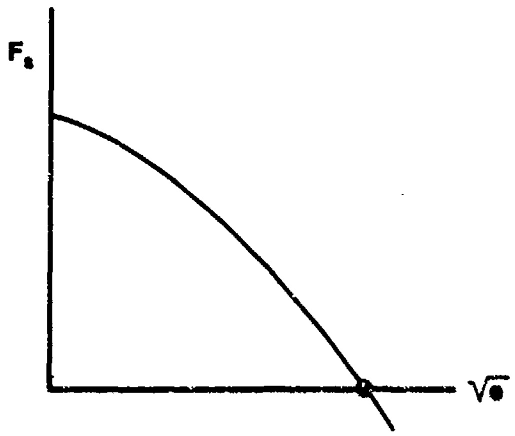
5.12 Given the curve shown below, show the effect of:

A. Shifting the cg FWD and retrimming to trim velocity.

B. Increasing C_{h_δ} and retrimming to trim velocity.

C. Adding a downspring and retrimming to trim velocity.

D. Adding a bobweight and retrimming to trim velocity.



5.13 Which changes in Problem 5.12 affect:

apparent stability actual stability

5.14 Read the question and answer true (T) or false (F).

T F If a body disturbed from equilibrium remains in the disturbed position it is statically unstable.

T F Longitudinal static stability and "gust stability" are the same thing.

T F Static longitudinal stability is a prerequisite for dynamic longitudinal stability.

T F Aircraft response in the X-Z plane (about the Y axis) usually cannot be considered as independent of the lateral directional motions.

T F Although C_{m_α} is a direct indication of longitudinal static stability, $C_{m_{C_L}}$ is relatively unimportant.

T F At the USAF Test Pilot School elevator TEU and nose pitching up are positive in sign by convention. CAREFUL.

T F Tail efficiency factor and tail volume coefficient are not normally considered constant.

T F With the cg forward, an aircraft is more stable and maneuverable.

T F There are no well defined static stability criteria.

T F To call a canard surface a horizontal stabilizer is a misnomer.

T F The most stable wing contribution to stability results from a low wing forward of the cg.

T F With the cg aft, an aircraft is less maneuverable and more stable.

T F A thrust line below the cg is destabilizing for either a prop or turbojet.

T F The normal force contribution of either a prop or turbojet is destabilizing if the prop or inlet is aft of the aircraft cg.

T F A positive value of downwash derivatives causes a tail-to the-rear aircraft to be less stable if $d\epsilon/d\alpha$ is less than 1.0 than it would be if d_ϵ/d_α were equal to zero.

T F Verifying adequate stability and maneuverability at established cg limits is a legitimate flight test function.

T F An aircraft is balanced if it is forced to a negative value of $C_{m_{CG}}$ for some useful positive value of C_L.

T F An aircraft is considered stable if $dC_{m_{cg}} / dC_L$ is positive.

T F The slope of the $C_{m_{cg}}$ versus C_L curve of an aircraft is a direct measure of "gust stability."

T F Aircraft center of gravity position is only of secondary importance when discussing longitudinal static stability.

T F Due to the advance control system technology such as fly-by-wire, a basic knowledge of the requirements for natural aircraft stability is of little use to a sophisticated USAF test pilot.

T F Most contractors would encourage an answer of true (T), to the above question.

T F For an aircraft with a large vertical cg travel, the chordwise force contribution of the wing to stability probably cannot be neglected.

T F FWD and AFT cg limits are often determined from flight test.

T F A canard is a hoax.

T F The value of stick-fixed static stability is equal to cg minus h_n in percent MAC.

T F The stick-fixed static margin is stick-fixed stability with the sign reversed.

T F A slab tail (or stabilizer) is a more powerful longitudinal control than a two-piece elevator.

T F Elevator effectiveness and power are the same thing.

T F Elevator effectiveness is negative in sign for a canard configuration.

T F Static margin is negative for a statically stable canard configuration.

T F Elevator power is positive in sign for either a tail-to-the-rear or a canard configuration.

T F Upwash causes a canard to be more destabilizing.

T F The main effect on longitudinal stability when accelerating to supersonic flight is caused by the shift in wing ac from 25% MAC to about 50% MAC.

T F The elevator of a reversible control system is normally statically balanced.

T F C_{h_δ} is always negative and is known as the "restoring" moment coefficient.

T F C_{h_α} is always negative and is known as the "floating" moment coefficient.

T F With no pilot applied force a reversible elevator will "float" until hinge moments are zero.

T F A free elevator factor of one results in the stick-fixed and stick-free stability being the same.

T F Generally, freeing the elevator is destabilizing (tail-to-the-rear).

T F Speed stability and stick force gradient about trim are the same.

T F Speed stability and apparent stability are the same.

T F Speed stability and stick-free stability are the same.

T F Dynamic control balancing is making C_{h_δ} small or just slightly positive.

T F cg movement affects real and apparent stability (after retrimming).

T F Aerodynamic balancing affects real and apparent stability (after retrimming).

T F Downsprings and bobweights affect real and apparent stability (after retrimming).

T F In general, an aircraft becomes more stable supersonically which is characterized by an AFT shift in the neutral point.

T F Even though an aircraft is more stable supersonically (neutral point further AFT) it may have a speed instability.

T F The neutral point of the entire aircraft is analogous to the aerodynamic center of the wing by itself.

T F Aerodynamic balancing is "adjusting" or "tailoring" C_{h_δ} and C_{h_α}.

T F Neutral point is a constant for a given configuration and is never a function of C_L.

T F The cg location where $dF_s/dV = 0$ is the actual stick-free neutral point regardless of control system "gadgets."

T F The effects of elevator weight on hinge moment coefficient are normally eliminated by static balance.

T F A positive δ_e is a deflection causing a nose-up pitching moment.

T F A positive C_h is one defined as deflecting or trying to deflect the elevator in a negative direction.

T F $C_{h_{\alpha_t}}$ is normally negative (tail-to-the-rear).

T F $C_{h_{\delta_e}}$ is always negative (tail-to-the-rear and canard).

T F $C_{h_{\alpha_t}}$ and $C_{h_{\delta_e}}$ are under control of the aircraft designer and can be varied to "tailor" stick-free stability characteristics.

T F $C_{h_{\alpha_t}}$ and C_{h_α} are identical.

T F C_{h_δ} and $C_{h_{\delta_e}}$ are not the same.

T F dF_s/dV does not necessarily reflect actual stick-free stability characteristics.

T F There are many ways to alter an aircraft's speed stability characteristics.

T F A bobweight can only be used to increase stick-force gradient.

T F $C_{m_{\delta_e}}$ cannot be determined from flight test.

<u>ANSWERS</u>

5.1. b. $C_{m_{tail}} = +0.151$

c. cg = 0.226 or 22.6%

d. $C_{m_{tail}} = +.172$, 14% inc.

5.4. A. SM = 0.15

B. $C_{m_{\delta_e}} = .01$/deg or .57/rad

C. h = 10%

5.5. A. $h_n = 35\%$

B. $C_{n_{\delta_e}} = 0.01$/deg

C. $C_{m_\alpha} = -.05$/deg

5.6. $C_{m_{\delta_e}} = 0.1$/deg

5.9. B. 80% larger

5.10. E. $h_n = 0.25$

G. SM = 0.10

Bibliography

1. Etkin, B. Dynamics of Atmospheric Flight. New York: John Wiley & Sons, Inc., 1972.

2. Phillips, W.H. Appreciation and Prediction of Flying Qualities. NACA Report 927, National Advisory Committee for Aeronautics, Washington, DC, 1949.

3. Durbin, E.J. and Perkins, C.D., ed. AGARD Flight Test Manual, 2nd ed. Vol II, New York: Pergamon Press, 1962.

Volume II

CHAPTER 6

MANEUVERING FLIGHT

6.1 INTRODUCTION

The method used to analyze maneuvering flight will be to determine a stick-fixed maneuver point (h_m) and stick-free maneuver point (h'_m). These are analogous to their counterparts in static stability, the stick-fixed and stick-free neutral points. The maneuver points will also be derived in terms of the neutral points, and their relationship to cg location will be shown.

6.2 DEFINITIONS

(Also see definitions for Chapter 5, Longitudinal Static Stability.)

Acceleration Sensitivity - The ratio n/α is used to determine allowable maneuvering stick force gradients. It is defined in MIL-F-8785C as the "steady-state normal acceleration change per unit change in angle of attack for an incremental pitch control deflection at constant speed."

Centripetal Acceleration - The acceleration vector normal to the velocity vector that causes changes in direction (not magnitude) of the velocity vector.

Free Elevator Factor - F $= 1 - \tau \, C_{h_\alpha} / C_{h_\delta}$

A multiplier that accounts for the change in stability caused by freeing the elevator (allowing it to "float").

Pitch Damping - A stability derivative. $C_{m_q} = \frac{2U_0}{C} \frac{\partial C_m}{\partial Q}$
Damping that is generated by a pitch rate.

Stick-Fixed Maneuver Margin - The distance in percent MAC between the cg and the stick-fixed maneuver point $= h - h_m$.

Stick-Fixed Maneuver Point - h_m The cg location where $d\delta_e / dn = 0$.

Stick-Free Maneuver Margin - The distance, in percent MAC, between the cg and the stick-free maneuver point $= h - h'_m$.

Stick-Free Maneuver Point - h'_m = The cg location where $dF_s/dn = 0$.

6.3 ANALYSIS OF MANEUVERING FLIGHT

Maneuvering flight will be analyzed much in the same manner used in determining a flight test relationship in longitudinal stability. For stick-fixed longitudinal stability, the flight test relationship was determined to be

$$\frac{d\delta_e}{dC_L} = - \frac{dC_m/dC_L}{C_{m_{\delta_e}}} \tag{5.69}$$

This equation gave the static longitudinal stability of the aircraft in terms that could easily be measured in a flight test.

In maneuvering flight, a similar stick-fixed equation relating to easily measurable flight test quantities is desirable. Where in longitudinal stability, the elevator deflection was related to lift coefficient or angle of attack, in maneuvering flight, elevator deflection will relate to load factor, n.

To determine this expression, we will start with the aircraft's basic equations of motion. As in longitudinal static stability, the six equations of motion are the basis for all analysis of aircraft stability and control. In maneuvering an aircraft, the same equations will hold true. Recalling the pitching moment

$$G_y = \dot{Q}I_y - PR\,(I_z - I_x) + (P^2 - R^2)\,I_{xz} \tag{4.3}$$

and the fact that in static stability analysis we have no roll rate, yaw rate, or pitch acceleration, Equation 4.3 reduces to

$$G_y = 0$$

There are five primary variables that cause external pitching moments on an aircraft:

$$\mathcal{M} = f\,(U,\ \alpha,\ \dot{\alpha},\ Q,\ \delta_e) \tag{6.1}$$

If any or all of these variables change, there will be a change of total pitching moment that will equal the sum of the partial changes of all the variables. This is written as

$$\Delta \mathcal{M} = \frac{\partial \mathcal{M}}{\partial U} \Delta U + \frac{\partial \mathcal{M}}{\partial \alpha} \Delta\alpha + \frac{\partial \mathcal{M}}{\partial \alpha} \Delta\alpha + \frac{\partial \mathcal{M}}{\partial Q} \Delta Q + \frac{\partial \mathcal{M}}{\partial \delta_e} \Delta\delta_e \qquad (6.2)$$

Since in maneuvering flight, ΔU and $\Delta\alpha$ are zero, Equation 6.2 becomes

$$\Delta \mathcal{M} = \frac{\partial \mathcal{M}}{\partial \alpha} \Delta\alpha + \frac{\partial \mathcal{M}}{\partial Q} \Delta Q + \frac{\partial \mathcal{M}}{\partial \delta_e} \Delta\delta_e = 0 \qquad (6.3)$$

and since $= qSc\ C_m$, then

$$\frac{\partial \mathcal{M}}{\partial \alpha} = qSc \frac{\partial C_m}{\partial \alpha} = qSc\ C_{m_\alpha} \qquad (6.4)$$

$$\frac{\partial \mathcal{M}}{\partial Q} = qSc \frac{\partial C_m}{\partial Q} \qquad (6.5)$$

$$\frac{\partial \mathcal{M}}{\partial \delta_e} = qSc \frac{\partial C_m}{\partial \delta_e} = qSc\ C_{m_{\delta_e}} \qquad (6.6)$$

Substituting these values into Equation 6.3 and multiplying by $1/qSc$,

$$C_{m_\alpha} \Delta\alpha + \frac{\partial C_m}{\partial Q} \Delta Q + C_{m_{\delta_e}} \Delta\delta_e = 0 \qquad (6.7)$$

The derivative $\partial C_m/\partial Q$ is carried instead of C_{m_q} since the compensating factor c $2U_0$ is not used at this time.

Solving for the change in elevator deflection $\Delta\delta_e$,

$$\Delta\delta_e = \frac{-C_{m_\alpha} \Delta\alpha - (\partial C_m/\partial Q)\ \Delta Q}{C_{m_{\delta_e}}} \qquad (6.8)$$

The analysis of Equation 6.8 may be continued by substituting in values for $\Delta\alpha$ and ΔQ. The final equation obtained should be in the form of some

flight test relationship. Since maneuvering is related to load factor, the elevator deflection required to obtain different load factors will define the stick-fixed maneuver point. The immediate goal then is to determine the change in angle of attack, $\Delta\alpha$, and change in pitch rate, ΔQ, in terms of load factor, n.

6.4 THE PULL-UP MANEUVER

In the pull-up maneuver, the change in angle of attack of the aircraft, $\Delta\alpha$, may be related to the lift coefficient of the aircraft. In the pull-up with constant velocity, the angle of attack of the whole aircraft will be changed since the aircraft has to fly at a higher C_L to obtain the load factor required. The change in C_L required to maneuver at high load factors at a constant velocity comes from two sources: (1) load factor increase and (2) elevator deflection. Although often ignored because of its small value when compared to total C_L, the change in lift with elevator deflection $C_{L_{\delta_e}} \Delta\delta_e$ will be included for a more general analysis.

Referring to Figure 6.1, the aircraft is in equilibrium at some C_{L_0} corresponding to some α_0 before the elevator is deflected to initiate the pull-up. If the elevator is considered as a flap, its deflection will affect the lift curve as follows. When the elevator is deflected upward, the lift curve shifts downward and does not change slope. This says that a certain amount of lift is initially lost when the elevator is deflected upward. The loss in lift because of elevator deflection is designated $C_{L_{\delta_e}} \Delta\delta_e$. The increase in down-loading continues to pitch upward and increase its angle of attack until it reaches a new C_L and an equilibrium load factor.

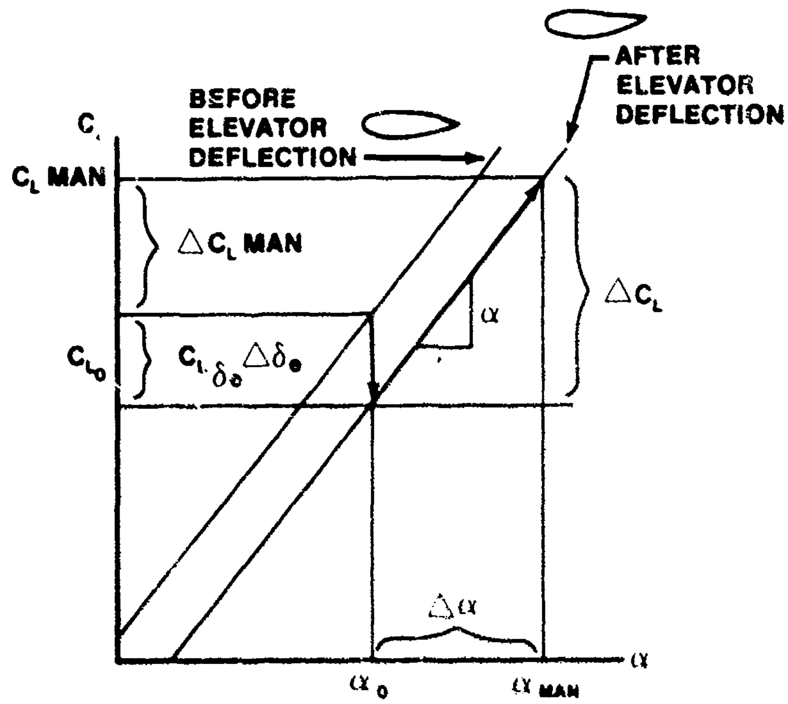


FIGURE 6.1. LIFT COEFFICIENT VERSUS ANGLE OF ATTACK

In other words, a pitch rate is initiated and α increases until a maneuvering lift coefficient $C_{L_{MAN}}$ is reached for the deflected elevator δ_e. The change in angle of attack is $\Delta\alpha$. The change in C_L has come partially from the deflected elevator and mainly from the pitching maneuver. The change in C_L due to the maneuver is from C_{L_0} to $C_{L_{MAN}}$. Since it did not change the the lift curve, and including the change in lift caused by elevator deflection, the expression for $\Delta\alpha$ becomes

$$C_L = a\alpha \tag{6.9}$$

$$\Delta C_L = a\Delta\alpha$$

$$\Delta C_L = \Delta C_{L_{MAN}} - C_{L_{\delta_e}} \Delta\delta_e = a\Delta\alpha \tag{6.10}$$

$$\Delta\alpha = \frac{1}{a}\left[\Delta C_{L_{MAN}} - C_{L_{\delta_e}}\Delta\delta_e\right] \qquad (6.11)$$

To put Equation 6.11 in terms of load factor, $\Delta C_{L_{MAN}}$ must be defined. This is the change in lift coefficient from the initial condition to the final maneuvering condition. This change can occur from one g flight to some other load factor or it can start at two or three g's and progress to some new load factor. If C_L is at one g then

$$C_L = \frac{W}{qS} \qquad (6.12)$$

and

$$C_{L_0} = \frac{n_0 W}{qS} \qquad (6.13)$$

where n_0 is the initial load factor. Similarly,

$$C_{L_{MAN}} = \frac{nW}{qS} \qquad (6.14)$$

where n is the maneuvering load factor.

$$\Delta C_{L_{MAN}} = C_{L_{MAN}} - C_{L_0} = \frac{nW}{qS} - \frac{n_0 W}{qS} = C_L\left(n - n_0\right) = C_L \Delta n \qquad (6.15)$$

Finally substituting Equation 6.15 into Equation 6.11

$$\Delta\alpha = \frac{1}{a}\left[C_L \Delta n - C_{L_{\delta_e}}\Delta\delta_e\right] \qquad (6.16)$$

Equation 6.16 is now ready for substitution into Equation 6.8.

An expression for ΔQ in Equation .8 will be derived using the pull-up maneuver analysis.

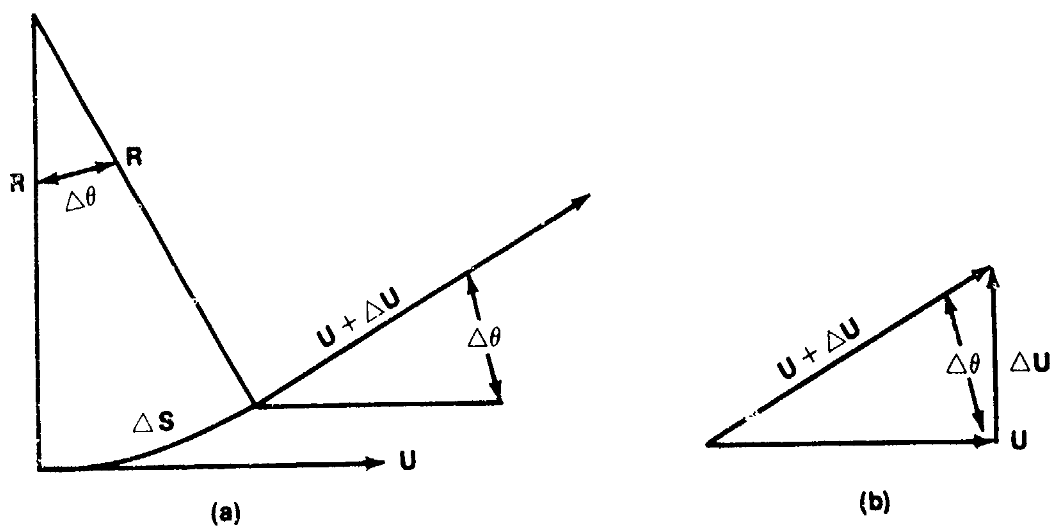


FIGURE 6.2. CURVILINEAR MOTION

Referring to Figure 6.2(a)

$$\Delta\theta = \frac{\Delta S}{R} \tag{6.17}$$

$$\frac{d\theta}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t}\frac{1}{R} \tag{6.18}$$

$$\frac{d\theta}{dt} = \frac{U}{R} = Q \tag{6.19}$$

From Figure 6.2(b)

$$\frac{\Delta U}{U} = \Delta\theta \quad \text{(small angles where } \tan\theta \approx \theta\text{)} \tag{6.20}$$

$$\frac{d\theta}{dt} = \lim_{\Delta t \to 0} \frac{\Delta U}{\Delta t}\frac{1}{U} = \frac{1}{U}\frac{dU}{dt} \tag{6.21}$$

Combining Equations 6.21 and 6.19

$$\frac{dU}{dt} = \frac{U^2}{R} \tag{6.22}$$

which may be recognized as the equation for the centripetal acceleration of a particle moving in a circle of radius R at constant velocity U. The force (or change in lift, ΔL) required to achieve this centripetal acceleration can be derived from (F = ma). Thus,

$$\Delta L = \frac{W}{g} a = \frac{W}{g} \frac{U^2}{R} \tag{6.23}$$

The change in lift can be seen in Figure 6.3 to be

$$\Delta L = nW - n_0 W = W(n - n_0) \tag{6.24}$$

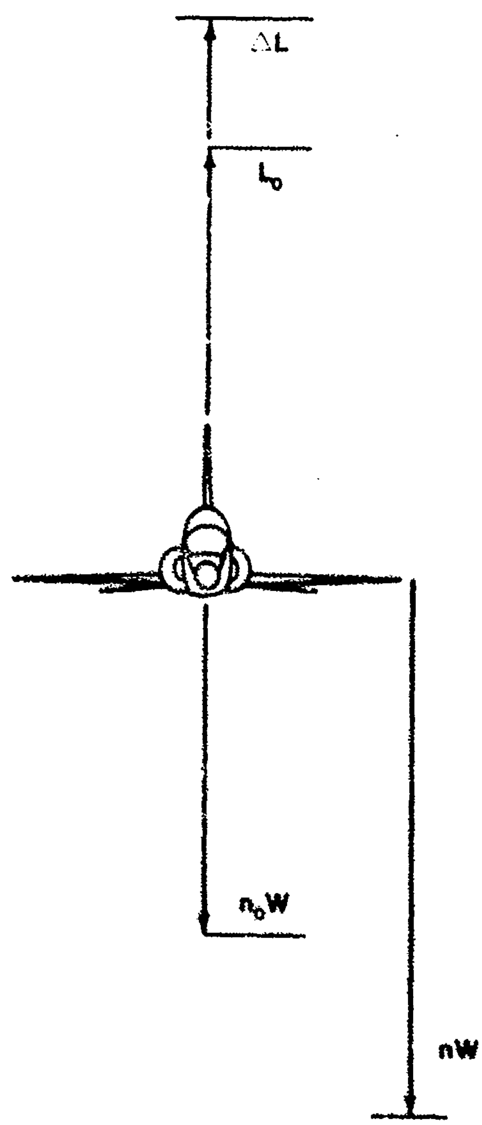


FIGURE 6.3. WINGS LEVEL PULL-UP

Again, the change may take place from any original load factor and is not limited to the straight and level flight condition ($n_0 = 1$). Therefore, for a constant velocity maneuver at U_0, Equations 6.23 and 6.24 give

$$W\,(n - n_0) = \frac{WU_0}{g}\left(\frac{U_0}{R} - \frac{U_0}{R_0}\right) \tag{6.25}$$

Using Equation 6.19 and the definition of ΔQ

$$\frac{U_0}{R} - \frac{U_0}{R_0} = Q - Q_0 = \Delta Q \tag{6.26}$$

Equation 6.25 can be written

$$\Delta Q = \frac{g}{U_0}\,(n - n_0) = \frac{g}{U_0}\,\Delta n \tag{6.27}$$

Now Equations 6.27 and 6.16 may be substituted into Equation 6.8.

$$\Delta\delta_e = \frac{-C_{m_\alpha}\,\frac{1}{a}\left[C_L\,\Delta n - C_{L_{\delta_e}}\,\Delta\delta_e\right]}{C_{m_{\delta_e}}} - \frac{\frac{\partial C_m}{\partial Q}\,\frac{g}{U_0}\,\Delta n}{C_{m_{\delta_e}}} \tag{6.28}$$

From longitudinal static stability,

$$C_{m_\alpha} = a\,(h - h_n) \tag{6.29}$$

Also to help further in reducing the equation to its simplest terms,

$$U_0^2 = \frac{2\,W}{\rho S C_L} \tag{6.30}$$

and

$$\frac{\partial C_m}{\partial Q} = \frac{c}{2U_0}\,C_{m_q} \tag{6.31}$$

Substituting Equations 6.31, 6.30, and 6.29 into Equation 6.28 results in

$$\frac{\Delta\delta_e}{\Delta n} = \frac{aC_L}{C_{m_\alpha} C_{L_{\delta_e}} - C_{m_{\delta_e}} a} \left(h - h_n + \frac{\rho Sc}{4m} C_{m_q}\right) \tag{6.32}$$

Equation 6.32 is now in the form that will define the stick-fixed maneuver point for the pull-up. The definition of the maneuver point, h_m, is the cg position at which the elevator deflection per g goes to zero. Taking the limit of Equation 6.32,

$$\lim_{\Delta n \to 0} \frac{\Delta\delta_e}{\Delta n} = \frac{d\delta_e}{dn} \tag{6.33}$$

or

$$\frac{d\delta_e}{dn} = \frac{aC_L}{C_{m_\alpha} C_{L_{\delta_e}} - C_{m_{\delta_e}} a} \left(h - h_n + \frac{\rho Sc}{4m} C_{m_q}\right) \tag{6.34}$$

Setting Equation 6.34 equal to zero will give the cg position at the maneuver point $h = h_m$

$$h_m = h_n - \frac{\rho Sc}{4m} C_{m_q} \tag{6.35}$$

Solving Equation 6.35 for h_n and substituting into Equation 6.34,

$$\frac{d\delta_e}{dn} = \frac{C_L}{C_{m_\alpha} C_{L_{\delta_e}} - C_{m_{\delta_e}} a} (h - h_m) \tag{6.36}$$

where we now define $h_m - h$ as the stick-fixed maneuver margin.

The significant points to be made about Equation 6.36 are:

1. The derivative $d\delta_e/dn$ varies with the maneuver margin. The more forward the cg, the more elevator will be required to obtain the limit load factor. That is, as the cg moves forward, more elevator deflection is necessary to obtain a given load factor.

2. The higher the C_L, the more elevator will be required to obtain the limit load factor. That is, at low speeds (high C_L) more elevator deflection is necessary to obtain a given load factor than is required to obtain the same load factor at a higher speed (lower C_L).

3. The derivative $d\delta_e/dn$ should be linear with respect to cg at a constant C_L (Figure 6.4).

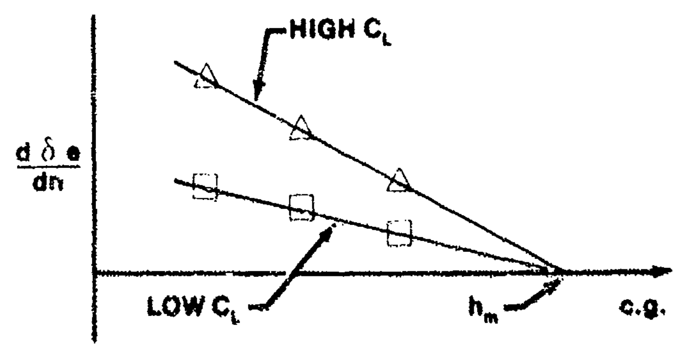


FIGURE 6.4. ELEVATOR DEFLECTION PER G

Another approach to solving for the maneuver point h_m is to return to the original stability equation from longitudinal static stability.

$$\frac{dC_m}{dC_L} = h - \frac{X_{ac}}{c} + \frac{dC_m}{dC_{L_{Fus}}} - \frac{a_t}{a_w} V_H \eta_t \left(1 - \frac{d\varepsilon}{d\alpha}\right) \tag{6.43}$$

The effect of pitch damping on aircraft stability will be determined and added to Equation 6.43. Recalling the relationship

$$\frac{\partial C_m}{\partial Q} = \frac{c}{2U_0} C_{m_q} \tag{6.31}$$

from equations of motion, Equation 6.37 can be written

$$\Delta C_m = \frac{c}{2U_0} C_{m_q} \Delta Q \tag{6.37}$$

Substituting the value obtained for ΔQ from Equation 6.27

$$\Delta C_m = \frac{cg}{2U_0^{\,2}} C_{m_q} \Delta n \tag{6.38}$$

Substituting

$$\Delta n = \frac{\Delta C_{L_{MAN}}}{C_L}$$

from Equation 6.15 and Equation 6.12

$$C_L = \frac{W}{qS} \tag{6.12}$$

into Equation 6.38 gives

$$\Delta C_m = \frac{\rho Sc}{4m} C_{m_q} \Delta C_{L_{MAN}} \tag{6.39}$$

$$\lim_{\Delta C_L \to 0} \frac{\Delta C_m}{\Delta C_{L_{MAN}}} = \frac{dC_m}{dC_L}_{\substack{\text{Pitch} \\ \text{Damping}}} = \frac{\rho Sc}{4m} C_{m_q} \tag{6.40}$$

This term may now be added to Equation 5.43. If the sign of C_{m_q} is negative, then the term is a stabilizing contribution to the stability equation. C_{m_q} will be analyzed further.

$$\frac{dC_m}{dC_L} = h - \frac{X_{ac}}{c} + \frac{dC_m}{dC_{L_{Fus}}} - \frac{a_t}{a_w} V_H \eta_t \left(1 - \frac{d\varepsilon}{d\alpha}\right) + \frac{\rho Sc}{4m} C_{m_q} \tag{6.41}$$

The maneuver point is found by setting dC_m/dC_L equal to zero and solving for the cg position where this occurs.

$$h_m = \frac{X_{ac}}{c} - \frac{dC_m}{dC_{L_{Fus}}} + \frac{a_t}{a_w} V_H \eta_t \left(1 - \frac{d\varepsilon}{d\alpha}\right) - \frac{\rho Sc}{4m} C_{m_q} \tag{6.42}$$

The first three terms on the right side of Equation 6.42 may be identified as the expression for the neutral point h_n. If this substitution is made in Equation 6.42, Equation 6.35 is again obtained.

$$h_m = h_n - \frac{\rho Sc}{4m} C_{m_q} \tag{6.35}$$

The derivative C_{m_q} found in Equations 6.34 and 6.35 needs to be examined before proceeding with further discussion.

The damping that comes from the pitch rate established in a pull-up comes from the wing, tail, and fuselage components. The tail is the largest contributor to the pitch damping because of the long moment arm. For this reason, it is usually used to derive the value of C_{m_q}. Sometimes an empirical value of 10% is added to account for damping of the rest of the aircraft, but often the value for the tail alone is used to estimate the derivative. The effect of the tail may be calculated from Figure 6.5.

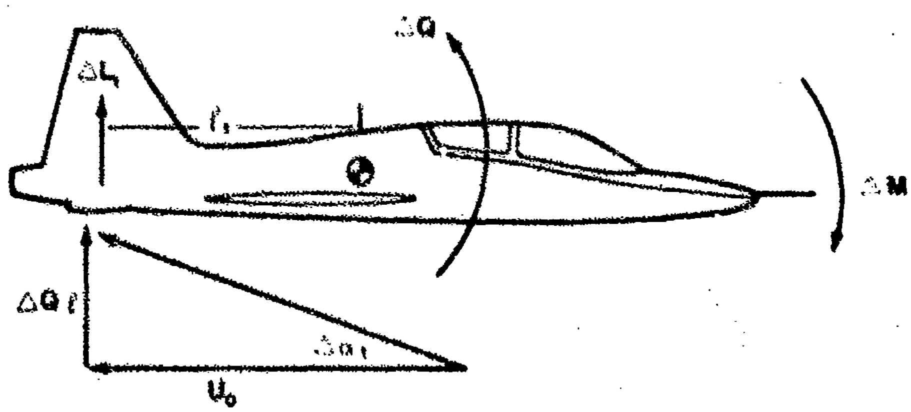


FIGURE 6.5. PITCH DAMPING

The pitching moment effect on the aircraft from the downward moving horizontal stabilizer is

$$\Delta \mathcal{M} = - \ell_t \Delta L_t = q_w S_w c_w \Delta C_m \tag{6.43}$$

where

$$\Delta L_t = q_t S_t \Delta C_{L_t} \tag{6.44}$$

Solving for ΔC_m,

$$\Delta C_m = -\left(\frac{q_t}{q_w}\right) \frac{\ell_t S_t}{c_w S_w} \Delta C_{L_t} \tag{6.45}$$

The combination $\ell_t S_t / c_w S_w$ can be recognized as the tail volume coefficient, V_H. The term q_t/q_w is the tail efficiency factor, η_t. Equation 6.45 may then be written

$$\Delta C_m = - V_H \eta_t \Delta C_{L_t} \tag{6.46}$$

which can be further refined to

$$\Delta C_m = - V_H \eta_t a_t \Delta \alpha_t \tag{6.47}$$

From Figure 6.5, the change in angle of attack at the tail caused by the pitch rate will be

$$\Delta \alpha_t = \tan^{-1} \frac{\Delta Q \ell_t}{U_0} \approx \Delta Q \frac{\ell_t}{U_0} \tag{6.48}$$

Substituting Equation 6.48 into 6.47

$$\Delta C_m = - a_t V_H \eta_t \frac{\ell_t}{U_0} \Delta Q \tag{6.49}$$

Taking the limit of Equation 6.49 gives

$$\frac{\partial C_m}{\partial Q} = - a_t V_H \eta_t \frac{\ell_t}{U_0} \tag{6.50}$$

Equation 6.50 shows that the damping expression $\partial C_m/\partial Q$ is an inverse function of airspeed (i.e., this term is greater at lower speeds). Solving for C_{m_q} using Equation 6.31

$$C_{m_q} = \frac{2U_0}{c}\frac{\partial C_m}{\partial Q} = -2a_t V_H \eta_t \frac{\ell_t}{c} \tag{6.51}$$

The damping derivative is not a function of airspeed, but rather a value determined by design considerations only (subsonic flight). The pitch damping derivative may be increased by increasing S_t or ℓ_t.

When this value for C_{m_q} is substituted into Equation 6.35

$$h_m = h_n + \frac{\rho S\, a_t\, \eta_t\, \ell_t V_H}{2m} \tag{6.52}$$

The following conclusions are apparent from Equation 6.52:

1. The maneuver point should always be behind the neutral point. This is verified since the addition of a pitch rate increases the stability of the aircraft (C_{m_q} is negative in Equation 6.41). Therefore, the stability margin should increase.

2. Aircraft geometry is influential in locating the maneuver point aft of the neutral point.

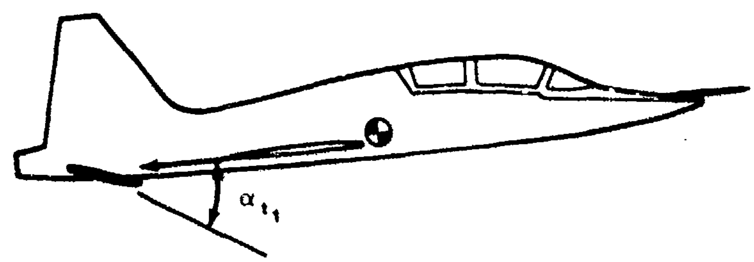
3. As altitude increases, the distance between the neutral point and maneuver point decreases.

4. As weight decreases at any given altitude, the maneuver point moves further behind the neutral point and the maneuver stability margin increases.

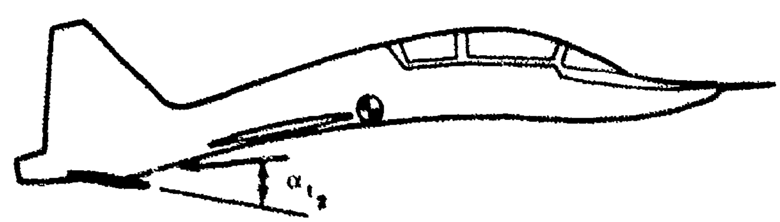
5. The largest variation between maneuver point and neutral point occurs with a light aircraft flying at sea level.

6.5 AIRCRAFT BENDING

Before the pull-up analysis is completed, one more subject should be covered. One of the assumptions made early in the equations of motion course was that the aircraft was a rigid body. In reality, all aircraft bend when a load is applied. The bigger the aircraft, the more they bend. The effect on the aircraft bending is shown in Figure 6.6.



RIGID AIRCRAFT UNDER HIGH LOAD FACTOR



NONRIGID AIRCRAFT UNDER HIGH LOAD FACTOR

FIGURE 6.6. AIRCRAFT BENDING

As the non-rigid aircraft bends, the angle of attack, α_t of the horizontal stablizer decreases. In order to keep the aircraft at the same overall angle of attack, the original angle of attack of the tail must be reestablished. This requires an increase in the elevator (slab) deflection or an additional $\Delta\delta_e$ per load factor.

6.6 THE TURN MANEUVER

The subject of maneuvering in pull-ups has already been presented. While it is the easiest method for a test pilot to perform, it is also the most time

consuming. Therefore, most maneuvering data is collected by turning. There are several methods used to collect data in a turn.

In order to analyze the maneuvering turn, Equation 6.8 is recalled

$$\Delta\delta_e = - \frac{C_{m_\alpha} \Delta\alpha \; (\partial C_m/\partial Q) \; \Delta Q}{C_{m_{\delta_e}}} \tag{6.8}$$

The expression for $\Delta\alpha$ in Equation 6.16 derived for the pull-up maneuver, is also applicable to the turning maneuver.

$$\Delta\alpha = \frac{1}{a}\left(C_L \Delta n - C_{L_{\delta_e}} \Delta\delta_e\right) \tag{6.16}$$

Such is not the case for the ΔQ expression in Equation 6.8. Another expression (other than Equation 6.27) for ΔQ pertaining to the turn maneuver, must be developed.

Referring to Figure 6.7, the lift vector will be balanced by the weight and centripetal acceleration. One component ($L \cos \phi$) balances the weight and the other ($L \sin \phi$) results in the centripetal acceleration.

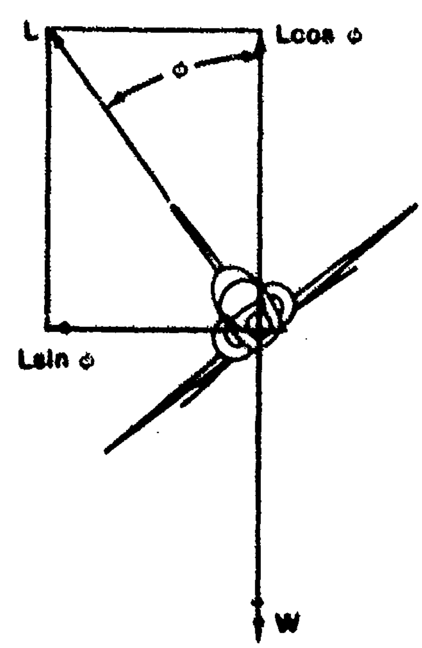


FIGURE 6.7. FORCES IN THE TURN MANEUVER

$$L \sin \phi = \frac{W}{g}\frac{U^2}{R} \tag{6.53}$$

For a level turn,

$$L \cos \phi = W \tag{6.54}$$

$$n = L/W = \frac{1}{\cos \phi} \tag{6.55}$$

Now, dividing Equation 6.53 by Equation 6.54 and rearranging terms

$$\frac{U}{R} = \frac{g}{U}\frac{\sin \phi}{\cos \phi} \tag{6.56}$$

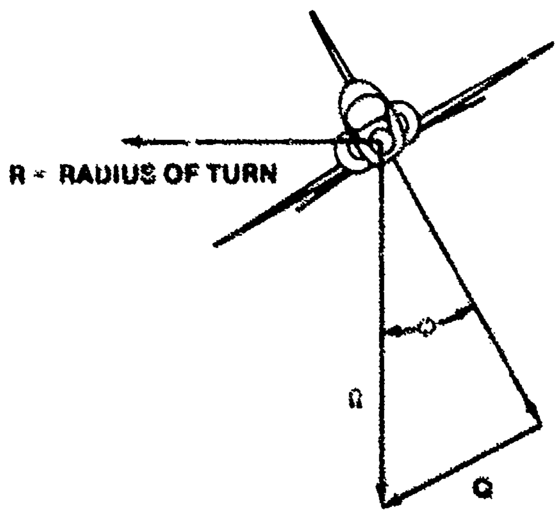


FIGURE 6.8. AIRCRAFT IN THE TURN MANEUVER

Referring to Figure 6.8 where pitch rate is represented by a vector along the wings and yaw rate a vector vertically through the center of gravity, the following relationships can be derived.

$$\Omega = \frac{U}{R} \tag{6.57}$$

$$Q = \Omega \sin \phi \tag{6.58}$$

$$Q = \frac{U}{R} \sin \phi \tag{6.59}$$

Substituting Equation 6.56 into Equation 6.59

$$Q = \frac{g}{U} \frac{\sin^2 \phi}{\cos \phi} \tag{6.60}$$

From trigonometry,

$$Q = \frac{g}{U} \left(\frac{1 - \cos^2 \phi}{\cos \phi} \right) \tag{6.61}$$

$$Q = \frac{g}{U} \quad \frac{1}{\cos \phi} - \cos \phi \tag{6.62}$$

Substituting Equation 6.55 into Equation 6.62 gives

$$Q = \frac{g}{U} \left(n - \frac{1}{n} \right) \tag{6.63}$$

When maneuvering from initial conditions of n_0 to n, the ΔQ equation becomes,

$$\Delta Q = Q - Q_0 = \frac{g}{U_0} \left(n - \frac{1}{n} \right) - \frac{g}{U_0} \left(n_0 - \frac{1}{n_0} \right) \tag{6.64}$$

$$\Delta Q = \frac{g}{U_0} (n - n_0) \left(1 + \frac{1}{nn_0} \right) \tag{6.65}$$

The general expression for ΔQ in Equation 6.65 and the value of $\Delta\alpha$ in Equation 6.16 may now be substituted into Equation 6.8 to determine $\Delta\delta_e$

$$\Delta\delta_e = - C_{m_\alpha} \frac{1}{a} \frac{\left(C_L \Delta n - C_{L_{\delta_e}} \Delta\delta_e \right)}{C_{m_{\delta_e}}} - \frac{\frac{\partial C_m}{\partial Q} \frac{g}{U_0} (\Delta n) \left(1 + \frac{1}{nn_0} \right)}{C_{m_{\delta_e}}} \tag{6.66}$$

Substituting Equation 6.31

$$\frac{\partial C_m}{\partial Q} = \frac{c}{2U_0} C_{m_q} \tag{6.31}$$

into Equation 6.66 and rearranging gives

$$\Delta\delta_e = \frac{C_{m_\alpha} C_L \Delta n + C_{m_q} a \frac{cg}{2U_0^2} (\Delta n)\left(1 + \frac{1}{nn_0}\right)}{C_{m_\alpha} C_{L_{\delta_e}} - C_{m_{\delta_e}} a} \tag{6.67}$$

Now, from longitudinal static stability,

$$C_{m_\alpha} = a\,(h - h_n)$$

and

$$U_0^2 = \frac{2\,W}{\rho S C_L}$$

Using these relationships, Equation 6.67 can be written

$$\frac{\Delta\delta_e}{\Delta n} = \frac{aC_L}{C_{m_\alpha} C_{L_{\delta_e}} - C_{m_{\delta_e}} a}\left[(h - h_n) + \frac{\rho Sc}{4m} C_{m_q} \left(1 + \frac{1}{nn_0}\right)\right] \tag{6.68}$$

Taking the limit of $\Delta\delta_e/\Delta n$ as $\Delta n \longrightarrow 0$ in Equation 6.68

$$\frac{d\delta_e}{dn} = \frac{aC_L}{C_{m_\alpha} C_{L_{\delta_e}} - C_{m_{\delta_e}} a}\left[(h - h_n) + \frac{\rho Sc}{4m} C_{m_q} \left(1 + \frac{1}{n^2}\right)\right] \tag{6.69}$$

The maneuver point is determined by setting $d\delta_e/dn$ equal to zero and solving for the cg position at this point.

$$h_m = h_n - \frac{\rho Sc}{4m} C_{m_q} \left(1 + \frac{1}{n^2}\right) \tag{6.70}$$

The maneuver point in a turn differs from the pull-up by the factor $\left(1 + 1/n^2\right)$. This means that at high load factors the turn and pull-up maneuver points will be very nearly the same. If Equation 6.70 is solved for h_n and substituted back into Equation 6.69 the result is

$$\frac{d\delta_e}{dn} = \frac{aC_L}{C_{m_\alpha} C_{L_{\delta_e}} - C_{m_{\delta_e}} a} (h - h_m) \qquad (6.71)$$

The term $d\delta_e/dn$ is not the same for both pull-up and turn since h_m in Equation 6.71 for turns includes the factor $(1 + 1/n^2)$ and is different from the h_m found for the pull-up maneuver. The conclusions reached for Equations 6.36 and 6.52 apply to Equations 6.71 and 6.72 as well.

$$h_m = h_n + \frac{\rho S a_t \eta_t \ell_t V_H}{2m} (1 + 1/n^2) \qquad (6.72)$$

6.7 SUMMARY

Before looking further into the stick-free maneuverability case, it would be well to review the development in the preceding paragraphs and relate it to the results of Chapter 5.

The basic approach to longitudinal stability was centered around finding a value for dC_m/dC_L. It was found that a negative value for this derivative meant that the aircraft was statically stable. The derivative was analyzed for the stick-fixed case first and then the stick-free case. The cg position where this derivative was zero was defined as the neutral point. Static margin was defined as the difference between the neutral point and the cg location. The stick-free case was determined by

$$\left.\frac{dC_m}{dC_L}\right|_{\substack{\text{Stick-Free}\\ \text{Aircraft}}} = \left.\frac{dC_m}{dC_L}\right|_{\substack{\text{Stick-Fixed}\\ \text{Aircraft}}} + \Delta \left.\frac{dC_m}{dC_L}\right|_{\substack{\text{Effect of}\\ \text{Free Elev}}} \qquad (5.82)$$

The free elevator case was merely the basic stability of the aircraft with the effect of freeing the elevator added to it.

When the maneuvering case was introduced, it was shown that there was a new derivative to be discussed, but the basic stability of the aircraft would not change - only the effect of pitch rate was added to it.

$$\left.\frac{dC_m}{dC_L}\right|_{\substack{\text{Stick-Fixed}\\ \text{Aircraft Pitching}}} = \left.\frac{dC_m}{dC_L}\right|_{\substack{\text{Stick-Fixed}\\ \text{Aircraft}}} + \Delta \left.\frac{dC_m}{dC_L}\right|_{\substack{\text{Effect of the}\\ \text{Pitch Rate}}} \tag{6.73}$$

For the stick-free case, the following must be true,

$$\left.\frac{dC_m}{dC_L}\right|_{\substack{\text{Stick-Free}\\ \text{Aircraft Pitching}}} = \left.\frac{dC_m}{dC_L}\right|_{\substack{\text{Stick-Fixed}\\ \text{Aircraft}}} + \Delta \left.\frac{dC_m}{dC_L}\right|_{\substack{\text{Effect of}\\ \text{Free Elev}}} + \Delta \left.\frac{dC_m}{dC_L}\right|_{\substack{\text{Effect of}\\ \text{Pitch Rate}}} \tag{6.74}$$

$$\left.\frac{dC_m}{dC_L}\right|_{\substack{\text{Stick-Free}\\ \text{Aircraft Pitching}}} = \left.\frac{dC_m}{dC_L}\right|_{\substack{\text{Stick-Free}\\ \text{Aircraft}}} + \Delta \left.\frac{dC_m}{dC_L}\right|_{\substack{\text{Effect of}\\ \text{Pitch Rate}}} \tag{6.75}$$

NOTE: This "Effect of Pitch Rate" term is not necessarily the same as the corresponding term in Equation 6.73.

6.8. STICK-FREE MANEUVERING

The first analysis of stick-free maneuvering requires a review of longitudinal static stability. It was determined in Chapter 5 that the effect of freeing the elevator was to multiply the tail term by the free elevator factor F which equaled $\left(1 - \tau\, C_{h_\alpha}/C_{h_\delta}\right)$. Consequently, in the maneuvering case, to find the stick-free maneuver point the tail effect of stick-fixed maneuvering must be multiplied by this free elevator factor. Recalling Equation 6.42 from the stick-fixed maneuvering discussion,

$$h_m = \frac{X_{ac}}{c} - \frac{dC_m}{dC_{L_{Fus}}} + \frac{a_t}{a_w} V_H\, \eta_t \left(1 - \frac{d\varepsilon}{d\alpha}\right) - \frac{\rho Sc}{4m} C_{m_q} \tag{6.42}$$

Multiplying the tail terms by F,

$$h'_m = \frac{X_{ac}}{c} - \frac{dC_m}{dC_{L_{Fus}}} + \frac{a_t}{a_w} V_H \eta_t \left(1 - \frac{d\varepsilon}{d\alpha}\right)F - \frac{\rho Sc}{4m} C_{m_q} F \qquad (6.76)$$

The first three terms on the right are the expression for stick-free neutral point, h'_n. Thus,

$$h'_m = h'_n - \frac{\rho Sc}{4m} C_{m_q} F \qquad (6.77)$$

This is the stick-free maneuver point in terms of the stick-free neutral point for the pull-up case. It may be extended to the turn case by using the term for the pitch rate of the tail in a turn.

$$h'_m = h'_n - \frac{\rho Sc}{4m} C_{m_q} F \left(1 + \frac{1}{n^2}\right) \qquad (6.79)$$

These equations do not give a flight test relationship, so it is necessary to derive one from stick forces, as was done in longitudinal static stability. The method used will be to relate the stick-force-per-g to the stick-free maneuver point. Starting with the relationship of stick force, gearing, and hinge moment that was derived in Chapter 5,

$$F_s = - GH_e \qquad (5.91)$$

$$H_e = q\, S_e\, c_e\, C_h \qquad (5.92)$$

$$F_s = - Gq\, S_e\, c_e\, C_h \qquad (5.93)$$

The change in stick-force for a change in load factor becomes,

$$\frac{\Delta F_s}{\Delta n} = - Gq\, S_e\, c_e \frac{\Delta C_h}{\Delta n} \qquad (6.79)$$

where

$$\Delta C_h = C_{h_\alpha} \Delta\alpha_t + C_{h_\delta} \Delta\delta_e \qquad (6.80)$$

6.8.1 Stick-Free Pull-up Maneuver

ΔC_h must be written in terms of load factor and substituted back into Equation 6.79. This will require defining $\Delta\alpha_t$ and $\Delta\alpha_e$ in terms of load factor. The change in angle of attack of the tail comes partly from the change in angle of attack of the wing due to downwash and partly from the pitch rate.

$$\Delta\alpha_t = \Delta\alpha \left(1 - \frac{d\varepsilon}{d\alpha}\right) + \Delta Q \frac{\ell_t}{U_0} \qquad (6.81)$$

Where $\Delta\alpha$ and ΔQ in the above equation are

$$\Delta\alpha = \frac{1}{a} \left(C_L \Delta n - C_{L_{\delta_e}} \Delta\delta_e\right) \qquad (6.16)$$

$$\Delta Q = \frac{g}{U_0} \Delta n \qquad (6.27)$$

Recall that

$$\frac{\Delta\delta_e}{\Delta n} = \frac{aC_L}{C_{m_\alpha} C_{L_{\delta_e}} - C_{m_{\delta_e}} a} (h - h_m) \qquad (6.36)$$

Assuming $C_{L_{\delta_e}}$ is small enough to ignore, Equations 6.81 and 6.36 can be written

$$\Delta\alpha_t = \frac{C_L}{a} \left(1 - \frac{d\varepsilon}{d\alpha}\right) \Delta n + \frac{g}{U_0^2} \ell_t \Delta n \qquad (6.82)$$

$$\frac{\Delta\delta_e}{\Delta n} = -\frac{C_L}{C_{m_{\delta_e}}} (h - h_m) \tag{6.83}$$

Substituting Equations 6.82 and 6.83 into Equation 6.80, gives Equation 6.84

$$\frac{\Delta C_h}{\Delta n} = C_{h_a} \frac{C_L}{a} \left(1 - \frac{d\varepsilon}{d\alpha}\right) + C_{h_\alpha} \quad g \frac{\ell_t}{U_0^{\ 2}} - C_{h_\delta} \frac{C_L}{C_{m_{\delta_e}}} \quad (h - h_m) \tag{6.84}$$

Substituting,

$$U_0^{\ 2} = 2W/\rho SC_L \tag{6.30}$$

and the definition of control power,

$$C_{m_{\delta_e}} = - a_t \, V_H \, \eta_t \, \tau \tag{5.50}$$

into Equation 6.84 and factoring gives

$$\frac{\Delta C_h}{\Delta n} = -\frac{C_{h_\delta} C_L}{C_{m_{\delta_e}}} \left[\frac{C_{h_\alpha}}{C_{h_\delta} \, a} \left(1 - \frac{d\varepsilon}{d\alpha}\right) a_t \, V_H \, \eta_t \tau + \frac{C_{h_\alpha}}{C_{h_\delta}} \frac{\rho S \ell_t}{2m} a_t \, V_H \, \eta_t \tau + (h - h_m)\right] \tag{6.85}$$

From longitudinal stability,

$$h_n - h_n' = \frac{C_{h_\alpha}}{C_{h_\delta}} \frac{a_t}{a} \, V_H \, \eta_t \, \tau \left(1 - \frac{d\varepsilon}{d\alpha}\right) \tag{6.86}$$

Substituting Equations 6.86, 6.51, and 6.87 into Equation 6.85 gives Equation 6.88

$$C_{m_q} = - 2a_t \, V_H \, \eta_t \frac{\ell_t}{c} \tag{6.51}$$

$$F = 1 - \tau \quad \frac{C_{h_\alpha}}{C_{h_\delta}} \tag{6.87}$$

$$\frac{\Delta C_h}{\Delta n} = \frac{C_{h_\delta} C_L}{C_{m_{\delta_e}}} \left[(h'_n - h_n) + (1 - F) \frac{\rho Sc}{4m} C_{m_q} - h + h_m \right] \tag{6.88}$$

But

$$h_m = h_n - \frac{\rho Sc}{4m} C_{m_q} \tag{6.35}$$

Therefore

$$\frac{\Delta C_h}{\Delta n} = - \frac{C_{h_\delta} C_L}{C_{m_{\delta_e}}} \left[h - h'_n + \frac{\rho Sc}{4m} C_{m_q} F \right] \tag{6.89}$$

Substituting Equation 6.89 back into Equation 6.79 and taking the limit gives

$$\frac{dF_s}{dn} = G\, q\, S_e\, c_e \frac{C_{h_\delta} C_L}{C_{m_{\delta_e}}} \left[h - h'_n + \frac{\rho Sc}{4m} C_{m_q} F \right] \tag{6.90}$$

Defining the stick-free maneuver point h'_m as the cg position where dF_s/dn is equal to zero, Equation 6.90 reduces to

$$h'_m = h'_n - \frac{\rho Sc}{4m} C_{m_q} F \tag{6.77}$$

which was previously derived. Equation 6.90 can then be rewritten

$$\frac{dF_s}{dn} = G\, q\, S_e\, c_e \frac{C_{h_\delta} C_L}{C_{m_{\delta_e}}} (h - h'_m) \tag{6.91}$$

Equation 6.12 can be substituted into Equation 6.91

$$C_L = \frac{W}{qS} \tag{6.12}$$

which gives the final stick-force-per-g equation

$$\frac{dF_s}{dn} = G\, S_e\, c_e\, C_{h_\delta} \left(\frac{W}{S}\right)\left(\frac{1}{C_{m_{\delta_e}}}\right) (h - h'_m) \tag{6.92}$$

6.8.2 Stick-Free Turn Maneuver

The procedures set for determining the dF_s/dn equation and an expression for the stick-free maneuver point for the turning maneuver is practically identical to the pull-up case. For the turn condition, ΔQ is now

$$\Delta Q = \frac{g}{U_0}\Delta n \left(1 + \frac{1}{nn_0}\right) \tag{6.93}$$

The change in angle of attack of the tail, $\Delta\alpha_t$ becomes

$$\Delta\alpha_t = \frac{C_L}{a}\left(1 - \frac{d\varepsilon}{d\alpha}\right)\Delta n + \frac{g\ell_t}{U_0^{\,2}}\Delta n \left(1 + \frac{1}{nn_0}\right) \tag{6.94}$$

and

$$\frac{\Delta\delta_e}{\Delta n} = \frac{-C_L}{C_{m_{\delta_e}}}\left[h - h_n + \frac{\rho SC}{4m} C_{m_q}\left(1 + \frac{1}{nn_0}\right)\right] \tag{6.95}$$

Substituting Equations 6.94 and 6.95 into Equation 6.80 and performing the same factoring and substitutions as in the pull-up case

$$\frac{\Delta C_h}{\Delta n} = -\frac{C_{h_\delta} C_L}{C_{m_{\delta_e}}}\left[h - h_n' + \frac{\rho Sc}{4m} C_{m_q} F \left(1 + \frac{1}{nn_0}\right)\right] \tag{6.96}$$

Substituting Equation 6.96 into Equation 6.79 and taking the limit as $\Delta n \to 0$,

$$\frac{dF_s}{dn} = GqS_e c_e \frac{C_{h_\delta} C_L}{C_{m_{\delta_e}}}\left[h - h_n' + \frac{\rho Sc}{4m} C_{m_q} F \left(1 + \frac{1}{n^2}\right)\right] \tag{6.97}$$

Solving for the stick-free maneuver point,

$$h_m' = h_n' - \frac{\rho Sc}{4m} C_{m_q} F \left(1 + \frac{1}{n^2}\right) \tag{6.98}$$

Further substitution puts Equation 6.97 into the following form:

$$\frac{dF_s}{dn} = G \left(S_e c_e C_{h_\delta}\right)\left(\frac{W}{S}\right)\left(\frac{1}{C_{m_{\delta_e}}}\right)(h - h_m') \qquad (6.99)$$

Again, the turning stick-force-per-g Equation 6.99 appears identical to the stick-free pull-up equation. However, the expression for the maneuver point h_m' is different.

The term in the first parenthesis represents the hinge moment of the elevator and the aircraft size. The second term in parenthesis is wing loading, and the third term in parenthesis is the reciprocal of elevator power. The last term is the negative value of the stick-free maneuver margin. The following conclusions are drawn from this equation:

1. The stick-force-per-g appears to vary directly with the wing loading. However, weight also appears inversely in h_m'. Therefore, the full effect of weight cannot be truly analyzed since one effect could cancel the other.

2. Since airspeed does not appear in the equation, the stick-force-per-g will be the same at all airspeeds for a fixed cg.

$$h_m' = h_n' - \frac{\rho Sc}{4m} C_{m_q} F \qquad (6.77)$$

From Equation 6.77 come the following conclusions:

1. The difference between the stick-fixed and stick-free maneuver point is a function of the free elevator factor, F.

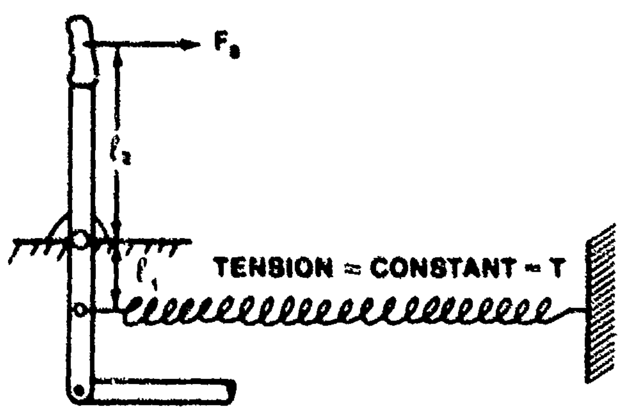
2. The stick-free maneuver point, h_m', varies directly with altitude, becoming closer to the stick-free neutral point, the higher the aircraft flies.

The location of the stick-free maneuver point occurs where $dF_s/dn = 0$. It is difficult to fly an aircraft with this type gradient. Consequently, military specifications limit the minimum value of dF_s/dn to three pounds per g.

The forward cg may be limited by stick-force-per-g. The maximum value is limited by the type aircraft (bomber, fighter, or trainer), i.e., heavier gradients in bomber types and lighter ones in fighters.

6.9 EFFECTS OF BOBWEIGHTS AND DOWNSPRINGS

The effect of bobweights and downsprings on the stick-free maneuver point and stick-force gradients are of interest. The result of adding a spring or bobweight to the control system adds an incremental force to the system. The effect of the spring is different from the effect of the bobweight. The spring exerts a constant force on the stick no matter what load factor is applied. The bobweight exerts a force on the stick proportional to the load factor.



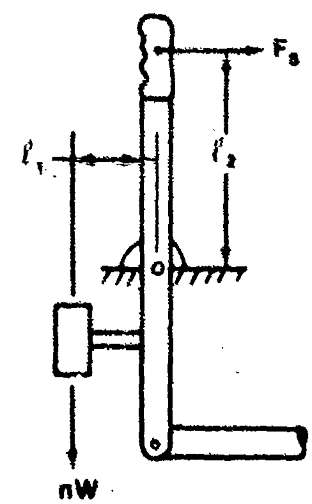


FIGURE 6.9. DOWNSPRING AND BOBWEIGHT

Adding incremental forces for the downspring and bobweight of Equation 5.93 gives

Downspring

$$F_s = - G\, q\, S_e\, c_e\, C_h + T \frac{\ell_1}{\ell_2} \qquad (6.100)$$

Bobweight

$$F_s = -G\,q\,S_e\,c_e\,C_h + nW\frac{\ell_1}{\ell_2} \tag{6.101}$$

When the derivative is taken with respect to load factor, the effect on dF_s/dn of the spring is zero. The stick force gradient is not affected by the spring nor is the stick-free maneuver point changed.

Downspring

$$\frac{dF_s}{dn} = -G\,q\,S_e\,c_e\frac{dC_h}{dn} + 0 \tag{6.102}$$

For the bobweight, the stick-force gradient dF_s/dn is affected and

Bobweight

$$\frac{dF_s}{dn} = -G\,q\,S_e\,c_e\frac{dC_h}{dn} + W\frac{\ell_1}{\ell_2} \tag{6.103}$$

Consequently, the addition of the bobweight (positive) increases the stick force gradient, moves the stick-free maneuver point aft, and shifts the allowable cg spread aft (the minimum and maximum cg positions as specified by force gradients are moved aft). See Figure 6.10.

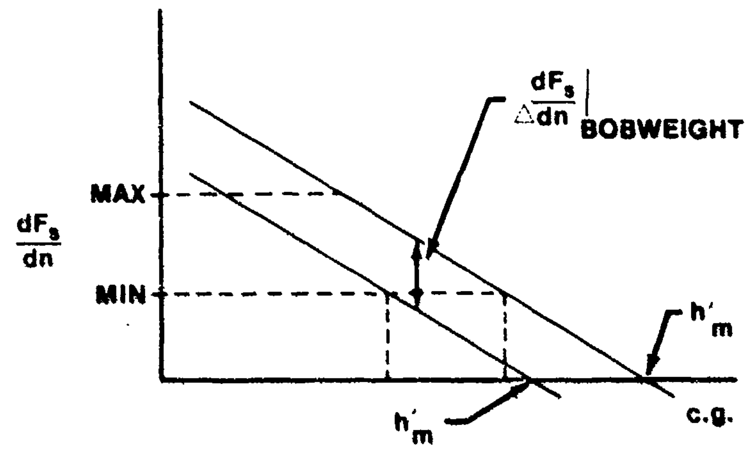


FIGURE 6.10. EFFECTS OF ADDING A BOBWEIGHT

Recall from Chapter 5, Longitudinal Static Stability, (Reference Figure 5.51), that downsprings and bobweights may be used to decrease stick forces. The bobweight may also decrease the stick force gradient. If a bobweight is used in this configuration, a downspring may be needed to counter-balance the bobweight for non-maneuvering (1g) flight conditions, i.e, to preserve the original speed stability before the bobweight was added.

6.10 AERODYNAMIC BALANCING

Aerodynamic balancing is used to affect the stick force gradient and stick-free maneuver point. Aerodynamic balancing or varying values of C_{h_α} and C_{h_δ} affects the following stick-free equations:

$$\frac{dF_s}{dn} = G\left(Sc\ \ c_e\ C_{h_\delta}\right)\frac{W}{S}\left(\frac{1}{C_{m_{\delta_e}}}\right)(h - h_m') \tag{6.99}$$

$$h'_m = h'_n - \frac{\rho Sc}{4m} C_{m_q} F \tag{6.77}$$

$$F = 1 - \tau \frac{C_{h_\alpha}}{C_{h_\delta}} \tag{6.87}$$

Decreasing C_{h_δ} and/or increasing C_{h_α} by using such aerodynamic balancing devices as an overhang balance or a lagging balance tab, does the following:

1. The free elevator factor F decreases
2. The stick-free maneuver point h'_m moves forward
3. The maneuver margin term $(h - h'_m)$ decreases
4. The stick force gradient decreases
5. The forward and aft cg limits move forward

Increasing C_{h_δ} and/or decreasing C_{h_α} by using a convex trailing edge or a leading balance tab does the following:

1. The free elevator factor F increases
2. The stick-free maneuver point h'_m moves aft
4. The maneuver margin term $(h - h'_m)$ increases
5. The forward and aft cg limits move aft

6.11 CENTER OF GRAVITY RESTRICTIONS

The restrictions on the aircraft's center of gravity location may be examined by referring to the mean aerodynamic chord in Figure 6.11.

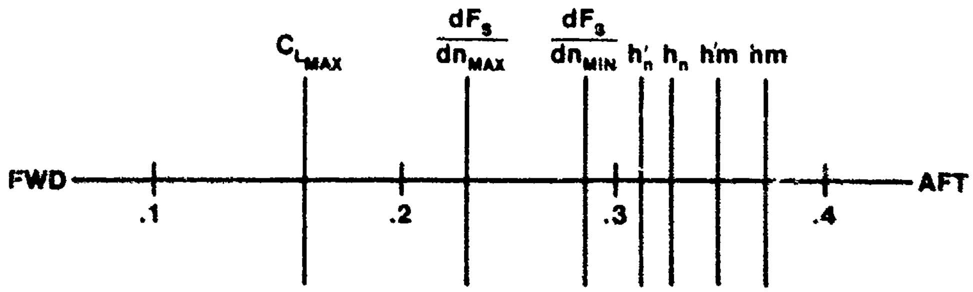


FIGURE 6.11. RESTRICTIONS TO CENTER OF GRAVITY LOCATIONS

The forward cg travel is normally limited by:

1. Maximum stick-force-per-g gradient, dF_s/dn
2. Elevator required to flare and land at $C_{L_{MAX}}$ in ground effect or
3. Elevator required to raise the nose for takeoff at the proper airspeed

The aft cg travel is normally limited by:

1. Minimum stick-force-per-g, dF_s/dn, or
2. Stick-free neutral point (power on), h_n'

Additional considerations:

1. The stick-free neutral and maneuver points are located ahead of their respective stick-fixed points.
2. The stick-free maneuver point, h_m', can be moved aft with a bobweight, but not with a downspring.

3. The desired aft cg location may be unsatisfactory because it lies aft of the cg position giving minimum stick force gradient. The requirements for a bobweight or a particular aerodynamic balancing then exists to increase the minimum stick force gradient aft of the desired aft cg position.

The equations which pertain to maneuvering flight are repeated below:

Pull-up, Stick-fixed

$$h_m = h_n - \frac{\rho Sc}{4m} C_{m_q} \tag{6.35}$$

$$\frac{d\delta_e}{dn} = \frac{aC_L}{C_{m_\alpha} C_{L_{\delta_e}} - C_{m_{\delta_e}} a} \quad h - h_m \tag{6.36}$$

Pull-up, Stick-free

$$h'_m = h'_n - \frac{\rho Sc}{4m} C_{m_q} F \tag{6.77}$$

$$\frac{dF_s}{dn} = G S_e c_e C_{h_\delta} \frac{W}{S} \frac{1}{C_{m_{\delta_e}}} \quad h - h'_m \tag{6.92}$$

$$\frac{dF_s}{dn} = - G q S_e c_e \frac{dC_h}{dn} + W \frac{l_1}{l_2} \tag{6.103}$$

Turn, Stick-fixed

$$h_m = h_n - \frac{\rho Sc}{4m} C_{m_q} \left(1 + \frac{1}{n^2}\right) \tag{6.70}$$

$$\frac{d\delta_e}{dn} = \frac{a C_L}{C_{m_\alpha} C_{L_{\delta_e}} - C_{m_{\delta_e}} a} \quad h - h_m \tag{6.71}$$

Turn, Stick-free

$$h'_m = h'_n - \frac{\rho Sc}{4m} C_{m_q} F \left(1 + \frac{1}{n^2}\right) \tag{6.98}$$

$$\frac{dF_s}{dn} = G \left(S_e \, c_e \, C_{h_\delta}\right)\left(\frac{W}{S}\right)\left(\frac{1}{C_{m_{\delta_e}}}\right)(h - h_m') \qquad (6.99)$$

6.12 MANEUVERING FLIGHT TESTS

The purpose of maneuvering flight is to determine the stick force versus load factor gradients and the forward and aft center of gravity limits for an aircraft in accelerated flight conditions.

To maneuver an aircraft longitudinally from its equilibrium condition, the pilot must apply a force, F_s, on the stick to deflect the elevator an increment, $\Delta\delta_e$. The requirements that must be met during longitudinal maneuvering are covered in MIL-F-8785C, Section 3.2.2.

6.12.1 Military Specification Requirements

MIL-F-8785C specifies the allowable stick-force-per-g gradient during maneuvering flight. It also specifies that the force gradients be approximately linear with pull forces required to maintain or increase normal acceleration. The pilot must also have sufficient aircraft response without excessive cockpit control movement. These requirements and associated requirements of lesser importance provide the legitimate background for good aircraft handling qualities in maneuvering flight.

The backbone of any discussion of maneuvering flight is stick-force-per-g. The amount of force that the pilot must apply to maneuver his aircraft is an important parameter. If the force is very light, a pilot could overstress or overcontrol his aircraft with very little resistance from the aircraft. The T-38, for instance, has a 5 lb/g gradient at 25,000 feet, Mach 0.9, and 20% MAC cg position. With this condition, a ham-fisted pilot could pull 10 g's with only 50 pounds of force and bend or destroy the aircraft. The designer could prevent this possibility by making the pilot exert 100 lb/g to maneuver. This would be highly unsatisfactory for a fighter type aircraft, but perhaps about right for a cargo type aircraft. The mission and type of aircraft must therefore be considered in deciding upon acceptable stick-force-per-g. Furthermore, the force gradient at any normal load factor must be within 50% of the average gradient over the limit load factor. If it took

10 pounds to achieve a 4 g turn, it would be unacceptable for the pilot to reach the limit load factor of 7.33 g's with only a little additional force.

The position of the aircraft's cg is a critical factor in stick-force-per-g consideration. The fore and aft limits of cg position may therefore be established by maneuvering requirements.

6.12.2 Flight Test Methods

There are four general flight test methods for determining maneuvering flight characteristics such as stick force gradients, maneuver points, and permissible cg locations. The names given to these methods vary among test organizations, so make certain that everyone involved is speaking the same language when discussing a particular test method.

6.12.2.1 Stabilized g Method. This method requires holding a constant airspeed and varying the load factor. Establish a trim shot at the test altitude, note the power setting, and climb the aircraft to the upper limit of the altitude band ($\pm 2{,}000$ feet). Reset trim power and roll the aircraft slowly into a 15° bank while lowering the nose slowly. At 15° of bank, the stick force required to maintain the condition is only slightly more than friction and breakout. Record data when the aircraft has been stabilized on an airspeed and bank angle. The attitude indicator should be used to establish the bank angle. Increase the bank angle to 30° and record data when stable. Obtain stabilized data points at 45° and 60° also.

Above 60° the bank angle should be increased so as to obtain 0.5 g increments in load factor. Stabilize at each 0.5 g increment, and record data. Terminate the test when heavy buffet or the limit load factor is reached. Above 2.0 g's only slight increases in bank angle are needed to obtain 0.5 g increments. Bank angle required can be approximated from the relationship $\cos \phi = 1/n$ (Figure 6.12).

Little altitude is lost at the lower bank angles up to approximately 60°, and thus more time may be spent stabilizing the aircraft. At 60° of bank and beyond, altitude is being lost rapidly; therefore every effort should be made to be on speed and well stabilized as rapidly as possible in order to stay within the allowable altitude block (test altitude ±2,000 feet). If the lower altitude limit is approached before reaching limit load factor, climb to the upper limit and continue the test. It is unnecessary to obtain data at precise values of target g since a good spread is all that is necessary. Realistically, data should be obtained within ±0.2 g of target.

$$\cos\phi = \frac{W}{nW} = \frac{1}{n}$$

$$\text{OR} \quad n = \frac{1}{\cos\phi}$$

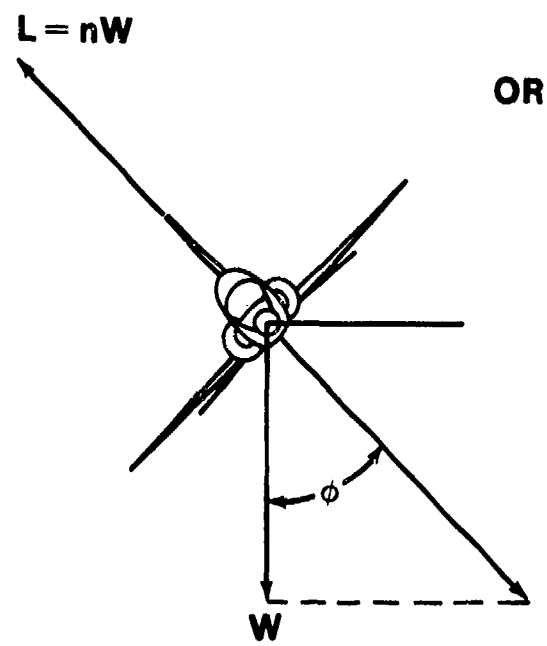


FIGURE 6.12. LOAD FACTOR VERSUS BANK ANGLE RELATIONSHIP

The method of holding airspeed constant within a specified altitude band is recommended where Mach is not of great importance. In regions where Mach may be a primary consideration, every effort should be made to hold Mach rather than airspeed, constant. If power has only a minor effect on the maneuvering stability and trim, altitude loss and the resulting Mach change may be minimized by adding power as load factor is increased. At times, constant Mach is held at the sacrifice of varying airspeed and altitude. For constant Mach tests, a sensitive Mach meter is required or a programmed airspeed/altitude schedule is flown. The stabilized g method is usually used for testing bomber and cargo aircraft and fighters in the power approach configuration.

6.12.2.2 Slowly Varying g Method. Trim the aircraft as before at the desired altitude. Note the power and fly to the upper limit of the altitude band ($\pm$2,000 feet). Reset power at the trimmed value and record data. Increase load factor and bank angle slowly holding airspeed constant until heavy buffet or limit load factor is reached. The rate of g onset should be approximately 0.2 g per second. Airspeed is of primary importance and should be held to within $\pm$2 knots of aim airspeed. Take care not to reverse stick forces during the maneuver.

If the airspeed varies excessively, or the lower altitude limit is approached, turn off the data recorder and repeat the test up to heavy buffet onset or limit load factor.

The greatest error made in this method is bank angle control when beyond 60° of bank. Excessive bank causes the aircraft to traverse the g increments too quickly to be able to accurately hold airspeed. Good bank control is important to obtain the proper g rate of 0.2 g per second. An error is induced in this method since the aircraft is in a descent rather than level flight. Stick forces to obtain a specific g will be less than in level flight. Fortunately, this error is in the conservative direction.

The slowly varying g method may be more applicable to fighter aircraft. Often a combination of the two methods is used in which the stabilized g method is followed up to 60° bank angle and then the slowly varying g method is used until heavy buffet or limit load factor is reached.

6.12.2.3 Constant g Method. Stabilize and trim at the desired altitude and maximum airspeed for the test. Establish a constant g turn. Record data and climb or descend to obtain a two to five knot per second airspeed bleed rate at the desired constant load factor. Normally climbs are used to obtain a bleed rate at low load factors and descents are used to obtain a bleed rate at high load factors. For high thrust-to-weight ratio aircraft at low altitudes, the maneuver may have to be initiated at reduced power to avoid rapidly traversing the altitude band. Maintaining the aim load factor is the primary requirement while establishing the bleed rate is secondary. Keep the aircraft within the altitude band of $\pm 2,000$ feet. Note the airspeed as the aircraft flies out of the altitude band. Return to the altitude band and start at an airspeed above the previously noted airspeed so that continuity of g and airspeed can be maintained. Note airspeed at buffet onset and the g break (when aim load factor can no longer be maintained). The buffet and stall flight envelope is determined or verified by this test method. Repeat the maneuver 0.5 g increments at high altitudes and 1 g increments at low altitudes.

6.12.2.4 Symmetrical Pull Up Method. Trim the aircraft at the desired test altitude and airspeed. Climb to an altitude above the test altitude using power as required. Reset trim power and push over into a dive. The dive angle and lead airspeed are functions of the target load factor.

Maneuver the aircraft to a lead point that will place it at a given constant load factor while passing through the test altitude at the test airspeed. Two methods may be used which yield the same results. The idea behind both methods is to minimize the number of variables.

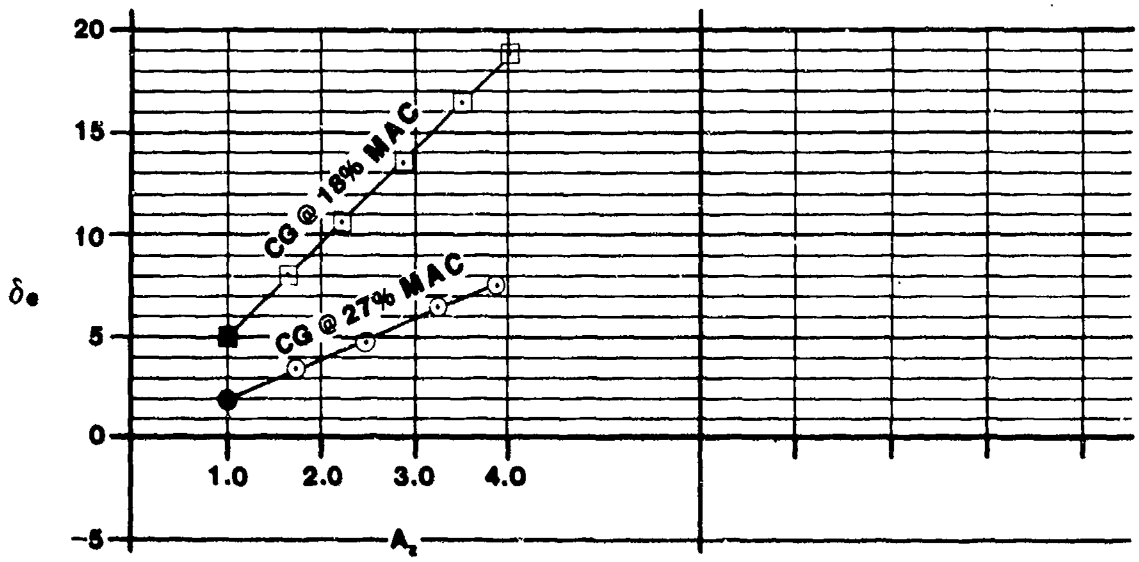
a. Method A - Using a variable dive angle and fixed lead airspeed, smoothly increase back pressure so that the airspeed is stabilizing as you pass through 10° nose low. For a given g and constant lead airspeed, there is a specific dive angle which will allow you to stabilize the airspeed as you pass through 10° of dive.

b. Method B - Using a constant dive angle and adjusting lead airspeed, smoothly apply back pressure to establish the target g. If the proper lead airspeed is used, the airspeed will stabilize as the target g is established.

Using either method, the aircraft should pass through level flight ($\pm 10^{\circ}$ from horizontal) just as the airspeed reaches the trim airspeed with aim g loading and steady stick forces. Be sure to freeze the stick. Achieving the trim airspeed through level flight, $\pm 10^{\circ}$, and holding steady stick forces to give a steady pitch rate are of primary importance. The variation in altitude ($\pm$1,000 feet) at the pull up is less important. The g loading need not be exact ($\pm$0.2g) but must be steady. Record data as the aircraft passes through level flight $\pm 10^{\circ}$.

PROBLEMS

6.1. Given the following data, find the stick fixed maneuver point:



6.2. A. Compute C_{m_q} for the T-33A

ℓ_T = 16.5 ft

c = 6.7 ft

η_T = 1.0

a_T = 3.5 RAD^{-1}

S_T = 45.5 ft^2

S = 235 ft^2

B. Compute C_{m_q} for the F-4C

ℓ_T = 21 ft

c = 16.0 ft

η_T = 1.0

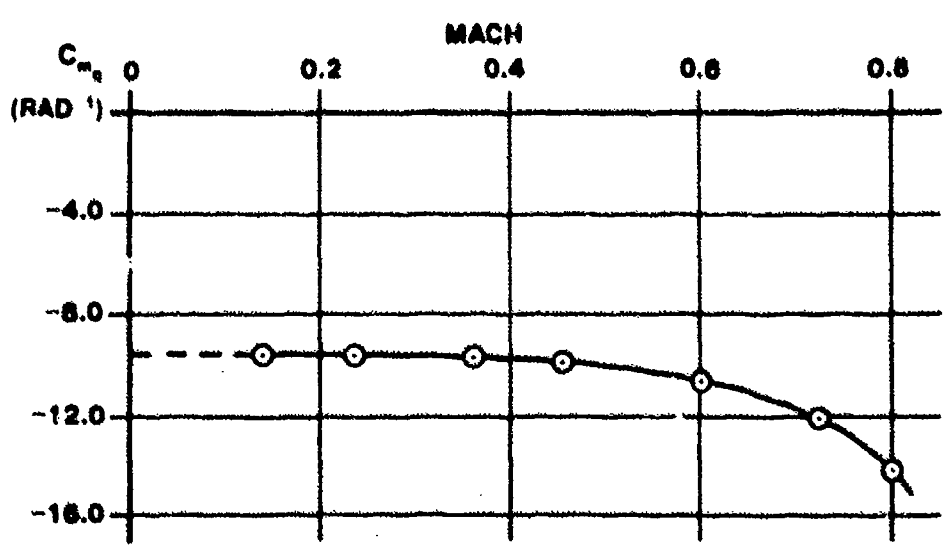
a_T = 3.0 RAD^{-1}

S_T = 96 ft^2

S = 530 ft^2

C. If C_{m_q} is in the range of -6.5 to -9 for most aircraft, which aircraft (T-33 or F-4C) would you anticipate might have maneuvering flight and dynamic damping problems?

D. How does C_{m_q} computed for the T-33A in 2.A on the last page compare to real wind tunnel data shown below?

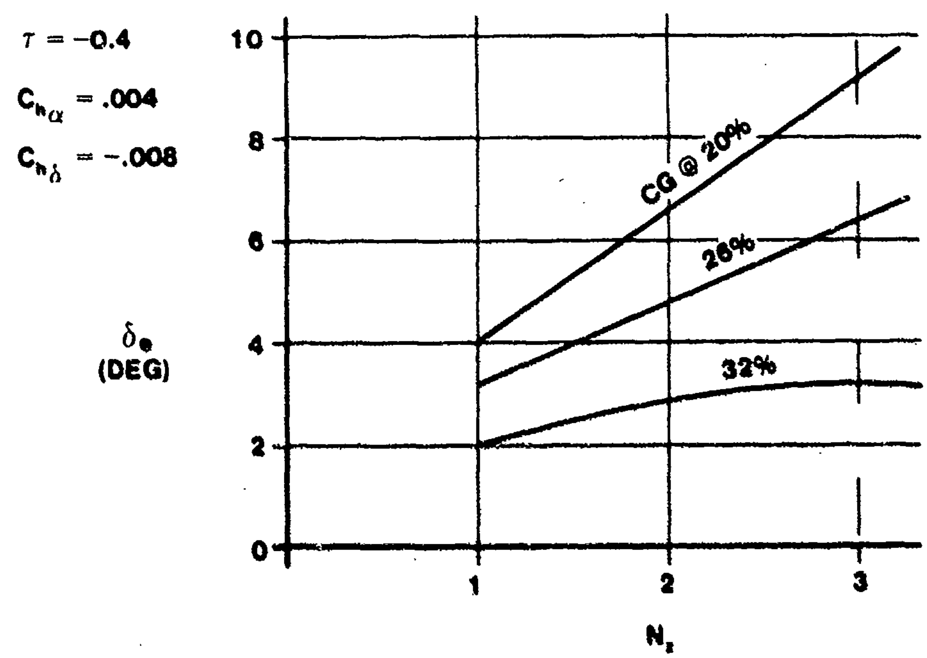


6.3. Given that an aircraft with a reversible flight control system has stick-fixed and stick-free neutral points of 30% and 28% respectively, the following is flight test data using the stabilized method (which is a technique involving stabilized turns).

$\tau = -0.4$

$C_{h_\alpha} = .004$

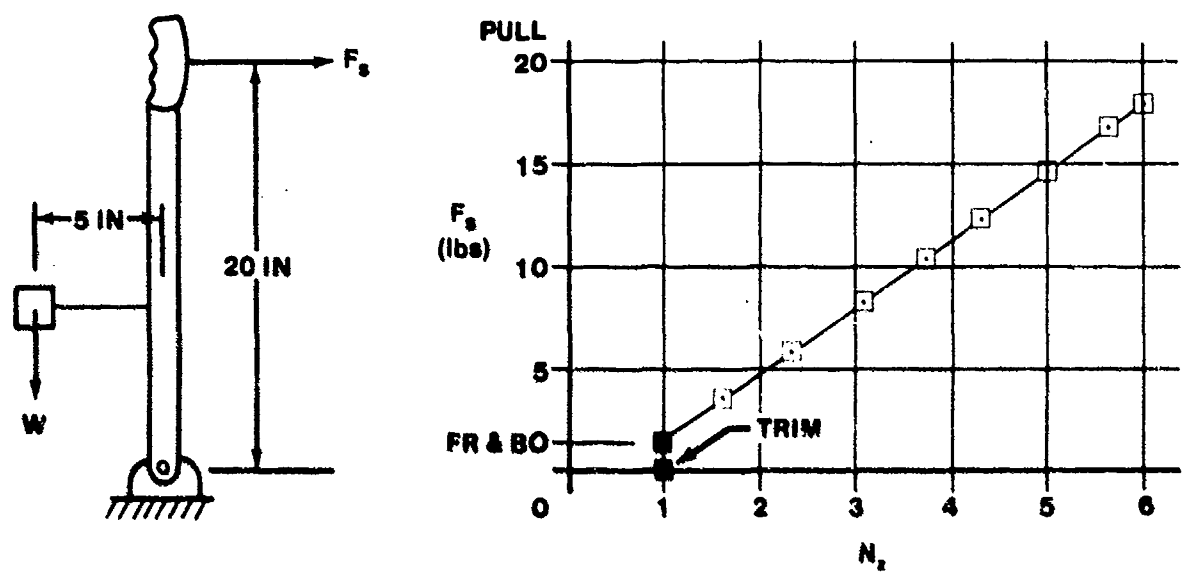
$C_{h_\delta} = -.008$



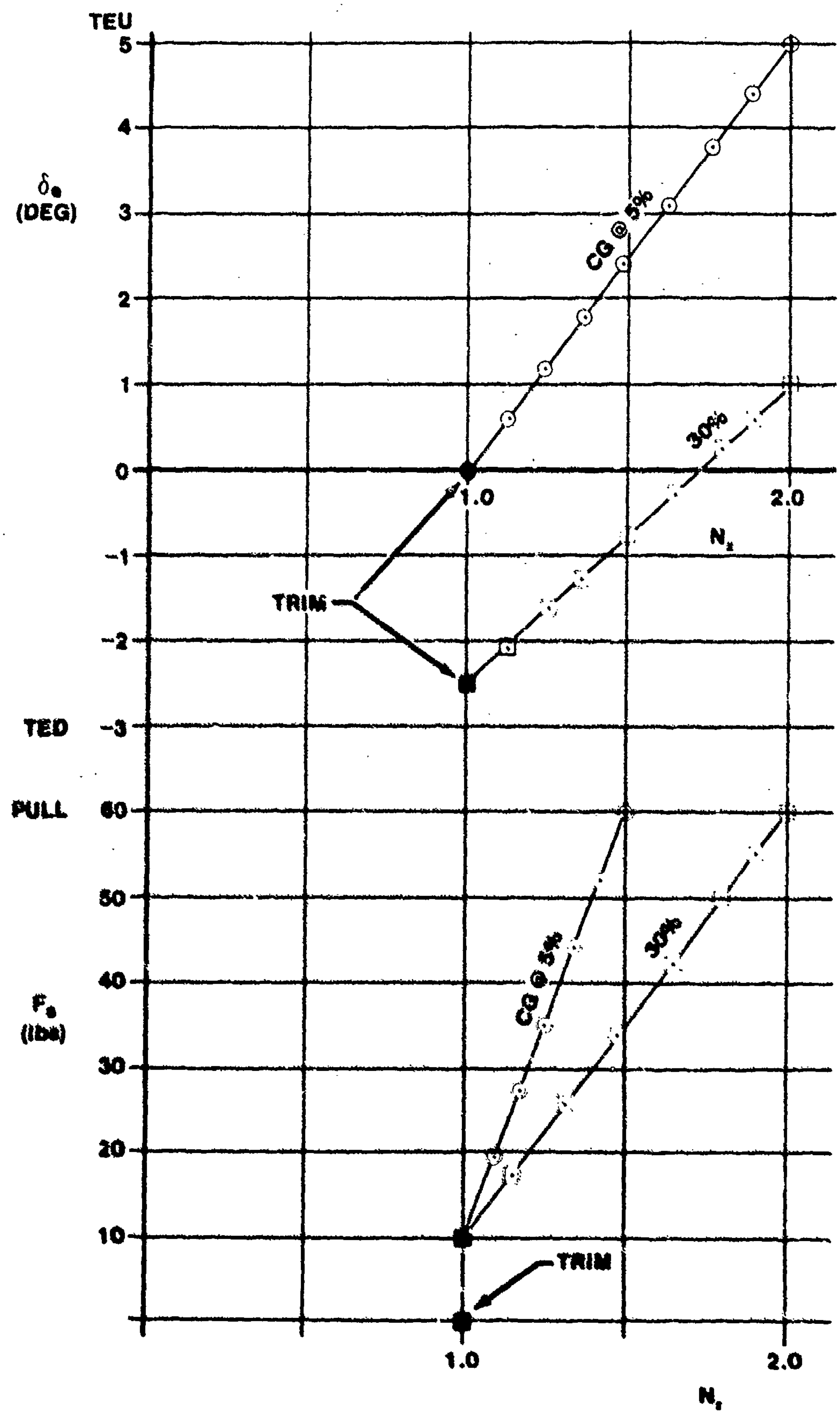
What is the value of the stick-free maneuver point at n = 3?

6.4. At the same trim and test point, a blue T-33 is in a stabilized 3 g pull up maneuver, and a red T-33 is in a stabilized 2 g steady turn maneuver. Both aircraft are standard T-33's (except for the color) and have identical gross weights. Which one has the higher-stick-force-per-g? Show why.

6.5. For a given set of conditions, an RF-4C had a stick force gradient as shown below. Compute the weight of the bobweight needed to increase this gradient to a minimum of five pounds per g.



6.6. Given the transport aircraft data on this page, find the stick-fixed and stick-free maneuver points. What are the stick-fixed and stick-free maneuver margins? What is the friction and breakout force?



6.7. Given the flight test data below:

A. Calculate the stick-free maneuver point.

B. Calculate the stick-fixed maneuver point.

C. If the minimum desired dF_s/dn is 3 lb/g, calculate the aft cg limit.

D. Calculate the maneuver margin at the aft cg limit.

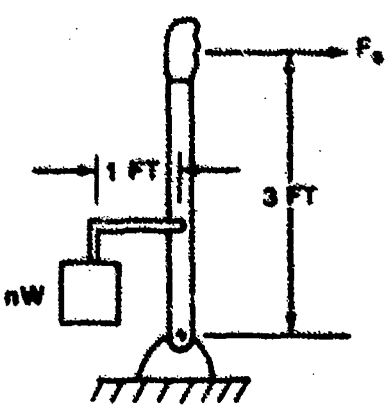
E. If the stick-fixed neutral point was 48% MAC and given the following data, calculate an estimate for C_{m_q}. NOTE: C_{m_q} is very sensitive to h_m locations so an estimate of C_{m_q} calculated from aircraft geometry is probably as good, or better, than this flight test derived value. Density at test altitude, ρ, is 0.002 slugs/ft^3.

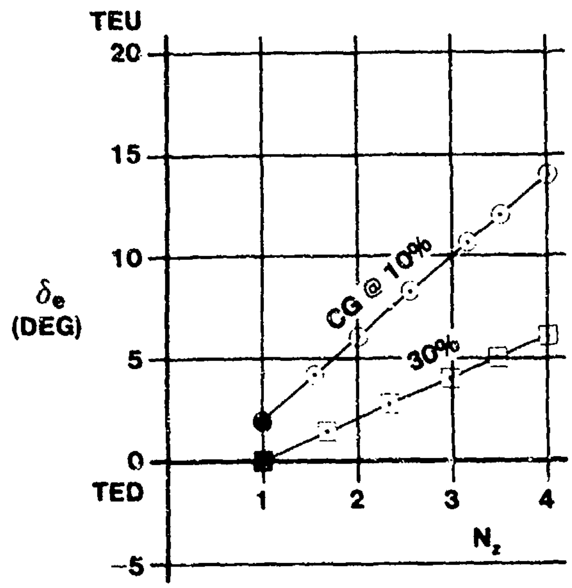
$S = 300\ ft^2$

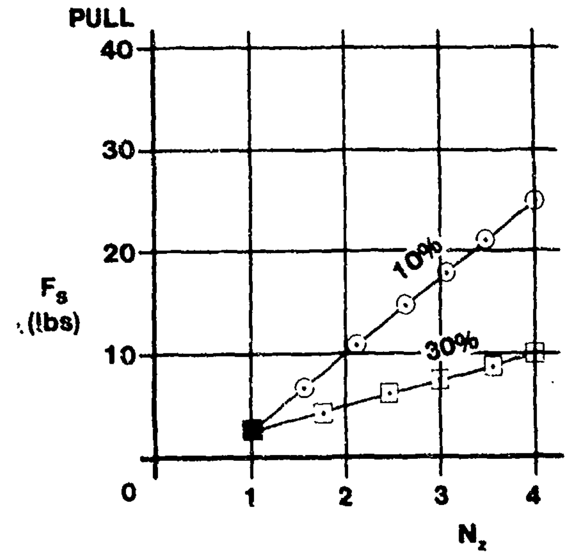
$c = 7\ ft$

$W = 18{,}000\ lbs$

F. A new internal fuel tank arrangement is planned for the aircraft which will move the aircraft cg to 40% MAC. If a minimum stick-force-per-g of 3 lb/g is required at this new cg location, what size bobweight is required as a "fix"?



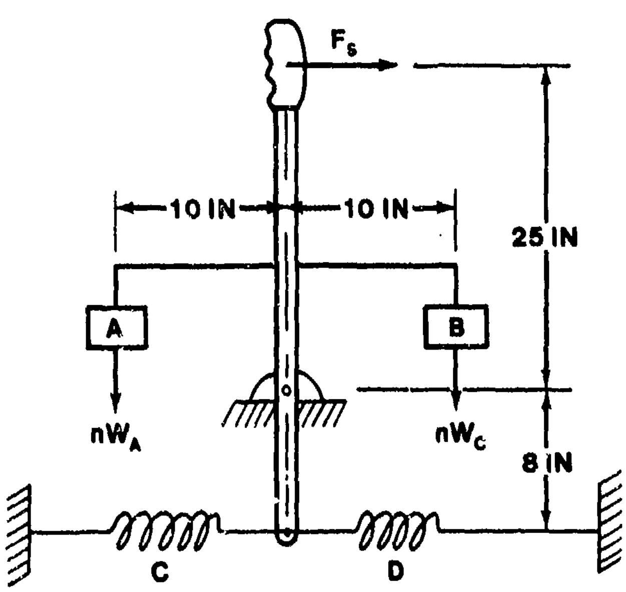




G. What downspring tension, T, is required to obtain the same minimum dF_s/dn determined in Part F.?

H. The forward cg limit is to remain at 10% MAC. What is the maximum stick-force-per-g the aircraft will have after bobweight installation ?

6.8. An aircraft has a stick-force-per-g of one lb/g at a cg of 40% MAC. It is desired that the aircraft have a three lb/g maneuvering stick force gradient at the same cg without changing the aircraft's speed stability. You are given the choice of bobweight A or B, and/or springs C and D. Which bobweights and springs should be used and what should their sizes and tensions be?



T_C AND T_D ARE CONSTANTS

6.9. Given the flight test data and MIL SPEC requirements shown below:

FIG. 1

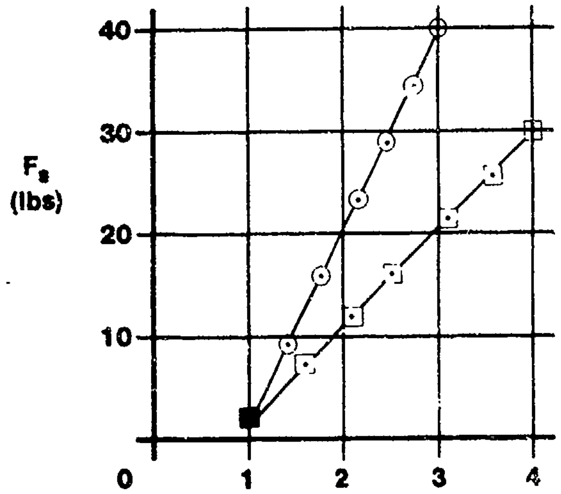
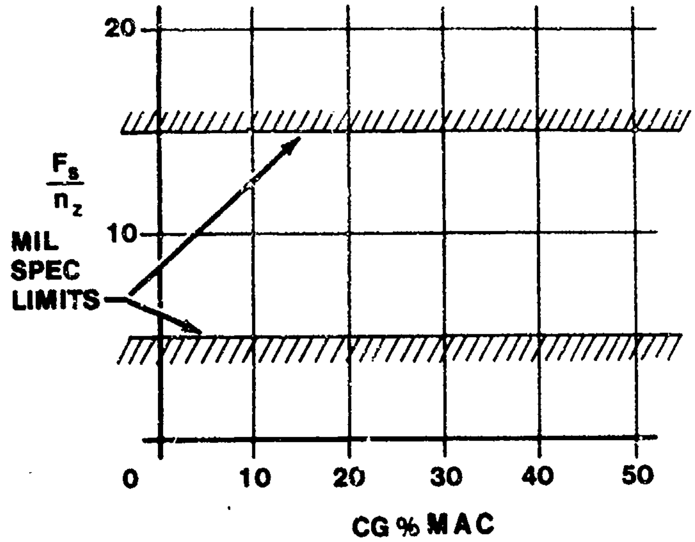


FIG. 2



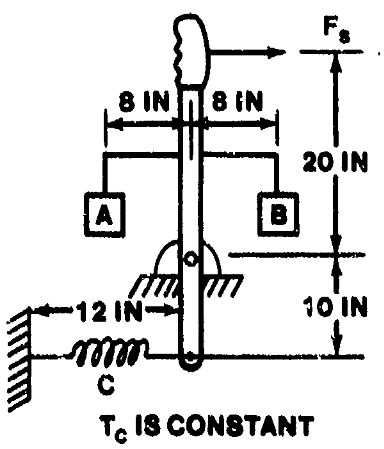
A. One set of data on Figure 1 was taken at FWD cg (15% MAC) and the other at an AFT cg (30% MAC). Label the curves FWD and AFT properly.

B. Determine the stick-free maneuver point.

C. What is the AFT cg limit to meet the minimum MIL SPEC stick force per g shown on Figure 2?

D. What is the FWD cg limit to meet the maximum MIL SPEC stick force per g?

E. Given the control stick geometry shown below and choice of either bobweight A, bobweight B, or upspring C, which weight or spring and what size weight or spring is needed to just meet the maximum MIL SPEC stick force per g requirement at cg of 10% MAC?



6.10. Read the question and circle the correct answer, true (T) or false (F).

T F The maneuver point should always be behind the neutral point.

T F C_{m_q} is always positive.

T F Pitch rate decreases stability.

T F The distance between the neutral point and the maneuver point is a function of aircraft geometry, altitude, and aircraft weight.

T F The additional elevator requirement under aircraft bending gives an increase in stability.

T F Maneuvering flight data can be collected in turns but not in pull-ups.

T F Theory says that stick-force-per-g is the same at all airspeeds for a given cg.

T F FWD cg position may be limited by a maximum value of dF_s/dn.

T F Imposing a minimum value of dF_s/dn as the MIL SPEC does prevent the permissible aircraft cg from being behind the maneuver point.

T F The effect of either a spring or a bobweight is the same on stick-force-per-g.

T F A downspring exerts a constant force on the stick independent of load factor.

T F A bobweight exerts a constant force on the stick independent of load factor.

T F A downspring effects maneuvering stick force gradient.

T F A bobweight effects maneuvering stick force gradient.

T F Aerodynamic balancing effects the stick-free maneuver point location.

T F The stick-free maneuver point is normally ahead of the stick-fixed maneuver point (tail-to-the-rear aircraft).

T F A downspring changes the location of the maneuver point.

T F In "second order differential equation (mass-spring-damper) terms," C_{m_α} is analogous to damping.

T F Although both stick-fixed and stick-free neutral points can be defined, only a stick-free maneuver point exists.

T F C_{m_q} can be obtained from maneuvering flight tests.

T F The wing is the largest contributor to pitch damping.

T F C_{m_α} is commonly called "pitch damping" in informed aeronautical circles.

T F In "second order differential equation (mass-spring-damper) terms," C_{m_q} is analogous to the spring.

T F In subsonic flight (no Mach effects) dC_m/dQ is constant.

T F In subsonic flight (no Mach effects), C_{m_q} is a function of velocity.

T F Maneuvering stick-force gradient data obtained from turning flight tests is identical to that obtained from pull-up flight tests.

T F dC_m/dQ and C_{m_q} are identical.

T F V_H is considered constant even though cg is allowed to vary.

T F Increasing stability decreases maneuverability.

T F $d\delta_e/dn$ is the same for both pull-up and turn maneuvers.

T F Aerodynamic balancing does not effect dF_s/dn.

T F FWD and AFT cg travel may be limited by maximum and minimum values of stick-force-per-g.

T F Maneuvering stick-force gradient and stick force-per-g are the same.

T F The same curve can be faired through maneuvering flight test data obtained by the pull-up and turn techniques.

ANSWERS

6.1. $h_m = 33\%$

6.2. A. $C_{m_q} = -8.2$ per rad for tail

$C_{m_q} = -9.0$ per rad for aircraft

B. $C_{m_q} = -1.9$ per rad - tail

$C_{m_q} = -2.1$ per rad - aircraft

6.3. $h'_m = 0.296$

6.4. Red has higher dF_s/dn

6.5. W = 6.8 lb

6.6. $h_m = 0.66$

$h'_m = 0.44$

cg	Maneuver margin Fixed	Maneuver margin Free
15%	0.51	0.29
30%	0.3[illegible]	0.14

Friction + Breakout = 10 lb

6.7. A. $h'_m = 0.40$

B. $h_m = 0.50$

C. Aft cg = 0.28

D. Man mar. fixed = 0.22
Man mar. free = 0.12

E. $C_{m_q} = -10.64$ per rad

F. $W = 9$ lb

G. Can't be done with spring.

H. $F_s/g = 10.3$ lb/g

6.8. $W_A = 5$ lb

Must be offset by $T_D = 6.25$ lb at $n = 1$

6.9. B. $h'_m = 0.43$

C. Aft lim = 37%

D. Fwd lim = 20%

E. $W_B = 10$ lb

CHAPTER 7

LATERAL-DIRECTIONAL STATIC STABILITY

7.1 INTRODUCTION

Your study of flying qualities to date has been concerned with the stability of the airplane flying in equilibrium on symmetrical flight paths. More specifically, you have been concerned with the problem of providing control over the airplane's angle of attack and thereby its lift coefficient, and with ensuring static stability of this angle of attack.

This course considers the characteristics of the airplane when its flight path no longer lies in the plane of symmetry. This means that the relative wind will make some angle to the aircraft centerline which we define as β. The motions which result from β being applied to the airplane are motion along the y-axis and motion about the x and z axes. These motions can be described by the following equations of aircraft lateral-directional motion

$$F_y = m\dot{v} + m\,r\,U - p\,w\,m \tag{7.1}$$

$$G_x = \dot{p}I_x + qr\,(I_z - I_y) - (\dot{r} + pq)\,I_{xz} \tag{7.2}$$

$$G_z = \dot{r}I_z + pq\,(I_y - I_x) + (qr - \dot{p})\,I_{xz} \tag{7.3}$$

where the right side of the equation represents the response of an aircraft to the applied forces and moments on the left side. These applied forces and moments are composed primarily of contributions from aerodynamic forces and moments, direct thrust, gravity, and gyroscopic moments. Since the aerodynamic forces and the moments are by far the most important, we shall consider the other contributions as negligible or as having been eliminated through proper design.

It has been shown in Equations of Motion that when operating under a small disturbance assumption, aircraft lateral-directional motion can be considered independent of longitudinal motion and can be considered as a function of the following variables

$$(Y, \mathcal{L}, \mathcal{N}) = f\,(\beta, \dot{\beta}, p, r, \delta_a, \delta_r) \tag{7.4}$$

The ensuing analysis is concerned with the question of lateral-directional static stability or the initial tendency of an airplane to return to stabilized flight after being perturbed in sideslip or roll. This will be determined by the values of the yawing and rolling moments ($\mathcal{N}$ and $\mathcal{L}$). Since the side force equation governs only the aircraft translatory response and has no effect on the angular motion, the side force equation will not be considered.

The two remaining aerodynamic functions can be expressed in terms of non-dimensional stability derivatives, angular rates and angular displacements

$$C_n = C_{n_\beta}\beta + C_{n_{\dot\beta}}\hat{\dot\beta} + C_{n_p}\hat{p} + C_{n_r}\hat{r} + C_{n_{\delta_a}}\delta_a + C_{n_{\delta_r}}\delta_r \qquad (7.5)$$

$$C_\ell = C_{\ell_\beta}\beta + C_{\ell_{\dot\beta}}\hat{\dot\beta} + C_{\ell_p}\hat{p} + C_{\ell_r}\hat{r} + C_{\ell_{\delta_a}}\delta_a + C_{\ell_{\delta_r}}\delta_r \qquad (7.6)$$

The analysis of aircraft lateral-directional motion is based on these two equations. A cursory examination of these two equations reveals that they are "cross-coupled." That is, C_{n_p} and $C_{n_{\delta_a}}$ are found in Equation 7.5, while C_{ℓ_r} and $C_{\ell_{\delta_r}}$ are present in the lateral Equation 7.6. It is for this reason that aircraft lateral and directional motions must be considered together - each one influences the other.

7.2 TERMINOLOGY

Since considerable confusion can arise if the terms sideslip and yaw are misunderstood, we shall define them before proceeding further.

Sideslip is defined as the angle the relative wind makes with the longitudinal axis of the airplane. From Figure 7.1 we see that the angle of sideslip, β, is equal to the arcsin (v/V), or for the small angles normally encountered in flight, $\beta \cong v/V$. By definition, β is positive when the relative wind is to the right of the geometric longitudinal axis of the airplane (i.e., when wind is in the right ear).

Yaw angle, ψ, is defined as the angular displacement of the airplane's longitudinal axis in the horizontal plane from some arbitrary direction taken as zero at some instant in time (Figure 7.1). Note that for a curved flight path, yaw angle does not equal sideslip angle. For example, in a 360° turn, the airplane yaws through 360°, but may not develop any sideslip during the maneuver, if the turn is perfectly coordinated.

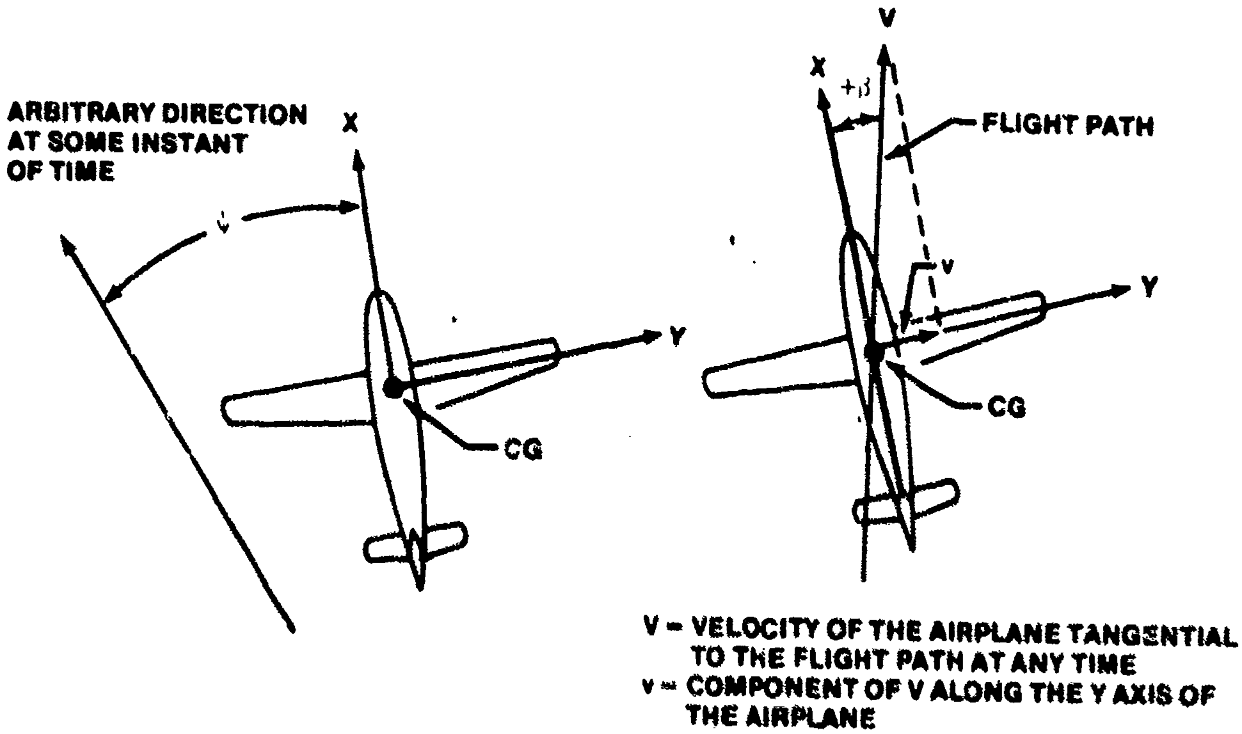


FIGURE 7.1. YAW AND SIDESLIP ANGLE

With these definitions of yaw and sideslip in mind, each of the stability derivatives comprising Equations 7.5 and 7.6 may be analyzed.

7.3 DIRECTIONAL STABILITY

In general, it is advantageous to fly an airplane at zero sideslip, and the easier it is for a pilot to do this, the better he will like the flying qualities of his airplane. The problem of directional stability and control, then, is first to ensure that the airplane will tend to remain in equilibrium at zero sideslip, and second to provide a control to maintain zero sideslip during maneuvers that introduce moments tending to produce sideslip. The stability derivatives which contribute to static directional stability are those comprising Equation 7.5. A summary of these derivatives is shown in Table 7.1.

TABLE 7.1
DIRECTIONAL STABILITY AND
CONTROL DERIVATIVES

DERIVATIVE	NAME	SIGN FOR A STABLE AIRCRAFT	CONTRIBUTING PARTS OF AIRCRAFT
C_{n_β}	Static Directional Stability or Weathercock Stability	(+)	Tail, Fuselage, Wing
$C_{n_{\dot\beta}}$	Lag Effects	(-)	Tail
C_{n_p}	Cross-Coupling	(+)	Wing, Tail
C_{n_r}	Yaw Damping	(-)	Tail, Wing, Fuselage
$C_{n_{\delta_a}}$	Adverse or Complimentary Yaw	"0" or slightly (-)	Lateral Control
$C_{n_{\delta_r}}$	Rudder Power	(+)	Rudder Control

7.3.1 C_{n_β} Static Directional Stability or Weathercock Stability

Static directional stability is defined as the initial tendency of an aircraft to return to, or depart from, its equilibrium angle of sideslip (normally zero) when disturbed. Although the static directional stability of an aircraft is determined through consideration of all the terms in Equation 7.5, C_{n_β} is often referred to as "static directional stability" because it is the predominant term.

When an aircraft is placed in a sideslip, aerodynamic forces develop which create moments about all three axes. The moments created about the z-axis tend to turn the nose of the aircraft into or away from the relative wind. The aircraft has positive directional stability if the moments created by a sideslip angle tend to align the nose of the aircraft with the relative wind.

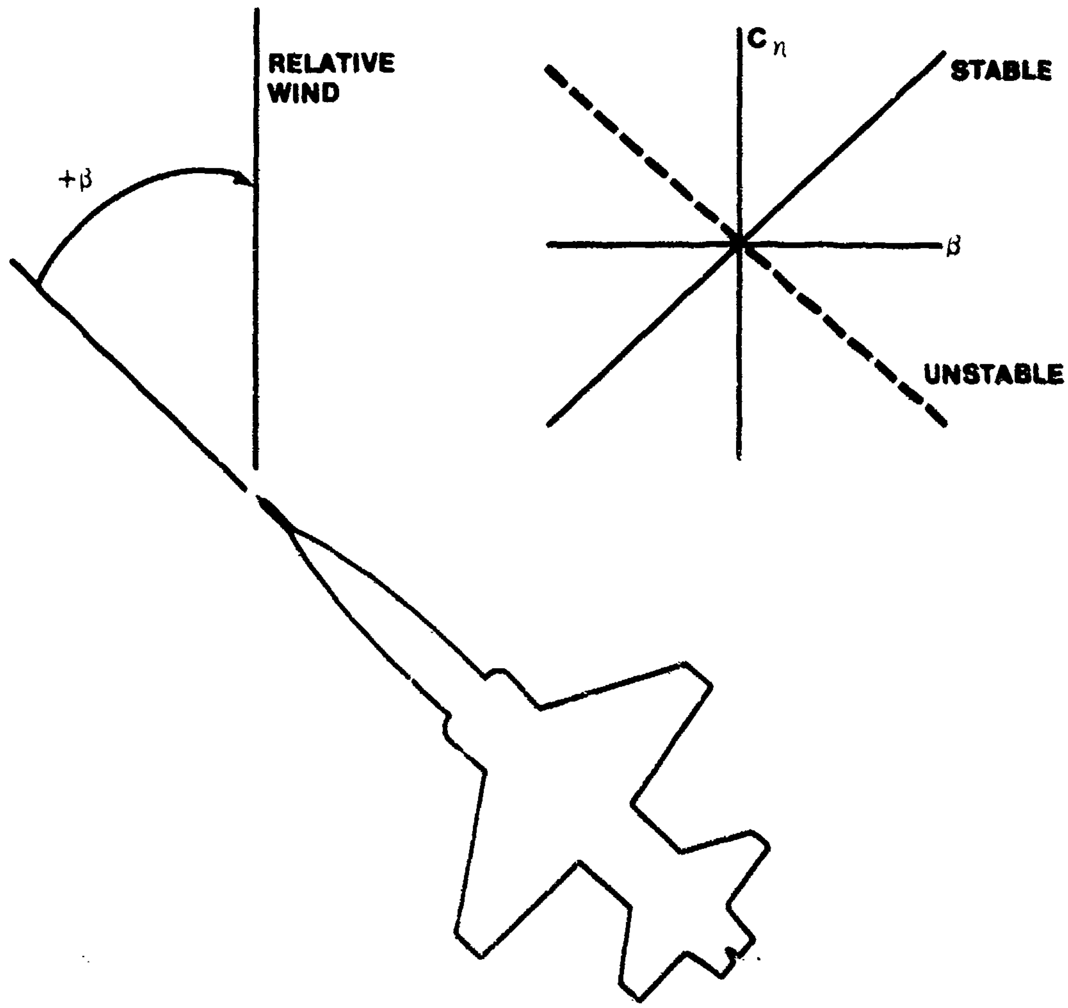


FIGURE 7.2. STATIC DIRECTIONAL STABILITY

In Figure 7.2 the aircraft is in a right sideslip. It is statically stable if it develops yawing moments that tend to align it with the relative wind, or in this case, right (positive) yawing moments. Therefore, an aircraft is statically directionally stable if it develops positive yawing moments with a positive increase in sideslip. Thus, the slope of a plot of yawing moment coefficient, C_n, versus sideslip, β, is a quantitative measure of the static directional stability that an aircraft possesses. This plot would normally be determined from wind tunnel results.

The total value of the directional stability derivative, C_{n_β}, at any sideslip angle, is determined primarily by contributions from the vertical tail, the fuselage, and the wing. These contributions will be discussed separately.

7.3.1.1 Vertical Tail Contribution to C_{n_β}. The vertical tail is the primary source of directional stability for virtually all aircraft. When the aircraft is yawed, the angle of attack of the vertical tail is changed. This change of angle of attack produces a change in lift on the vertical tail, and thus a yawing moment about the Z-axis.

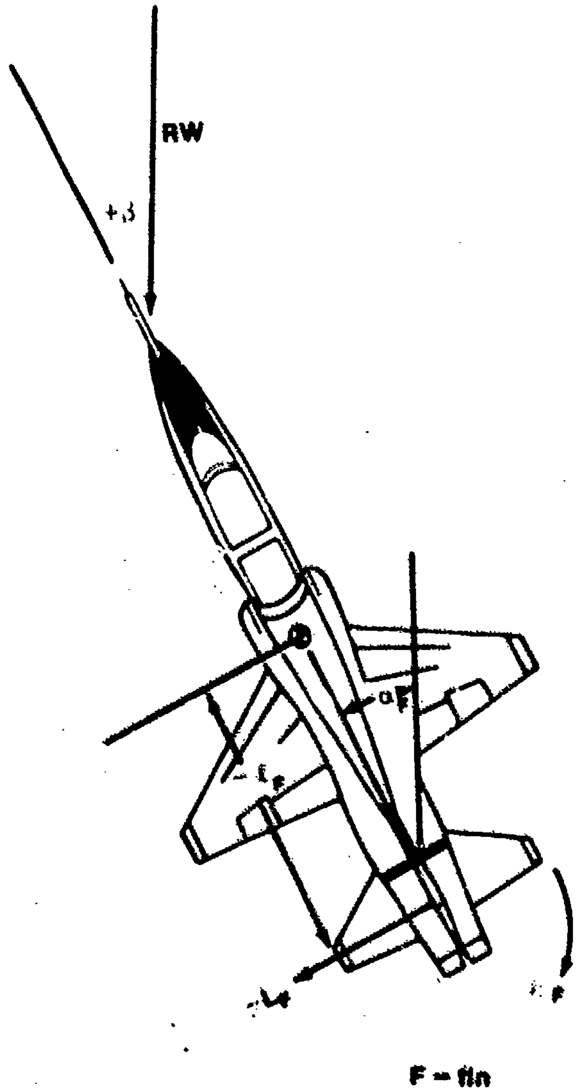


FIGURE 7.3. VERTICAL TAIL CONTRIBUTION TO C_{n_β}

Referring to Figure 7.3, the yawing moment produced by the tail is

$$n_F = (-\ell_F)\ (-L_F) = \ell_F\ L_F \tag{7.7}$$

The minus signs in this equation arise from the use of the sign convention adopted in the study of aircraft equations of motion. Forces to the left and distances behind the aircraft cg are negative.

As in other aerodynamic considerations, it is convenient to consider yawing moments in coefficient form so that static directional stability can be evaluated independent of weight, altitude and speed. Putting Equation 7.7 in coefficient form

$$C_{n_F} = \frac{\ell_F L_F}{q_W S_W b_W} = \frac{\ell_F C_{L_F} q_F S_F}{q_W S_W b_W} \quad \text{[where } q = 1/2\ \rho V^2 \text{ and } w = \text{wing]} \tag{7.8}$$

Vertical tail volume ratio, V_v, is defined as

$$V_v = \frac{S_F \ell_F}{S_W b_W} = \frac{(+)\ (-)}{(+)\ (+)} = (-) \text{ for tail to the rear aircraft} \tag{7.9}$$

$$= \frac{(+)\ (+)}{(+)\ (+)} = (+) \text{ for tail to the front aircraft}$$

Making this substitution into Equation 7.8

$$C_{n_F} = \frac{C_{L_F} q_F V_v}{q_W} \tag{7.10}$$

For a propeller-driven aircraft, q_W may be less than or greater than q_F. However, for a jet aircraft, these two quantities are normally equal. Thus, for a jet aircraft, $q_F/q_W = 1$ and Equation 7.10 becomes

$$C_{n_F} = C_{L_F}\ V_v \tag{7.11}$$

The lift curve for a vertical tail is presented in Figure 7.4.

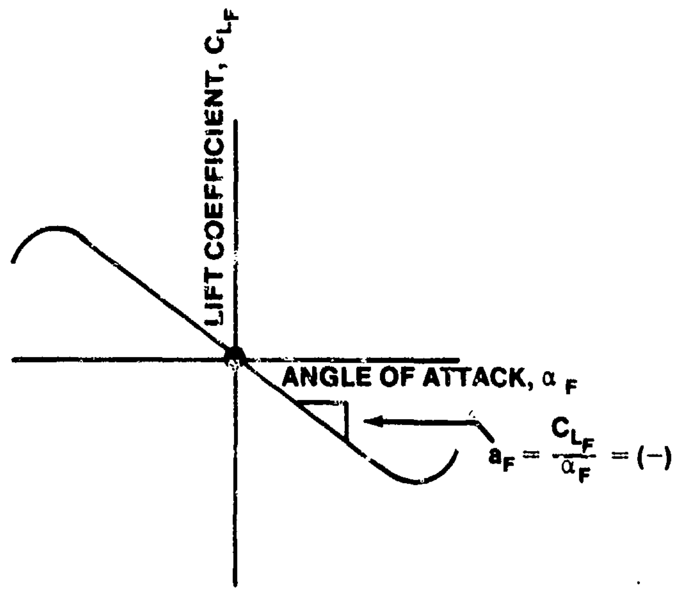


FIGURE 7.4. LIFT CURVE FOR VERTICAL TAIL

The negative slope is a result of the sign convention used (Figure 7.3). When the relative wind is displaced to the right of the fuselage reference line, the vertical tail is placed at a positive angle of attack. However, this results in a lift force to the left, or a negative lift. Thus, the sign of the lift curve slope of a vertical tail, a_F, will always be negative below the stall. Substituting $C_{L_F} = a_F \alpha_F$ into Equation 7.11 yields

$$C_{n_F} = a_F \alpha_F V_V \tag{7.12}$$

The angle of attack of the vertical tail, α_F, is not merely β. If the vertical tail were placed alone in an airstream, then α_F would be equal to β. However, when the tail is installed on an aircraft, changes in both magnitude and direction of the local flow at the tail take place. These changes may be caused by a propeller slipstream, or by the wing and the fuselage when the airplane is yawed. The angular deflection is allowed for by introducing the sidewash angle, σ, analogous to the downwash angle, ε. The value of σ is very difficult to predict, therefore suitable wind tunnel tests are

required. The sign of σ is defined as positive if it causes α_F to be less than β, which is normally the case since the fuselage tries to straighten the air which causes α_F to be less than β. Thus,

$$\sigma = \beta - \alpha_F \qquad (7.13)$$

Substituting α_F from Equation 7.13 into Equation 7.12

$$C_{n_F} = a_F V_V (\beta - \sigma) \qquad (7.14)$$

The contribution of the vertical tail to directional stability is found by examining the change in C_{n_F} with a change in sideslip angle, β.

(−) (−) (+) = (+) for tail to rear aircraft

$$\frac{\partial C_{n_F}}{\partial \beta} = \left[C_{n_{\beta\,(\text{Tail})}}\right]_{\text{Fixed}} = V_V \; a_F \overbrace{\underbrace{\left(1 - \frac{\partial \sigma}{\partial \beta}\right)}}\qquad (7.15)$$

(+) (−) (+) = (−) for tail to front aircraft

The subscript "fixed" is added to emphasize that, thus far, the vertical tail has been considered as a surface with no movable parts, i.e., the rudder is "fixed."

Equation 7.15 reveals that the vertical tail contribution to directional stability can only be changed by varying the vertical tail volume ratio, V_V, or the vertical tail lift curve slope, a_F. The vertical tail volume ratio can be changed by varying the size of the vertical tail, or its distance from the aircraft cg. The vertical tail lift curve slope can be changed by altering the basic airfoil section of the vertical tail, or by end plating the vertical fin. An end plate on the top of the vertical tail is a relatively minor modification, and yet it increases the directional stability of the aircraft

significantly at lower sideslip angles. This has been used on the T-38 (Figure 7.5). The entire stabilator on the F-104 acts as an end plate (Figure 7.6) and, therefore, adds greatly to the directional stability of the aircraft.

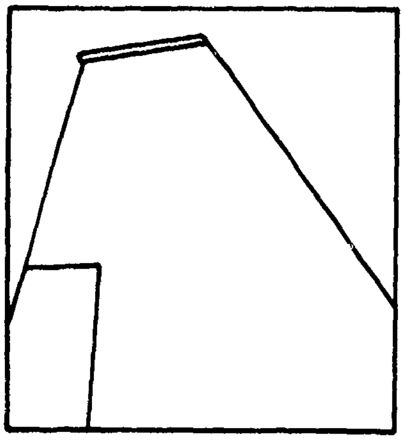


FIGURE 7.5. T-38 END PLATE

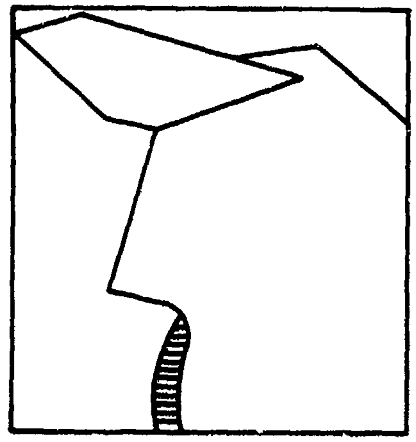


FIGURE 7.6. F-104 END PLATE

The end plate increases the effective aspect ratio of the vertical tail. As with any airfoil, this change in aspect ratio produces a change in the lift curve slope of the airfoil as shown in Figure 7.7.

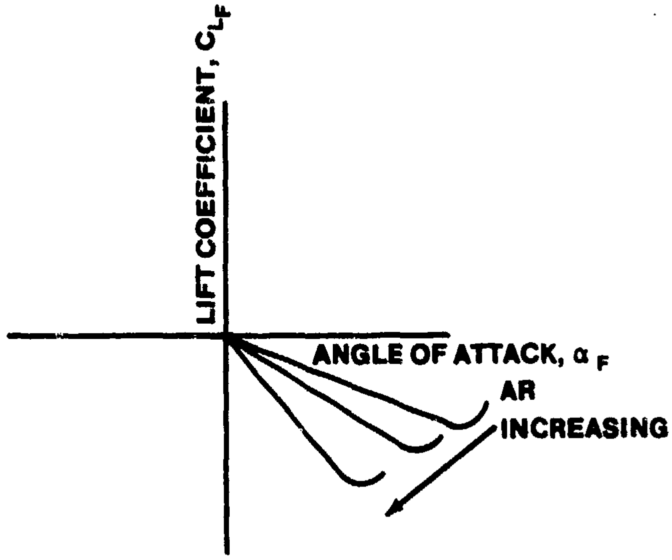


FIGURE 7.7. EFFECTS OF END PLATING

As the aspect ratio is increased, the α_F for stall is decreased. Thus, if the aspect ratio of the vertical tail is too high, the vertical tail will stall at low sideslip angles, and a large decrease in directional stability will occur.

7.3.1.2 Fuselage Contribution to C_{n_β}. The primary source of directional instability is the aircraft fuselage. This is so because the subsonic aerodynamic center of a typical fuselage usually lies ahead of the aircraft center of gravity. Therefore, a positive sideslip angle will produce a negative yawing moment about the cg causing C_{n_β} (fuselage) to be negative or destabilizing (Figure 7.8).

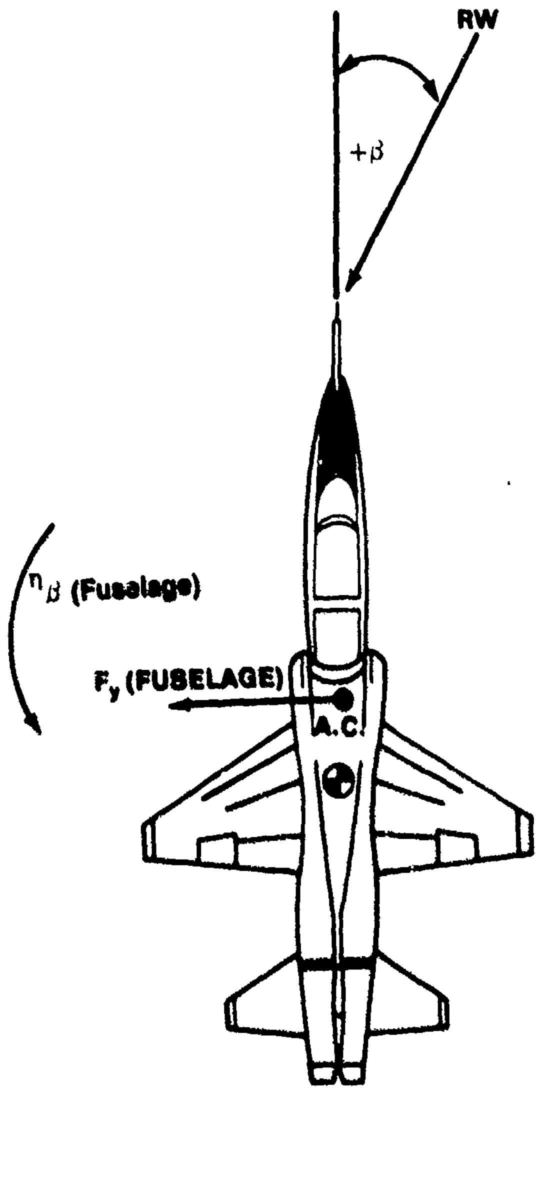


FIGURE 7.8. FUSELAGE CONTRIBUTION TO C_{n_β}

The destabilizing influence of the fuselage diminishes at large sideslip angles due to a decrease in lift as the fuselage stall angle of attack is exceeded and also due to an increase in parasite drag acting at the center of the equivalent parasite area which is located aft of the cg.

If the overall directional stability of an aircraft becomes too low, the fuselage-tail combination can be made more stabilizing by adding a dorsal fin or a ventral fin. A dorsal fin was added to the C-123, and a ventral fin was added to the F-104 to improve static directional stability.

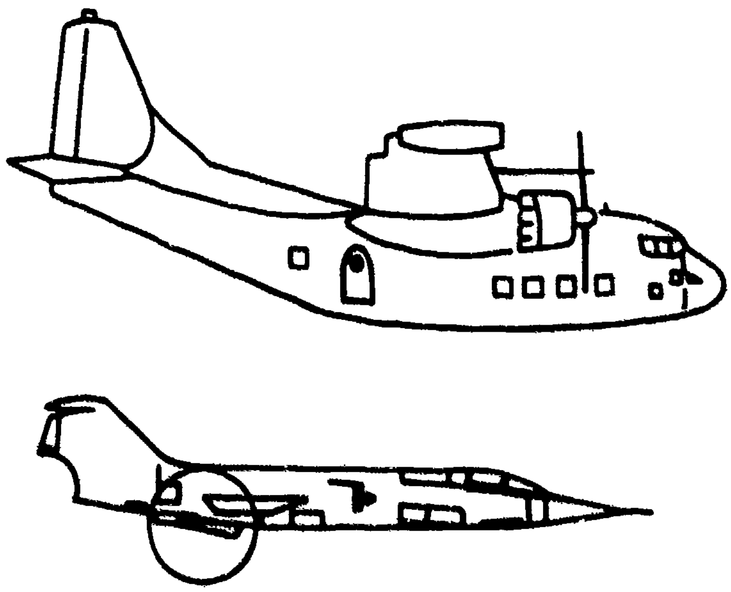


FIGURE 7.9. APPLICATIONS OF DORSAL AND VENTRAL FINS

The addition of a dorsal fin decreases the effective aspect ratio of the tail; therefore, a higher sideslip angle can be attained before the vertical fin stalls. Unfortunately this may occur at the expense of a loss in C_{L_F} (see Figure 7.9). However, this loss is usually more than compensated for by the increased area behind the cg. Thus, the overall lift of the fuselage-tail

combination is usually increased ($L_F = C_{L_F}\, q\, S$). Therefore, a dorsal fin greatly increases directional stability at large sideslip angles. Figure 7.10 shows the effect of adding a dorsal fin on directional stability.

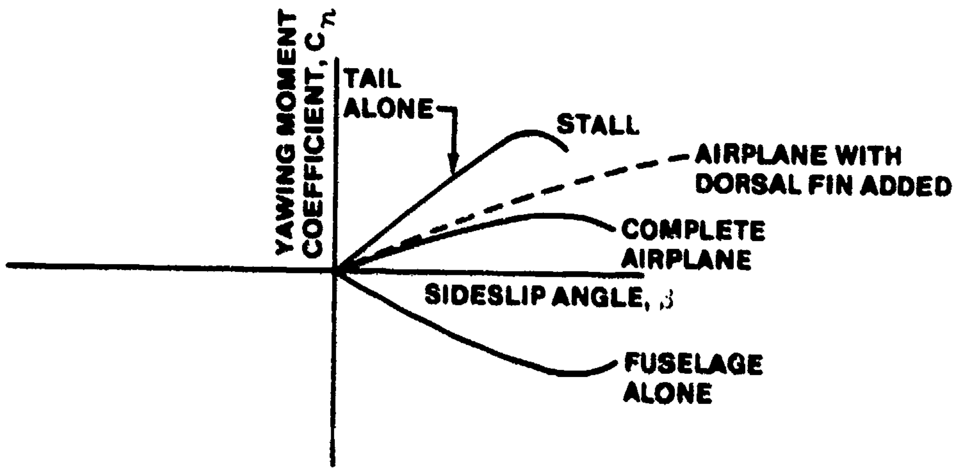


FIGURE 7.10. EFFECT OF ADDING A DORSAL FIN

The addition of a ventral fin is similar to adding another vertical tail. The net effect is an increased surface area and associated lift which produces a greater stabilizing moment.

Another design consideration which minimizes the destabilizing influence of the fuselage is nose shaping/modification. While these fore-body features are usually not put on primarily for directional stability, they do contribute. For example, the fore-body fences on the A-37 were incorporated to attain repeatable spin characteristics, but they also cause the nose to stall at smaller β than the same aircraft without the fences, thus diminishing the destabilizing influence of the fuselage (see Figure 7.11).

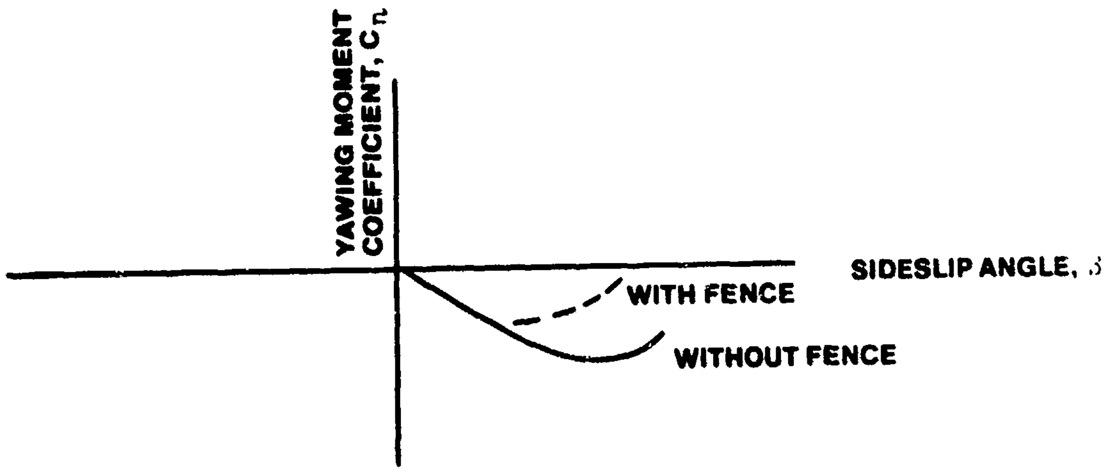


FIGURE 7.11. EFFECTS OF FORE-BODY SHAPING

7.3.1.3 Wing Contribution to C_{n_β}. The contribution of the wing to the airplane's static directional stability is usually small and is primarily a function of wing sweep (Λ). Straight wings make a slight positive contribution to static directional stability due to fuselage blanking in a sideslip. Effectively, the relative wind "sees" less of the downwind wing due to fuselage blanking. This reduces the lift of the downwind wing and thus reduces its induced drag. The difference in induced drag between the two wings tends to yaw the aircraft into the relative wind, which is stabilizing.

Swept back wings produce a greater positive contribution to static directional stability than do straight wings. In addition to fuselage blanking effects, it can be seen from Figure 7.12 that the component of free stream velocity normal to the upwind wing is significantly greater than on the downwind wing.

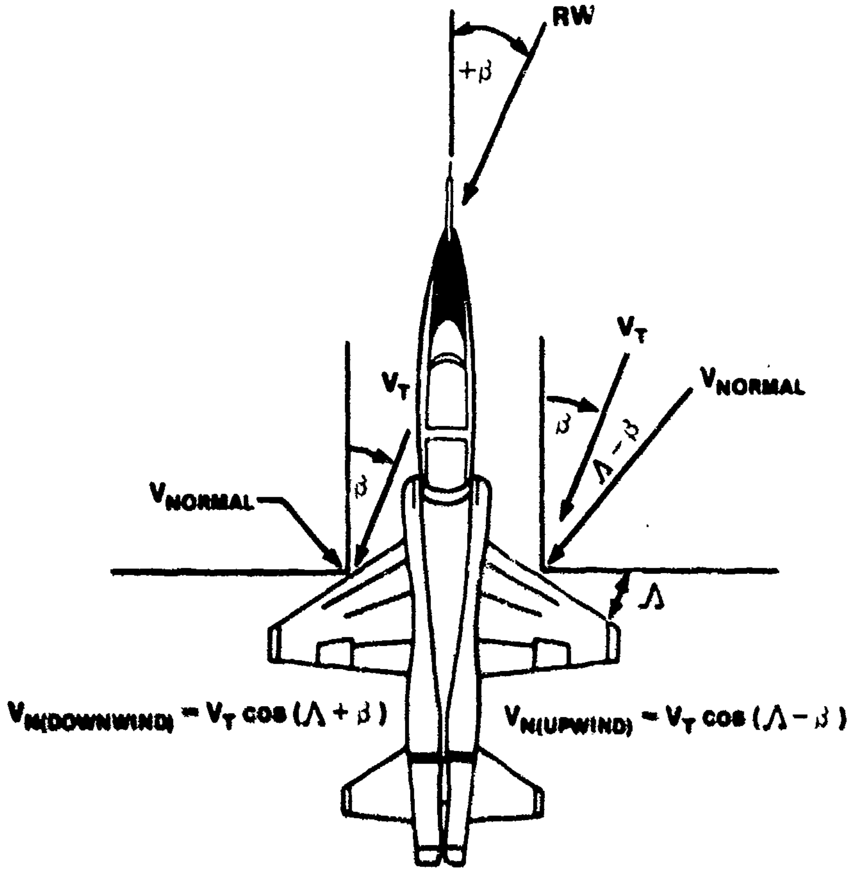


FIGURE 7.12. WING SWEEP EFFECTS ON C_{n_β}

The difference in normal components creates unbalanced lift and induced drag on the two wings, thus causing a stabilizing yawing moment. Similarly, a forward sweep angle would create an unstable contribution to static directional stability.

7.3.1.4 Miscellaneous Effects on C_{n_β}. The remaining contributors of significance to C_{n_β} are propellers, jet intakes, and engine nacelles.

A propeller can have large effects on an aircraft's static directional stability. The propeller contribution to directional stability arises from the side force component at the propeller disc created as a result of sideslip.

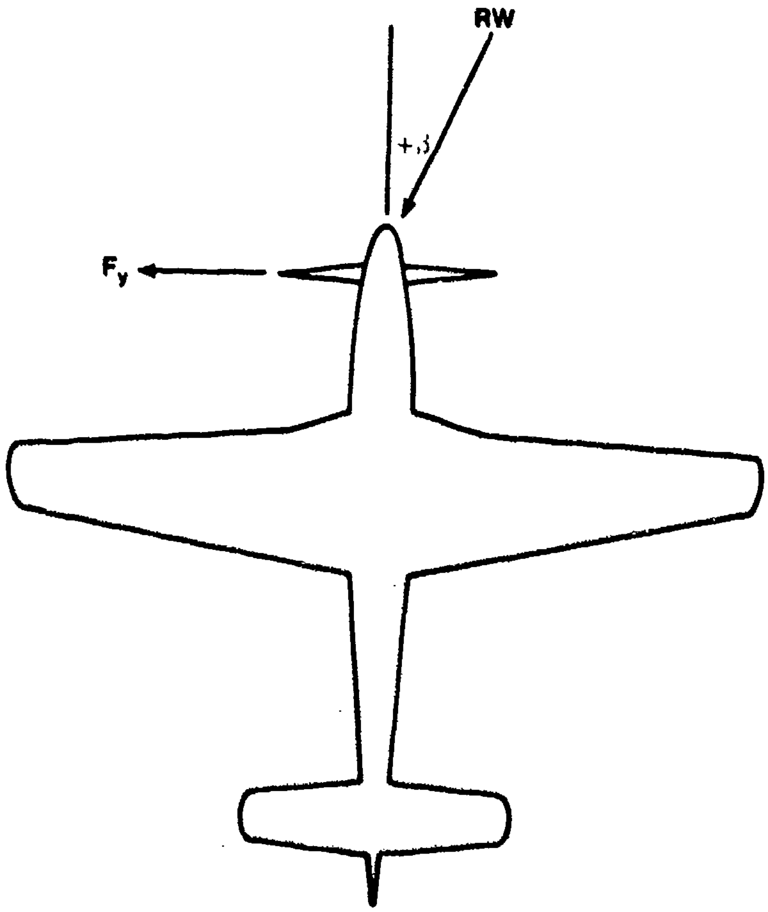


FIGURE 7.13. PROPELLER EFFECTS ON C_{n_β}

The propeller is destabilizing if a tractor and stabilizing if a pusher (Figure 7.13). Similarly, engine intakes have the same effects if they are located fore or aft of the aircraft cg.

Engine nacelles act like a small fuselage and can be stabilizing or destabilizing depending on whether their cp is located ahead or behind the cg. The magnitude of this contribution is usually small.

Aircraft cg movement is restricted by longitudinal static stability considerations. However, within the relatively narrow limits established by longitudinal considerations, cg movements have no significant effects on static directional stability.

7.3.1.5 C_{n_β} Summary. Figure 7.14 summarizes the relative magnitudes of the primary contributor to C_{n_β}.

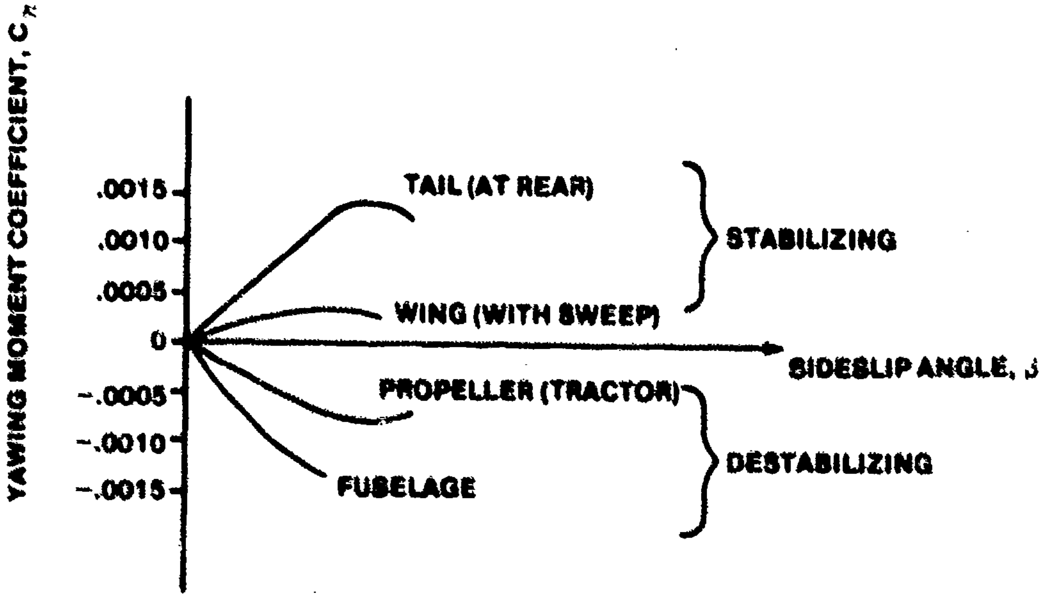


FIGURE 7.14. PRIMARY CONTRIBUTIONS TO C_{n_β}

7.3.2 $C_{n_{\delta_r}}$ Rudder Power

In most flight conditions, it is desired to maintain zero sideslip. If the aircraft has positive directional stability and is symmetrical, then it will tend to fly in this condition. However, yawing moments may act on the aircraft as a result of asymmetric thrust (one engine inoperative), slipstream rotation, or the unsymmetric flow field associated with turning flight. Under these conditions, sideslip angle can be kept to zero only by the application of a control moment. The control that provides this moment is the rudder. Recall from Equation 7.12 that

$$C_{n_F} = a_F \alpha_F V_v \qquad (7.12)$$

Differentiating with respect to δ_r

$$\frac{\partial C_{n_F}}{\partial \delta_r} = \frac{\partial C_n}{\partial \delta_r} = a_F V_v \frac{\partial \alpha_F}{\partial \delta_r} \qquad (7.16)$$

$\partial \alpha_F / \partial \delta_r$ is the equivalent change in effective vertical tail angle of attack per unit change in rudder deflection and is defined as rudder effectiveness, τ. This is a design parameter and ranges in value from zero (with no rudder) to one (in the case of an all moving vertical stabilizer surface). τ is a measure of how far one would have had to deflect the entire fin to get the same side force change that is obtained just by moving the rudder.
Substituting $\tau = \partial \alpha_F / \partial \delta_r$ into Equation 7.16.

$$\frac{\partial C_n}{\partial \delta_r} = C_{n_{\delta_r}} = a_F V_v \tau \qquad (7.17)$$

The derivative, $C_{n_{\delta_r}}$ is called "rudder power" and by definition, its algebraic sign is always positive. This is because a positive rudder deflection, $+\delta_r$ is defined as one that produces a positive moment about the cg, $+C_n$. The magnitude of the rudder power can be altered by varying the size of the vertical tail and its distance from the aircraft cg, by using different airfoils for the tail and/or rudder, or by varying the size of the rudder.

7.3.3 $C_{n_{\delta_a}}$ Yawing Moment Due to Lateral Control Deflection

The next two derivatives which will be studied $\left(C_{n_{\delta_a}} \text{ and } C_{n_p}\right)$ are called "cross derivatives," that is, a lateral input or rate generates a yaw (directional) moment. It is the existence of these cross derivatives that causes the rolling and yawing motions to be so closely coupled.

The first of these cross derivatives to be covered will be $C_{n_{\delta_a}}$, the yawing moment due to lateral control deflection. In order for a lateral control to produce a rolling moment, it must create an unbalanced lift condition on the wings. The wing with the most lift will also produce the most induced drag according to the equation $C_{D_i} = C_L^{\,2}/\pi AR\, e$. Also, any change in the profile of the wing due to a lateral control deflection will cause a change in profile drag. Thus, any lateral control deflection will produce a change in both induced and profile drag. The predominate effect will be dependent on the particular aircraft configuration and the flight condition. If induced drag predominates, the aircraft will tend to yaw away from the direction of roll (negative $C_{n_{\delta_a}}$). This phenomenon is known as "adverse yaw." The sign of $C_{n_{\delta_a}}$ for complimentary yaw is positive. Both ailerons and spoilers are capable of producing either adverse or complimentary yaw. In general, ailerons usually produce adverse yaw and spoilers usually produce proverse yaw. Many aircraft use differential horizontal stabilizer deflections for roll control. When deflected, the horizontal stabilizer on the downgoing side has a region of high pressure above it. This high pressure also acts on the side of the vertical stabilizer, which results in a yawing moment. This yawing moment is normally

proverse. To determine which condition will actually prevail, the particular aircraft configuration and flight condition must be analyzed. If design permits, it is desirable to have $C_{n_{\delta_a}} = 0$ or be slightly negative. A slight negative value may ease the pilot's turn coordination task by eliminating a need to cross control. The designs of some modern fighter-type aircraft make the pilot's task easier by keeping $C_{n_{\delta_a}} = 0$.

7.3.4 C_{n_p} <u>Yawing Moment Due to Roll Rate</u>

The second cross derivative is the yawing moment due to roll rate (C_{n_p}). Both the wing and vertical tail contribute to this derivative. In this discussion the aircraft will be considered with a roll rate, but no deflection of the control surfaces. It is important that this situation not be confused with yawing moments caused by control surface deflections. This is particularly true in flight tests where it may be difficult or impossible to separate them.

The wing contribution to C_{n_p} arises from two sources: profile drag and the tilting of the lift vectors.

As an aircraft is rolled, the angle of attack on the downgoing wing is increased, while the angle of attack on the upgoing wing is decreased. The increase in angle of attack exposes more of the downgoing wing to the relative wind. Therefore, the profile drag will be greater on the downgoing wing than on the upgoing wing. Thus, the profile drag results in a positive contribution to C_{n_p}.

Since the two wings are at different angles of attack during the roll, their lift vectors will be at different angles. The downgoing wing with a greater angle of attack will tend to have its lift vector tilted more forward. The upgoing wing with a reduced angle of attack will tend to have its lift vector tilted more aft (Figure 7.15).

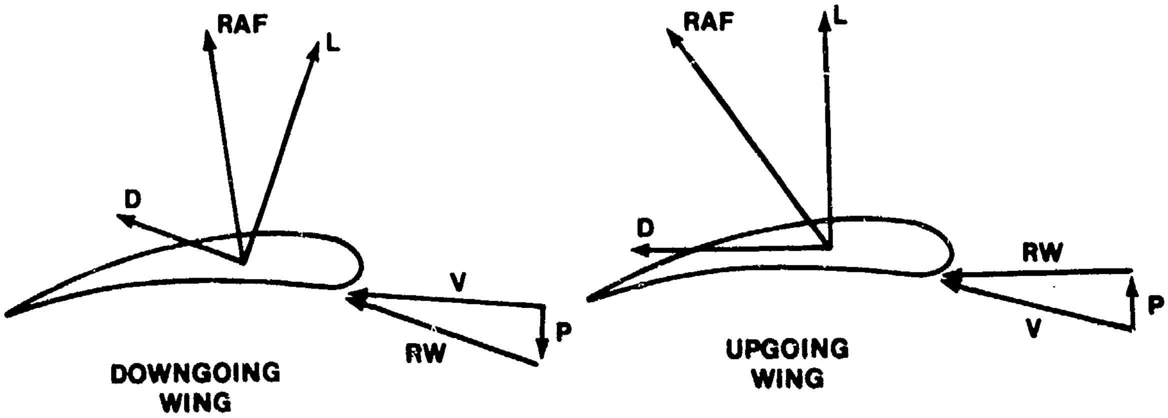


FIGURE 7.15. VECTOR TILT DUE TO ROLL RATE

For a right roll, the left wing will be pulled aft more than the right wing. This causes a negative contribution to C_{n_p}. This is true even though the magnitude of the resultant aerodynamic force may be greater on the downgoing wing than on the upgoing wing. The contribution caused by tilting of the lift vector is normally greater than the contribution due to profile drag. Therefore, the overall wing contribution to C_{n_p} is usually negative.

Rolling changes the angle of attack on the vertical tail as shown in Figure 7.16. This change in angle of attack on the vertical tail will generate a lift force. In the situation depicted in Figure 7.16, the change in angle of attack will generate a lift force, L_F, to the left. This will create a positive yawing moment. Thus, C_{n_p} for the vertical tail is positive.

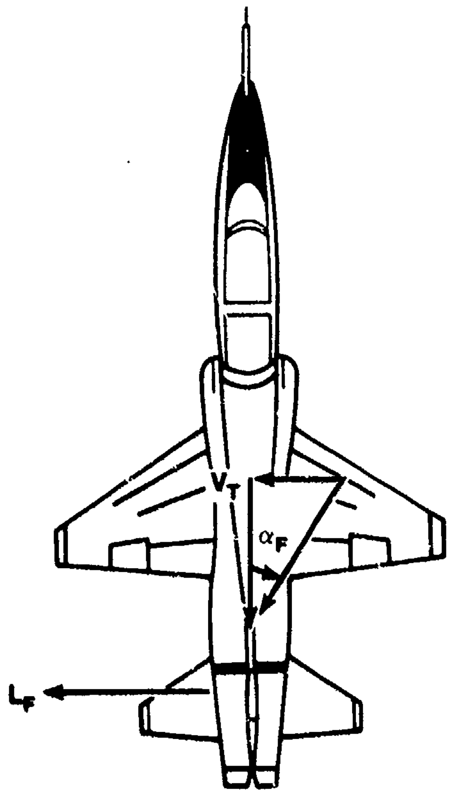


FIGURE 7.16. CHANGE IN ANGLE OF ATTACK OF THE VERTICAL TAIL DUE TO A RIGHT ROLL RATE

Considering both wing and tail, a slight positive value of C_{n_p} is desired to aid in Dutch roll damping.

7.3.5 C_{n_r} Yaw Damping

The derivative C_{n_r} is called yaw damping. It is strongly desired that C_{n_r} be negative. This is so because the forces generated when an airplane is

yawing about its center of gravity should develop moments which tend to oppose the motion.

Figure 7.17 summarizes the major contributors to C_{n_r}. In general, the fuselage contributes a negligible amount except when it is very large. The more important contributors are the wing and tail.

The tail contribution to C_{n_r} arises from the fact that there is change in angle of attack on the vertical tail whenever the aircraft is yawed. This change in α_F produces a lift force, L_F, that in turn produces a yawing moment that opposes the original yawing moment. The tail contribution to C_{n_r} accounts for 80-90% of the total "yaw damping" on most aircraft.

The wing contribution to C_{n_r} arises from the fact that in a yaw, the outside wing experiences an increase in both induced drag and profile drag due to the increased dynamic pressure on the wing. An increase in drag on the outside wing increases a yawing moment that opposes the original direction of yaw.

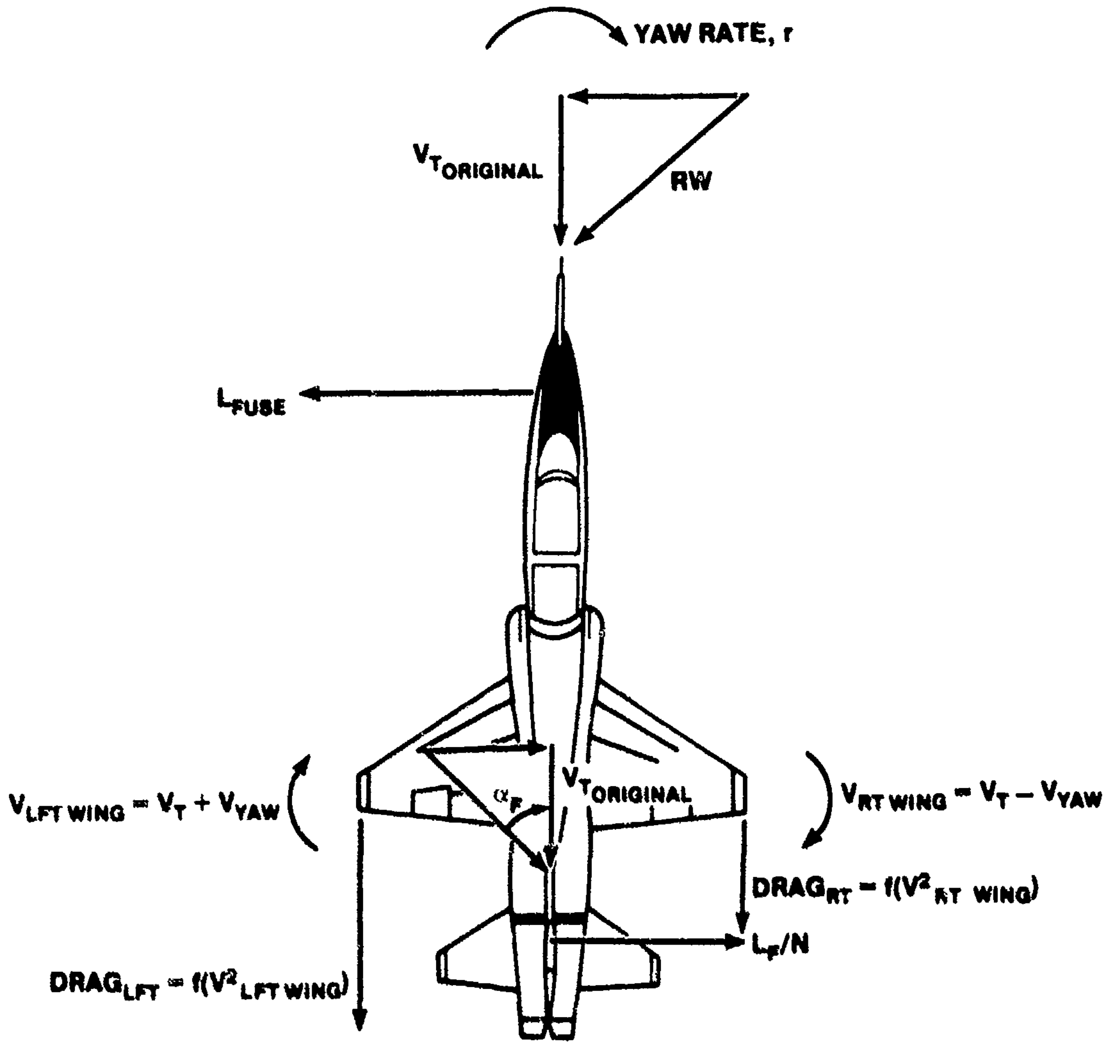


FIGURE 7.17. CONTRIBUTORS TO C_{n_r}

7.3.6 $C_{n_{\dot{\beta}}}$ Yaw Damping Due to Lag Effects in Sidewash

The derivative $C_{n_{\dot{\beta}}}$ is yaw damping due to lag effects in sidewash, σ. Very little can be authoritatively stated about the magnitude or algebraic sign of $C_{n_{\dot{\beta}}}$ due to the wide variations of opinion in interpreting the experimental data concerning it.

As an aircraft moves through a certain sideslip angle, the angle of attack of the vertical tail will be less than it would be if the aircraft were allowed to stabilize at that angle of sideslip. This is due to lag effects in sidewash which tends to straighten the flow over the tail. Since this phenomenon reduces the angle of attack of the vertical tail, it also reduces the yawing moment created by the vertical tail. This reduction in yawing moment is, effectively, a contribution to the yaw damping. Figure 7.18 illustrates description, "yaw damping due to lag effects in sidewash."

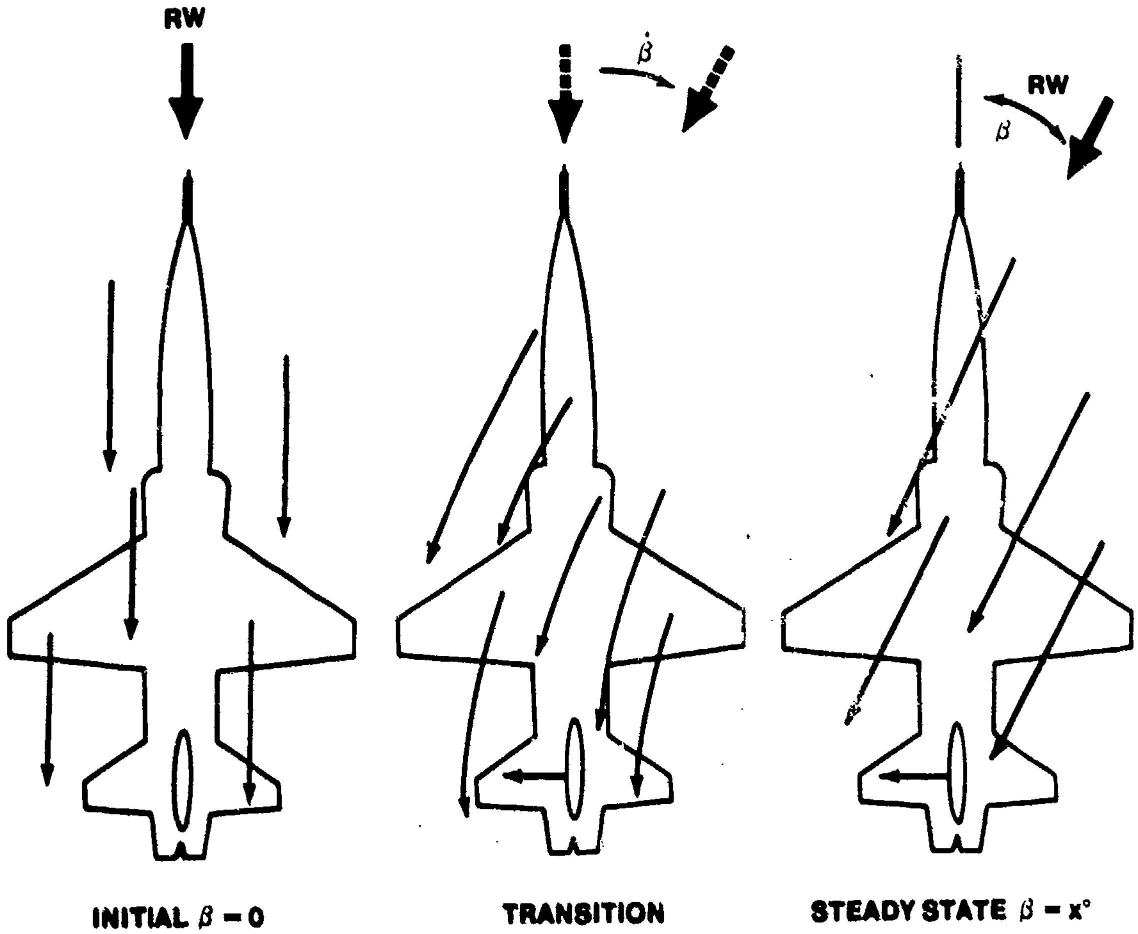


FIGURE 7.18. LAG EFFECTS

7.3.7 High Speed Effects on Static Directional Stability Derivatives

Since most of the directional stability derivatives are dependent on the lift produced by various surfaces, we can generalize the effects of Mach on these derivatives by reviewing the following relationship from supersonic aerodynamics. Thus the effectiveness of an airfoil decreases as the velocity increases supersonically (Figure 7.19).

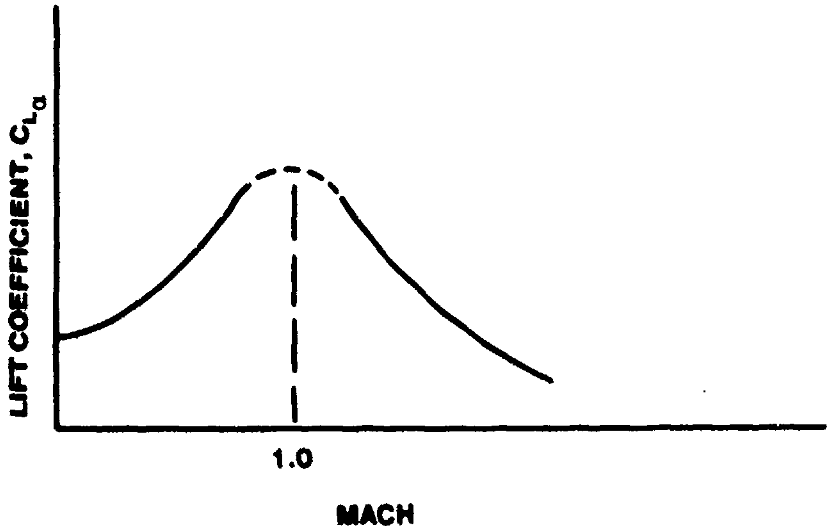


FIGURE 7.19. C_{L_α} VS M

7.3.7.1 C_{n_β}. Since $C_{n_{\beta_{Fin}}} = f\,(a_{Fin}, V_v)$ and $a_{Fin} = f\left(C_{L_{Fin}}\right)$ then for a given β, as Mach increases beyond Mach critical, the restoring moment generated by the tail diminishes. Unfortunately the wing-fuselage combination is destabilizing throughout the flight envelope. Thus, the overall C_{n_β} of the aircraft will decrease with increasing Mach, and in fact approaches zero at very high Mach (Figure 7.20).

The requirement for large values of C_{n_β} is compounded by the tendency of high speed aerodynamic designs to diverge in yaw due to roll coupling. This problem can be combated by designing an extremely large tail (F-111 and T-38), by endplating the tail (C-5 and T-38), by using ventral fins (F-11 and F-16), by using forebody strakes (SR-71), or by designing twin tails (F-15).

The F-111 employs ventral fins in addition to a sizeable vertical stabilizer to increase supersonic directional stability. The efficiency of

underbody surfaces is not affected by wing wake at high angles of attack, and supersonically, they are located in a high energy compression pattern.

Forebody strakes located radially along the horizontal center line in the x-y plane of the aircraft have also been employed effectively to increase directional stability at supersonic speeds. This increase in C_{n_β} by the employment of strakes is a results from a more favorable pressure distribution over the forebody and creates improved flow effects at the vertical tail by virtue of diminished flow circulation. Even small sideslip angles will produce fuselage blanking of the downwind strake, creating an unbalanced induced drag, and thus a stable contribution to C_{n_β}.

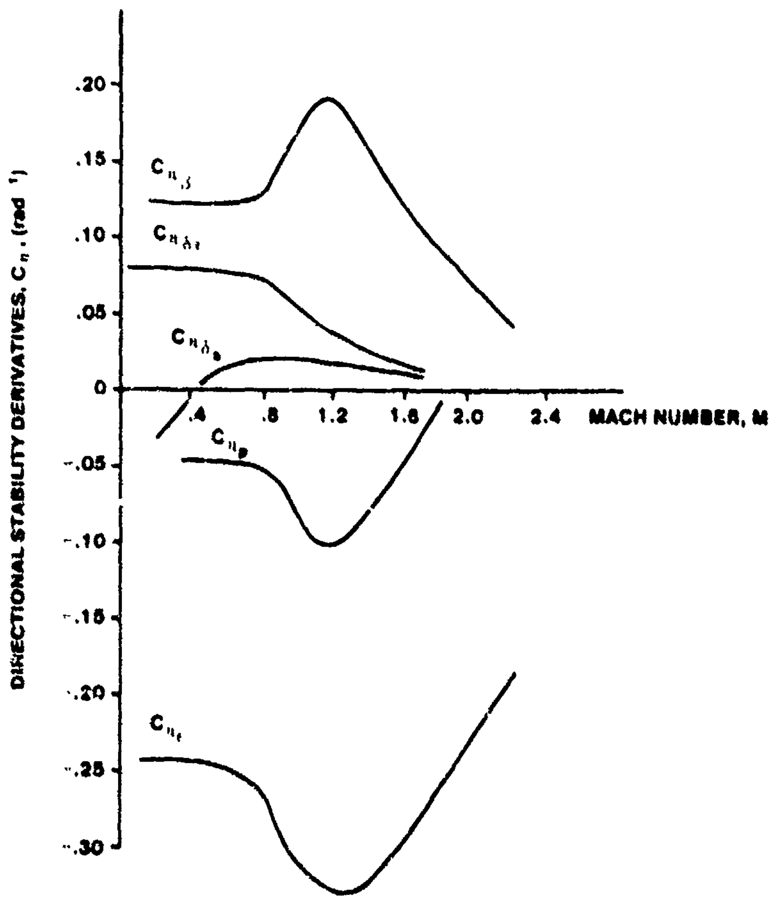


FIGURE 7.20. CHANGES IN DIRECTIONAL STABILITY DERIVATIVES WITH MACH (F-4C)

7.3.7.2 $C_{n_{\delta_r}}$. Flow separation will decrease the effectiveness of any trailing edge control surface in the transonic region. On most aircraft, however, this is offset by an increase in the C_{L_α} curve in the

transonic region. As a result, flight controls are usually the most effective in this region. However, as Mach continues to increase, the C_{L_α} curve decreases, and control surface effectiveness decreases. In addition, once the flow over the surface is supersonic, a trailing edge control cannot influence the pressure distribution on the surface itself, due to the fact that pressure disturbances cannot be transmitted forward in a supersonic environment. Thus, the rudder power will decrease as Mach increases above the transonic region.

7.3.7.3 $C_{n_{\delta_a}}$. For the same reasons discussed under rudder power, a given aileron deflection will not produce as much lift at high Mach as it did transonically. Therefore, induced drag will be less. In addition, the profile drag, for a given aileron deflection, increases with Mach. For some designs, such as roll spoilers or differential ailerons, these changes in drag will combine to cause proverse yaw.

7.3.7.4 C_{n_r}. Yaw damping depends on the ability of the wing and tail to develop lift. Thus, as Mach increases and the ability of all surfaces to develop lift decreases, yaw damping will also decrease.

7.3.7.5 C_{n_p}. The sign of C_{n_p} normally does not change with Mach. The wing contribution and the tail contribution both tend to decrease at high Mach. The exact response of the derivative to Mach varies greatly between different aircraft designs and with different lift coefficients.

7.3.7.6 C_{n_β}. The effect of Mach on this derivative is not precisely known.

7.3.8 Rudder Fixed Static Directional Stability (Flight Test Relationship)

Now that we have become familiar with the coefficients affecting directional stability, we will develop a flight test relationship to measure the static directional stability of the aircraft. The maneuver we use to determine C_{n_β} is the "steady straight sideslip" (Figure 7.21).

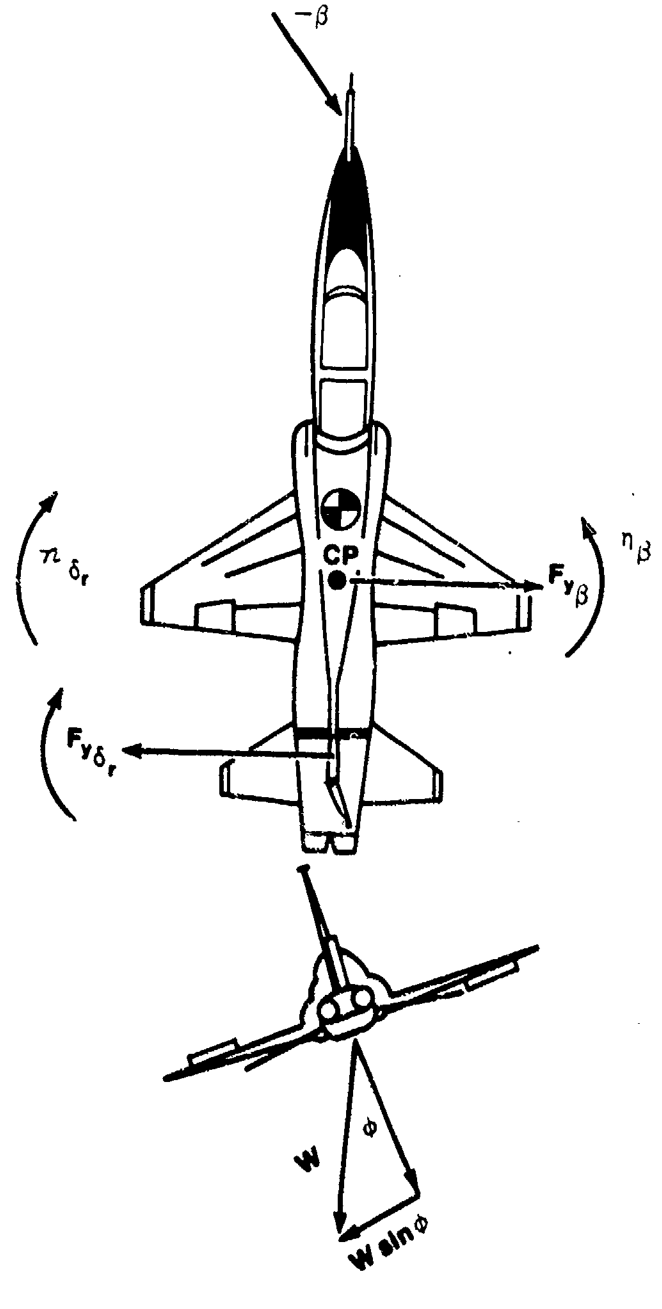


FIGURE 7.21. STEADY STRAIGHT SIDESLIP

Steady straight sideslip requires the pilot to balance the forces and moments generated on the airplane by the sideslip with appropriate lateral and directional control inputs. These control inputs are indicative of the sign (and relative magnitude) of the forces and moments generated.

As its name implies, steady straight sideslip means: $\Sigma F_{xyz} = \Sigma G_{xyz} = 0$. In addition, it implies that no rates are present and, therefore $p = q = r = \dot{\beta} = \dot{p} = \dot{r} = \dot{v} = 0$. Given this information and recalling the static directional equation of motion,

$$C_{n_\beta}\beta + \cancelto{0}{C_{n_{\dot\beta}}\hat{\dot\beta}} + \cancelto{0}{C_{n_p}\hat{p}} + \cancelto{0}{C_{n_r}\hat{r}} + C_{n_{\delta_a}}\delta_a + C_{n_{\delta_r}}\delta_r = \cancelto{0}{C_n}$$

Therefore,

$$C_{n_\beta}\beta + C_{n_{\delta_a}}\delta_a + C_{n_{\delta_r}}\delta_r = 0 \tag{7.18}$$

Solving for δ_r

$$\delta_r = \frac{C_{n_\beta}}{C_{n_{\delta_r}}}\beta - \frac{C_{n_{\delta_a}}}{C_{n_{\delta_r}}}\delta_a \tag{7.19}$$

and differentiating with respect to β

$$\frac{\partial\delta_r}{\partial\beta} = -\frac{C_{n_\beta}(\text{Fixed})}{C_{n_{\delta_r}}} - \frac{C_{n_{\delta_a}}}{C_{n_{\delta_r}}}\frac{\partial\delta_a}{\partial\beta} \tag{7.20}$$

The subscript "fixed" is added as a reminder that Equation 7.20 is an expression for the static directional stability of an aircraft if the rudder is <u>not</u> free to float.

Equation 7.20 can be further simplified by discarding the terms that are usually the smallest contributors to the expression. As we have already discovered C_{n_β} and $C_{n_{\delta_r}}$ are both usually large terms and normally dominate in the static directional equation of motion. On the other hand, if the air-flight control system is properly designed, $C_{n_{\delta_a}}$ should be zero or slightly negative. Therefore, if we assume that $C_{n_{\delta_a}}$ is significantly smaller than the other coefficients in the equation, then we are left with the following flight test relationship:

$$\frac{\partial \delta_r}{\partial \beta} = f\left(-\frac{C_{n_\beta}}{C_{n_{\delta_r}}}\right) \tag{7.21}$$

Since $C_{n_{\delta_r}}$ is a known quantity once an aircraft is built, then $\partial\delta_r/\partial\beta$ can be taken as a direct indication of the rudder fixed static directional stability of an aircraft. Moreover, $\partial\delta_r/\partial\beta$ can be easily measured in flight.

Since C_{n_β} has to be positive in order to have positive directional stability and $C_{n_{\delta_r}}$ is positive by definition, $\partial\delta_r/\partial\beta$ must be negative to obtain positive static directional stability.

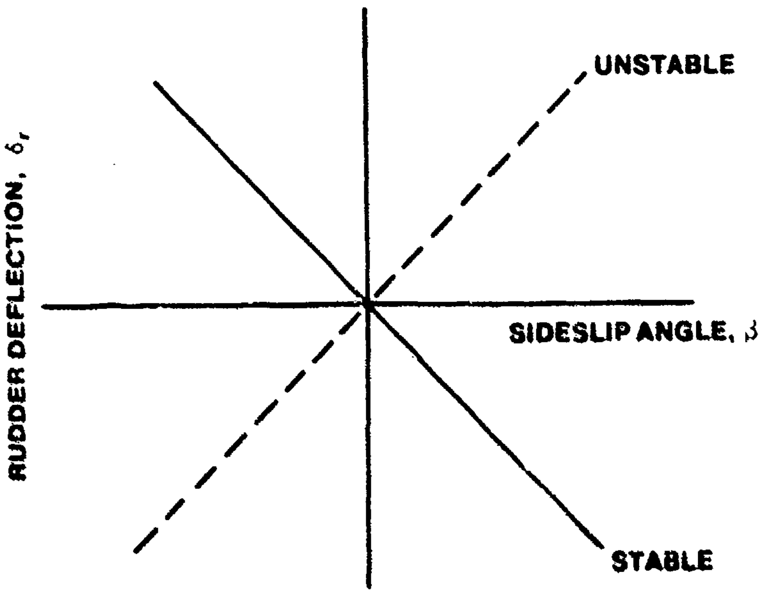


FIGURE 7.22. RUDDER DEFLECTION δ_r VS SIDESLIP

7.3.9 Rudder Free Directional Stability (Flight Test Relationship)

On aircraft with reversible control systems, the rudder is free to float in response to its hinge moments, and this floating can have large effects on the directional stability of the airplane. In fact, a plot of $\partial\delta_r/\partial\beta$ may be stable while an examination of the rudder free static directional stability reveals the aircraft to be unstable. Thus, if the rudder is free to float, there will be a change in the tail contribution to static directional stability. To analyze the nature of this change, recall that hinge moments are produced by the pressure distribution caused by angle of attack and control surface deflection.

Consider a conventional (tail-to-the-rear) aircraft with a reversible rudder. Figure 7.23 depicts the hinge moment on this rudder due to angle of attack only (i.e., $\delta_r = 0$). Note that α_F is positive with the relative wind from the right.

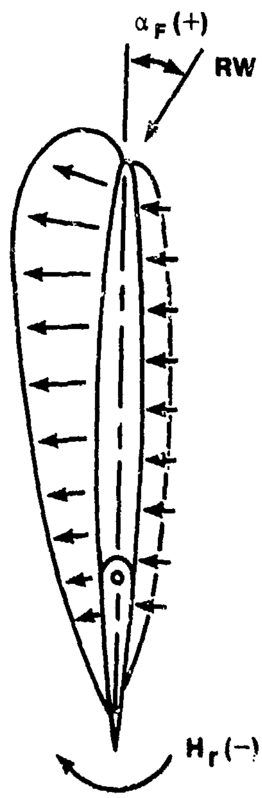


FIGURE 7.23. HINGE MOMENT DUE TO RUDDER ANGLE OF ATTACK

If the rudder control were released in this case, the hinge moment, H_r, would cause the rudder to rotate trailing edge left (TEL). This, in turn, would create a moment which would cause the nose of the aircraft to yaw to the left. Since our convention defines positive as a right yaw and anything that contributes to a right yaw is also defined as positive, then the hinge moment which causes the rudder to deflect TEL is NEGATIVE. Conversely, a positive H_r would cause the rudder to deflect TER.

Figure 7.24 depicts the hinge moment due to rudder deflection. This condition assumes $\alpha_F = 0$ before the rudder was deflected.

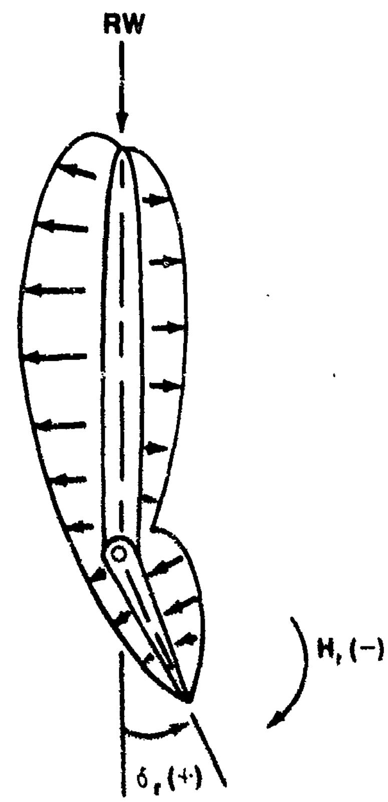


FIGURE 7.24. HINGE MOMENT DUE TO RUDDER DEFLECTION (TER)

This pressure distribution causes a hinge moment which tries to force the deflected surface back to its original position; that is, it tries to deflect the rudder TEL. We have already discovered that this moment is negative.

Combining the aerodynamic hinge moments for a given rudder deflection and a given rudder angle of attack, we find

$$H_r = \cancelto{0}{H_{r_0}} + \frac{\partial H_r}{\partial \alpha_F}\,\alpha_F + \frac{\partial H_r}{\partial \delta_r}\,\delta_r \tag{7.22}$$

In coefficient form

$$C_h = C_{h_{\alpha_F}}\,\alpha_F + C_{h_{\delta_r}}\,\delta_r \tag{7.23}$$

In the rudder free case, when the vertical tail is placed at some angle of attack, α_F, the rudder will start to "float." However, as soon as it deflects, restoring moments are set up, and an equilibrium floating angle will be reached where the floating tendency is just balanced by the restoring tendency. At this point $\Sigma H_r = 0$ which implies $C_h = 0$ (see Figure 7.25). Therefore,

$$C_{h_{\alpha_F}}\,\alpha_F + C_{h_{\delta_r}}\,\delta_{r\,(\mathrm{Float})} = 0 \tag{7.24}$$

or

$$C_{h_{\alpha_F}}\,\alpha_F = -\,C_{h_{\delta_r}}\,\delta_{r\,(\mathrm{Float})} \tag{7.25}$$

Thus,

$$\delta_{r\,(\mathrm{Float})} = -\,\frac{C_{h_{\alpha_F}}}{C_{h_{\delta_r}}}\,\alpha_F \tag{7.26}$$

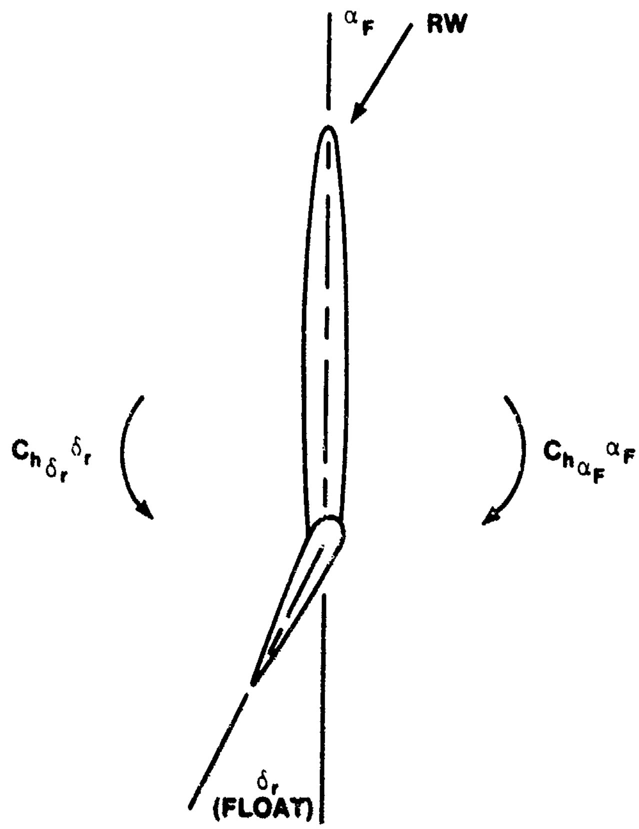


FIGURE 7.25. HINGE MOMENT EQUILIBRIUM (TEL)

With this background, it is now possible to develop a relationship that expresses the static directional stability of an aircraft with the rudder free to float.

Recall that

$$C_{n_F} = V_V \, a_F \, \alpha_F \tag{7.27}$$

and that

$$\alpha_F = \beta - \sigma \text{ (rudder fixed)} \tag{7.28}$$

But for rudder free, another factor $(\partial\alpha_F/\partial\delta_r)\cdot\delta_r$ must be added to the account for the $\Delta\alpha_F$ which will result from a floating rudder.

Therefore,

$$\alpha_F = \beta - \sigma + \frac{\partial\alpha_F}{\partial\delta_r}\,\delta_{r_{(Float)}} \tag{7.29}$$

Substituting into Equation 7.27

$$C_{n_F} = V_v\, a_F\left(\beta - \sigma + \frac{\partial\alpha_F}{\partial\delta_r}\,\delta_{r_{(Float)}}\right) \tag{7.30}$$

$$C_{n_{\beta(Free)}} = \frac{\partial C_{n_F}}{\partial\beta} = V_v\, a_F\left(1 - \frac{\partial\sigma}{\partial\beta} + \tau\,\frac{\partial\delta_{r_{(Float)}}}{\partial\beta}\right) \tag{7.31}$$

where $\tau = \partial\alpha_F/\partial\delta_r$ = rudder effectiveness

$$C_{n_{\beta(Free)}} = V_v\, a_F\left(1 - \frac{\partial\sigma}{\partial\beta}\right)\left[1 + \tau\,\frac{\partial\delta_{r_{(Float)}}}{\partial\beta}\left(\frac{1}{1 - \frac{\partial\sigma}{\partial\beta}}\right)\right] \tag{7.32}$$

Recalling that $\alpha_F = \beta - \sigma$, then $\partial\alpha_F/\partial\beta = 1 - \partial\sigma/\partial\beta$

$$C_{n_{\beta(\text{Free})}} = V_V a_F \left(1 - \frac{\partial\sigma}{\partial\beta}\right)\left[1 + \tau \frac{\partial\delta_{r(\text{Float})}}{\partial\beta}\left(\frac{\partial\beta}{\partial\alpha_F}\right)\right] \tag{7.33}$$

$$C_{n_{\beta(\text{Free})}} = V_V a_F \left(1 - \frac{\partial\sigma}{\partial\beta}\right)\left(1 + \tau \frac{\partial\delta_{r(\text{Float})}}{\partial\alpha_F}\right) \tag{7.34}$$

Recall that

$$\delta_{r(\text{Float})} = -\frac{C_{h_{\alpha_F}}}{C_{h_{\delta_r}}}\alpha_F \tag{7.35}$$

Therefore,

$$\frac{\partial\delta_{r(\text{Float})}}{\partial\alpha_F} = -\frac{C_{h_{\alpha_F}}}{C_{h_{\delta_r}}} \tag{7.36}$$

Thus, from Equation 7.34

$(-)\quad(-)\quad(+)\qquad 1 - (+)\frac{(-)}{(-)} = (+)$ for tail to rear aircraft

$$C_{n_{\beta(\text{Free})}} = V_V a_F \left(1 - \frac{\partial\sigma}{\partial\beta}\right)\left(1 - \tau \frac{C_{h_{\alpha_F}}}{C_{h_{\delta_r}}}\right) \tag{7.37}$$

$(+)\quad(-)\quad(+)\qquad 1 - (-)\frac{(+)}{(-)} = (-)$ for tail to front aircraft

It can be seen that this expression differs from Equation 7.15, the expression for rudder fixed directional stability by the term $\left(1 - \tau\, C_{h_{\alpha_F}} \Big/ C_{h_{\delta_r}}\right)$. Since this term will always result in a quantity less than one, it can be stated that the effect of rudder float is to reduce the slope of the static directional stability curve.

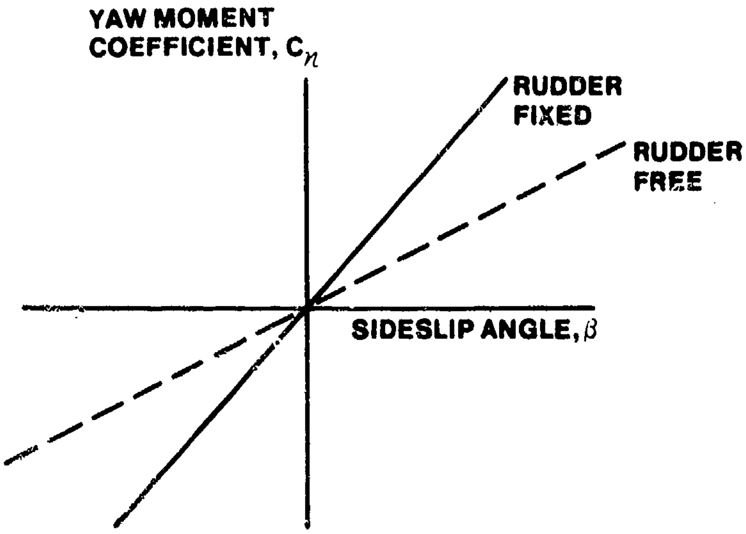


FIGURE 7.26. EFFECT OF RUDDER FLOAT ON DIRECTIONAL STABILITY

While Equation 7.37 is theoretically interesting, it does not contain parameters that are easily measured in flight. It is necessary, therefore, to develop an expression that will be useful in flight test work.

We have already seen that in a steady straight sideslip $\Sigma n = 0$. Therefore it follows that $\Sigma n_{\text{Hinge Pin}} = 0$. But we have also discovered that for a free floating system, as angle of attack is placed on the vertical fin, the rudder will tend to float and try to cancel some of this angle of attack until an equilibrium is reached. In a sideslip, therefore, the pilot must apply rudder force to oppose the aerodynamic hinge moment in order to keep the rudder deflected the desired amount to maintain the required β. This rudder force exerted by the pilot, F_r, acts through a moment arm and various gearing mechanisms, both of which are accounted for by some constant K.

Thus, in a steady straight sideslip

$$\Sigma n_{\text{Hinge Pin}} = F_r \cdot K + H_r = 0 \tag{7.38}$$

or

$$F_r = -G \cdot H_r \tag{7.39}$$

where

$$G = 1/K \text{ (definition)}$$

Recalling coefficient format

$$H_r = C_h \, q_r \, S_r \, c_r \tag{7.40}$$

From Equation 7.23

$$H_r = \left[q_r \, S_r \, c_r \, C_{h_{\alpha_F}} \, \alpha_F + C_{h_{\delta_r}} \, \delta_r \right] \tag{7.41}$$

Thus, Equation 7.39 becomes

$$F_r = - G q_r \, S_r \, c_r \left[C_{h_{\alpha_F}} \, \alpha_F + C_{h_{\delta_r}} \, \delta_r \right] \tag{7.42}$$

Applying Equation 7.24

$$F_r = - G q_r \, S_r \, c_r \left[- C_{h_{\delta_r}} \, \delta_{r_{(\text{Float})}} + C_{h_{\delta_r}} \, \delta_r \right] \tag{7.43}$$

$$F_r = - G q_r \, S_r \, c_r \, C_{h_{\delta_r}} \left[\delta_r - \delta_{r_{(\text{Float})}} \right] \tag{7.44}$$

The difference between where the pilot pushes the rudder, δ_r, and the amount it floats, $\delta_{r_{(\text{Float})}}$, is the free position of the rudder, $\delta_{r_{(\text{Free})}}$ (Figure 7.27).

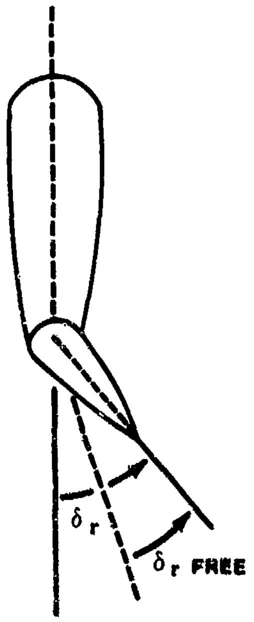


FIGURE 7.27. $\delta_{r_{Float}}$ VS $\delta_{r_{Free}}$

Therefore,

$$F_r = -Gq_r S_r c_r C_{h_{\delta_r}} \delta_{r_{(Free)}} \tag{7.45}$$

$$\frac{\partial F_r}{\partial \beta} = - Gq_r S_r c_r C_{h_{\delta_r}} \frac{\partial \delta_{r_{(Free)}}}{\partial \beta} \tag{7.46}$$

From Equation 7.21 it can be shown that

$$\frac{\partial \delta_{r_{(Free)}}}{\partial \beta} = - \frac{C_{n_{\beta_{(Free)}}}}{C_{n_{\delta_r}}} \tag{7.47}$$

Thus,

$$\frac{\partial F_r}{\partial \beta} = \overset{(+)}{G}\ \overset{(+)}{q_r}\ \overset{(+)}{S_r}\ \overset{(+)}{c_r}\ \overset{(-)}{\underset{(+)}{\frac{C_{h_{\delta_r}}}{C_{n_{\delta_r}}}}}\ \overset{(+)\text{ For Stability}}{C_{n_{\beta}}}{}_{(\text{Free})} \qquad (7.48)$$

Therefore,

$$\frac{\partial F_r}{\partial \beta} = (-) \text{ for stability, tail to front or rear.}$$

This equation shows that the parameter, $\partial F_r/\partial \beta$, can be taken as an indication of the rudder free static directional stability of an aircraft since all terms are either constant or set by design, except $C_{n_{\beta}(\text{Free})}$. Further, this equation constitutes a flight test relationship because $\partial F_r/\partial \beta$ can be readily measured in flight.

An analysis of the components of Equation 7.48 reveals that for static directional stability (i.e., $C_{n_\beta} = +$), the sign of $\partial F_r/\partial \beta$ should be negative (Figure 7.28).

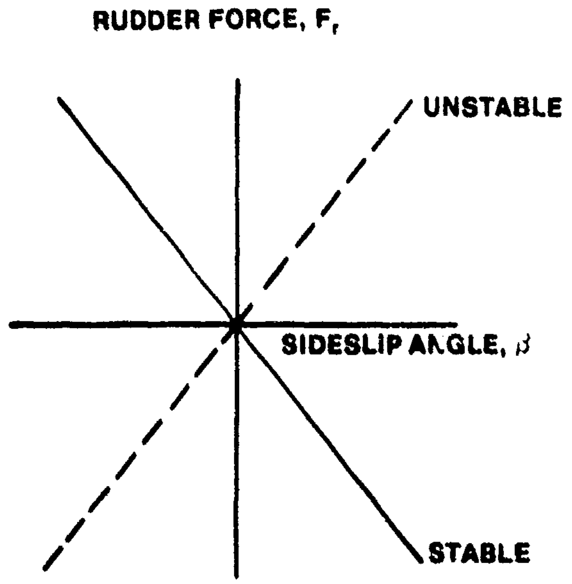


FIGURE 7.28. RUDDER FORCE VS SIDESLIP

7.4 STATIC LATERAL STABILITY

In our discussion of directional stability, the wings of the aircraft have been considered at some arbitrary angle to the vertical (angle of bank, ϕ), usually taken as zero, with no concern for the aerodynamic problem of holding this angle or for bringing the airplane into this attitude.

The problem of holding the wings level or of maintaining some angle of bank is one of control over the rolling moments about the airplane's longitudinal axis. The major control over the rolling moments is the ailerons, while secondary control can be obtained through control over the sideslip angle. Recalling the stability derivatives which contribute to static lateral stability, we see both of these factors present.

$$C_\ell = C_{\ell_\beta}\beta + C_{\ell_{\dot\beta}}\hat{\dot\beta} + C_{\ell_p}\hat{p} + C_{\ell_r}\hat{r} + C_{\ell_{\delta_a}}\delta_a + C_{\ell_{\delta_r}}\delta_r \qquad (7.6)$$

It can be seen that the rolling moment coefficient, C_ℓ, is not a function of bank angle, ϕ. In other words, a change in bank angle will produce no change in rolling moment. In fact, ϕ produces no moment at all. Thus, $C_{\ell_\phi} = 0$, and although it is analogous to C_{m_α} and C_{n_β}, it contributes lateral static stability analysis.

Bank angle, ϕ, does have an indirect effect on rolling moment. As the aircraft is rolled into a bank angle, a component of aircraft weight will act along the Y-axis and will thus produce an unbalanced force (Figure 7.29). This unbalanced force in the Y direction, F_y, will produce a sideslip, β, and as seen from Equation 7.6, this will influence the rolling moment produced.

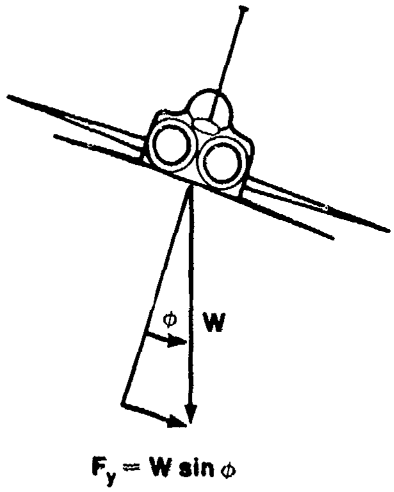


FIGURE 7.29. SIDE FORCE PRODUCED BY BANK ANGLE

Each stability derivative in Equation 7.6 will be discussed, and its contribution to aircraft stability will be analyzed. Table 7.2 summarizes these stability derivatives.

TABLE 7.2
LATERAL STABILITY AND CONTROL DERIVATIVES

DERIVATIVE	NAME	SIGN FOR A STABLE AIRCRAFT	CONTRIBUTING PARTS OF AIRCRAFT
C_{ℓ_β}	Dihedral Effect	(-)	Wing, Tail
$C_{\ell_{\dot\beta}}$	C_ℓ due to $\dot\beta$	(+)	Wing, Tail
C_{ℓ_p}	Roll Damping	(-)	Wing, Tail
C_{ℓ_r}	C_ℓ due to Yaw Rate	(+)	Wing, Tail
$C_{\ell_{\delta_a}}$	Lateral Control Power	(+)	Lateral Control
$C_{\ell_{\delta_r}}$	C_ℓ due to Rudder Deflection	(-)	Rudder

7.4.1 C_{ℓ_β} Dihedral Effect

C_{ℓ_β}, which is commonly referred to as "dihedral effect," is a measure of the tendency of an aircraft to roll when disturbed in sideslip. Although the static lateral stability of an aircraft is a function of all the derivatives in Equation 7.6, C_{ℓ_β} is the dominant term.

The algebraic sign of C_{ℓ_β} must be negative for stable dihedral effect (Figure 7.30). Consider an aircraft in wings level flight. If disturbed in bank to the right, the aircraft will develop a right sideslip (+β). If C_{ℓ_β} is negative, a rolling moment to the left (-) will result, and the initial tendency will be to return toward equilibrium.

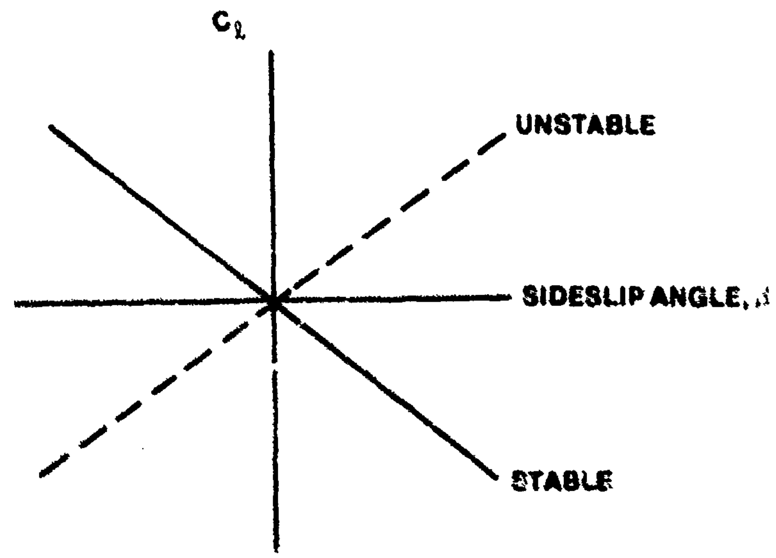


FIGURE 7.30. ROLLING MOMENT COEFFICIENT C_ℓ VS SIDESLIP

It is possible to have too much or too little dihedral effect. High values of dihedral effect give good spiral stability. If an aircraft has a large amount of dihedral effect, the pilot is able to pick up a wing with top rudder. This also means that in level flight, a small amount of sideslip will cause the aircraft to roll, and this can be annoying to the pilot. This is known as a high ϕ/β ratio. In multi-engine aircraft, an engine failure will

normally produce a large sideslip angle. If the aircraft has a great deal of dihedral effect, the pilot must supply an excessive amount of aileron force and deflection to overcome the rolling moment due to sideslip. Still another detrimental effect of too much dihedral effect may be encountered when the pilot rolls an aircraft. If an aircraft, in rolling to the right, tends to yaw to the left, the resulting sideslip, together with stable dihedral effect, creates a rolling moment to the left. This effect could significantly reduce the maximum roll rate available. The pilot wants a certain amount of dihedral effect, but not too much. The end result is usually a design compromise.

Both the wing and the tail contribute to C_{ℓ_β}. The various effects on C_{ℓ_β} can be classified as "direct" or "indirect." A direct effect actually produces some increment of C_{ℓ_β}, while an indirect effect merely alters the value of the existing C_{ℓ_β}.

The discrete wing and tail effects that will be considered are classified as shown in Table 7.3.

TABLE 7.3
EFFECTS ON C_{ℓ_β}

DIRECT	INDIRECT
Geometric Dihedral	Aspect Ratio
Wing Sweep	Taper Ratio
Wing-Fuselage Interference	Tip Tanks
Vertical Tail	Wing Flaps

7.4.1.1 Geometric Dihedral. Geometric dihedral, γ, is defined as shown in Figure 7.31, and is positive (dihedral) when the chord lines of the wingtip are above those at the wing root, and is negative (anhedral) when the tip chord lines are below the wing roots.

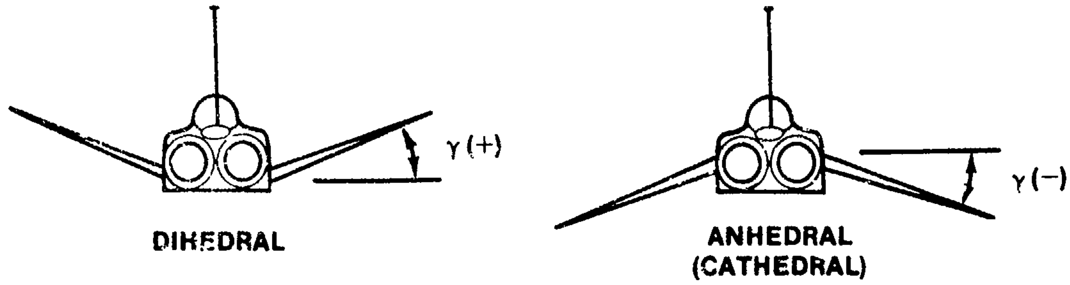


FIGURE 7.31. GEOMETRIC DIHEDRAL

To understand the effect of geometric dihedral on static lateral stability, consider Figure 7.32.

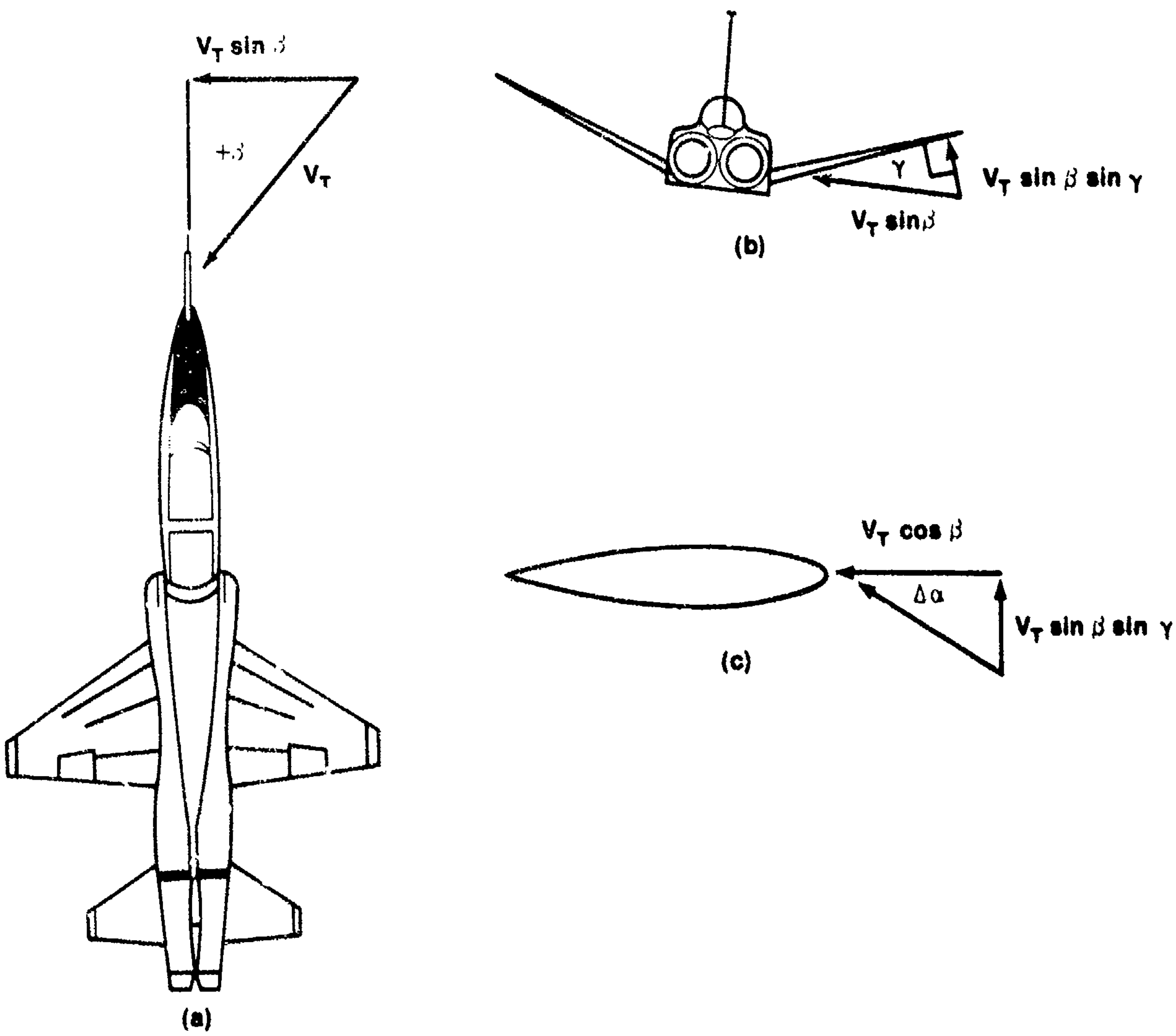


FIGURE 7.32. EFFECTS OF γ ON C_{ℓ_β}

It can be seen that when an aircraft is placed in a sideslip, positive geometric dihedral causes the component, $V_T \sin\beta \sin\gamma$ to be added to the lift producing component of the relative wind, $V_T \cos\beta$. Thus, geometric dihedral causes the angle of attack on the upwind wing to be increased by $\Delta\alpha$. To find this $\Delta\alpha$

$$\tan \Delta\alpha = \frac{V_T \sin\beta \sin\gamma}{V_T \cos\beta} = \tan\beta \sin\gamma \tag{7.49}$$

Making the small angle assumption,

$$\Delta\alpha = \tan\beta \sin\gamma \tag{7.50}$$

Conversely, the angle of attack on the downwind wing will be reduced. These changes in angle of attack tend to increase the lift on the upwind wing and decrease the lift on the downwind wing, thus producing a roll away from the sideslip. In Figure 7.32, a positive sideslip ($+\beta$) will increase the angle of attack on the upwind (right) wing, thus producing a roll to the left. Therefore, it can be seen that this effect produces a stable, or negative, contribution to C_{ℓ_β}.

7.4.1.2 Wing Sweep. The wing sweep angle, Λ, is measured from a perpendicular to the aircraft x-axis at the forward wing root, to a line connecting the quarter chord points of the wing. Wing sweep back is defined as positive.

Aerodynamic theory shows that the lift of a yawed wing is determined by the component of the free stream velocity normal to wing. That is, $L = 1/2\, C_L\, \rho V_N^2 S$ where, V_N is the normal velocity.

As was previously shown in our discussion of $C_{\ell_{\beta_{Wing}}}$, and as can be seen from Figure 7.33, the normal component of free stream velocity on the upwind wing on a swept wing aircraft is

$$V_N = V_T \cos(\Lambda - \beta) \tag{7.51}$$

Conversely, on the downwind wing,

$$V_N = V_T \cos(\Lambda + \beta) \tag{7.52}$$

Therefore, V_N is greater on the upwind wing. This causes the upwind wing to produce more lift and creates a roll away from the direction of the sideslip. In other words, a right sideslip will produce a roll to the left. Thus, aft wing sweep makes a stable contribution to C_{ℓ_β} and produces the same effect as positive geometric dihedral.

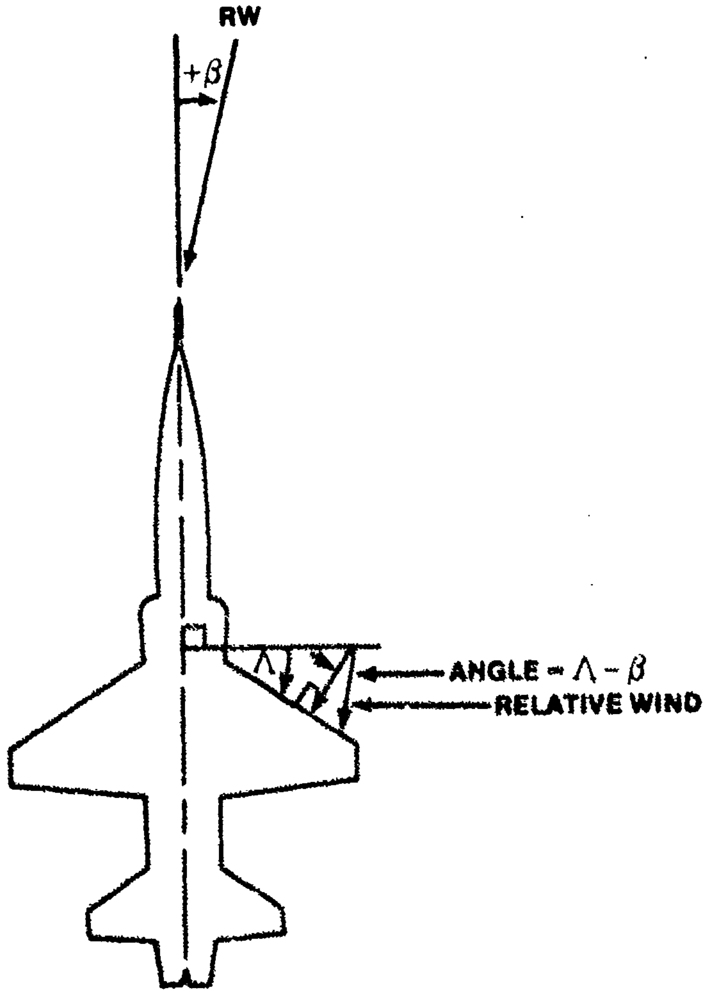


FIGURE 7.33. NORMAL VELOCITY COMPONENT ON SWEPT WING

To fully appreciate the effect of wing sweep on static lateral stability, it will be necessary to develop an equation relating the two.

$$L_{(\text{Upwind Wing})} = 1/2\ C_L \frac{S}{2} \rho V_N^2 \qquad (7.53)$$

$$L_{(\text{Upwind Wing})} = 1/2\ C_L \frac{S}{2} \rho\ [V_T \cos (\Lambda - \beta)]^2 \qquad (7.54)$$

Similarly,

$$L_{(\text{Downward Wing})} = 1/2\ C_L \frac{S}{2} \rho\ [V_T \cos (\Lambda + \beta)]^2 \qquad (7.55)$$

Thus,

$$\Delta L = 1/2\ C_L \frac{S}{2} \rho\ [V_T \cos (\Lambda - \beta)]^2 - 1/2\ C_L \frac{S}{2} \rho\ [V_T \cos (\Lambda + \beta)]^2 \qquad (7.56)$$

$$\Delta L = 1/2\ C_L \frac{S}{2} \rho\ V_T^2\ [\cos^2 (\Lambda - \beta) - \cos^2 (\Lambda + \beta)] \qquad (7.57)$$

Applying a trigonometric identity,

$$[\cos^2 (\Lambda - \beta) - \cos^2 (\Lambda + \beta)] = 2\beta \sin 2\Lambda \qquad (7.58)$$

Making the assumption of a small sideslip angle,

$$\sin 2\Lambda \sin 2\beta = 2\beta \sin 2\Lambda \qquad (7.59)$$

Therefore, Equation 7.57 becomes

$$\Delta L = 1/2 C_L \frac{S}{2} \rho\ V_T^2\ 2\beta \sin 2\Lambda = 1/2 C_L S\ \rho\ V_T^2\ \beta \sin 2\Lambda \qquad (7.60)$$

The rolling moment produced by this change in lift is

$$\mathcal{L} = -\Delta L \quad Y \tag{7.61}$$

Where Y is the distance from the wing cp to the aircraft cg. The minus sign arises from the fact that Equation 7.60 assumes a positive sideslip, $+\beta$, and for an aircraft with stable dihedral effect, this will produce a negative rolling moment

$$C_\ell = \frac{\mathcal{L}}{q_w S_w b_w} \tag{7.62}$$

$$C_\ell = -\frac{Y\, C_L\, S\, \rho\, V_T^2\, \beta \sin 2\Lambda}{\rho\, V_T^2\, Sb} = -\frac{C_L\, Y\, \beta}{b} \sin 2\Lambda \tag{7.63}$$

$$\frac{\partial C_\ell}{\partial \beta} = C_{\ell_\beta} = -\frac{Y}{b} C_L \sin 2\Lambda = -\text{CONST}\,(C_L \sin 2\Lambda) \tag{7.64}$$

where the constant will be on the order of 0.2. Equation 7.64 should not be used above $\Lambda = 45^\circ$ because highly swept wings are subject to leading edge separation at high angles of attack, and this can result in reversal of the dihedral effect. Therefore, it is best to use empirical results above $\Lambda = 45^\circ$.

Equation 7.64 shows that at low speeds (high C_L) sweepback makes a large contribution to stable dihedral effect. However, at high speeds (low C_L) sweepback makes a relatively small contribution to stable dihedral effect (Figure 7.34).

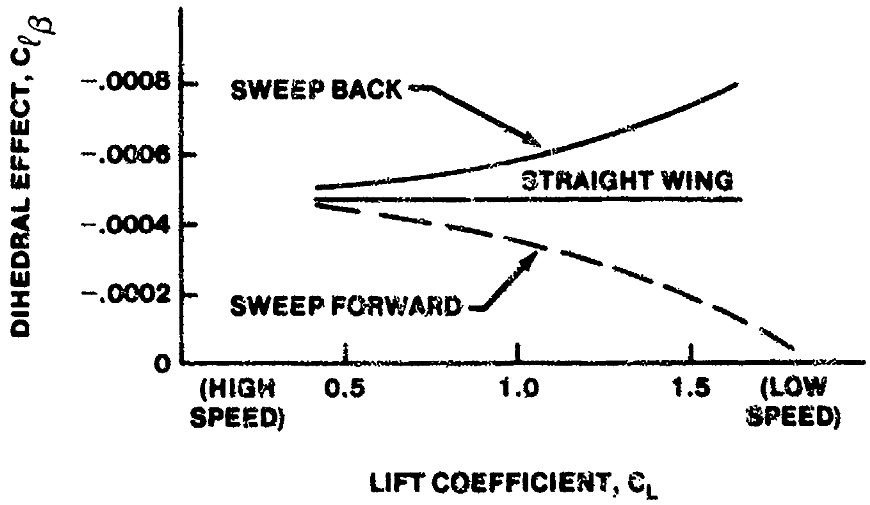


FIGURE 7.34. EFFECTS OF WING SWEEP AND LIFT COEFFICIENT ON DIHEDRAL EFFECT, C_{ℓ_β}

For forward swept wings, the sweep becomes more destabilizing at slow speeds and less destabilizing at high speeds. For angles of sweep on the order of 45°, the wing sweep contribution to C_{ℓ_β} may be on the order of $-1/5\ C_L$. For large values of C_L, this is a very large contribution, equivalent to nearly 10° of geometric dihedral. At very high angles of attack, such as during landing and takeoff, this effect can be very helpful to a swept wing fighter encountering downwash. Since the effect of sweepback varies with C_L, becoming extremely small at high speeds, it can help keep the proper ratio of C_{ℓ_β} to C_{n_β} at high speeds and reduce poor Dutch roll characteristics at these speeds.

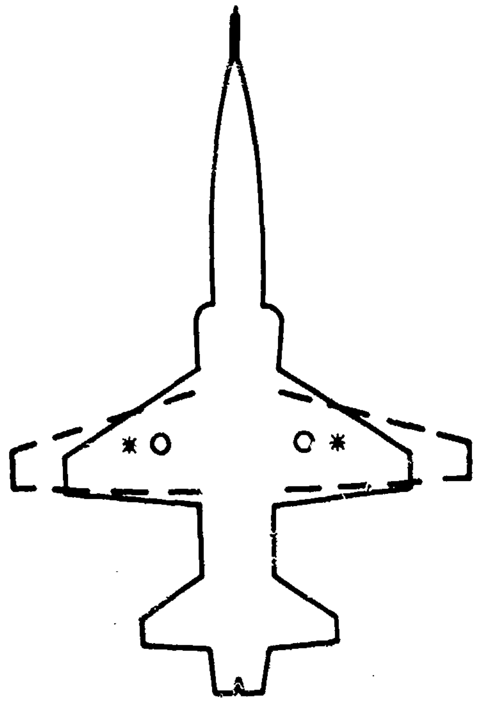


FIGURE 7.35. CONTRIBUTION OF ASPECT RATIO TO DIHEDRAL EFFECT

7.4.1.3 <u>Wing Aspect Ratio</u>. The wing aspect ratio exerts an indirect contribution to dihedral effect. On a high aspect ratio wing, the center of pressure is further from the cg than on a low aspect ratio wing. This results in high aspect ratio planforms having a longer moment arm and thus, greater rolling moments for a given asymmetric lift distribution (Figure 7.35). It should be noted that aspect ratio, in itself, does not create dihedral effect.

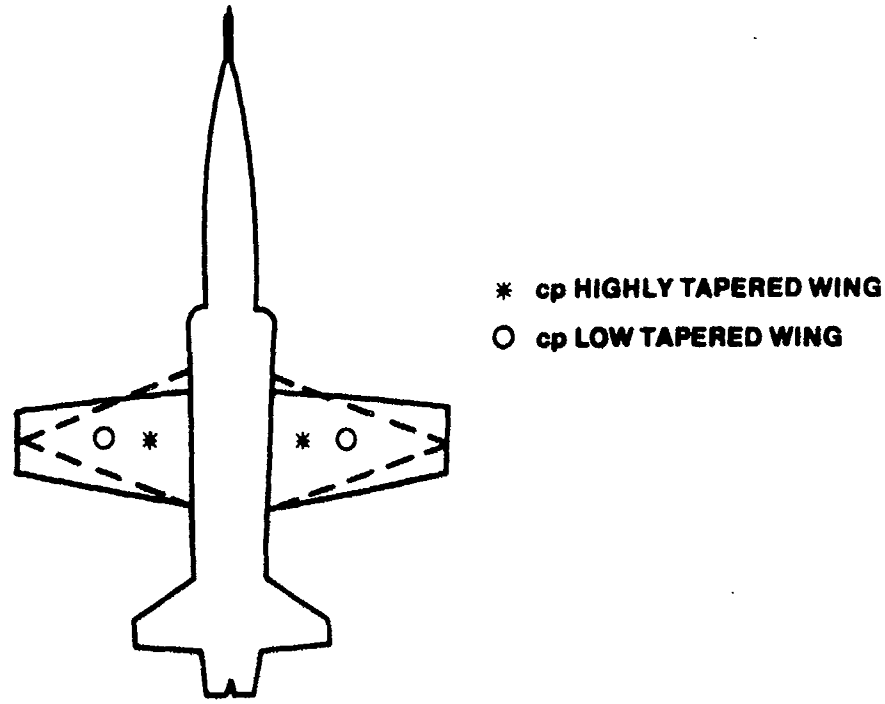


FIGURE 7.36. CONTRIBUTION OF TAPER RATIO TO DIHEDRAL EFFECT

7.4.1.4 <u>Wing Taper Ratio</u>. Taper ratio, λ, is the ratio of the tip chord to the root chord and is a measure of how fast the wing chord shortens. Therefore, the lower the taper ratio, the faster the chord shortens. On highly tapered wings, the center of pressure is closer to the aircraft cg than on untapered wings. This results in a shorter moment arm and thus, less rolling moment for a given asymmetric lift distribution (Figure 7.36). Taper ratio does not create dihedral effect but merely alters the magnitude of the existing dihedral effect. Thus it has an indirect contribution to dihedral effect.

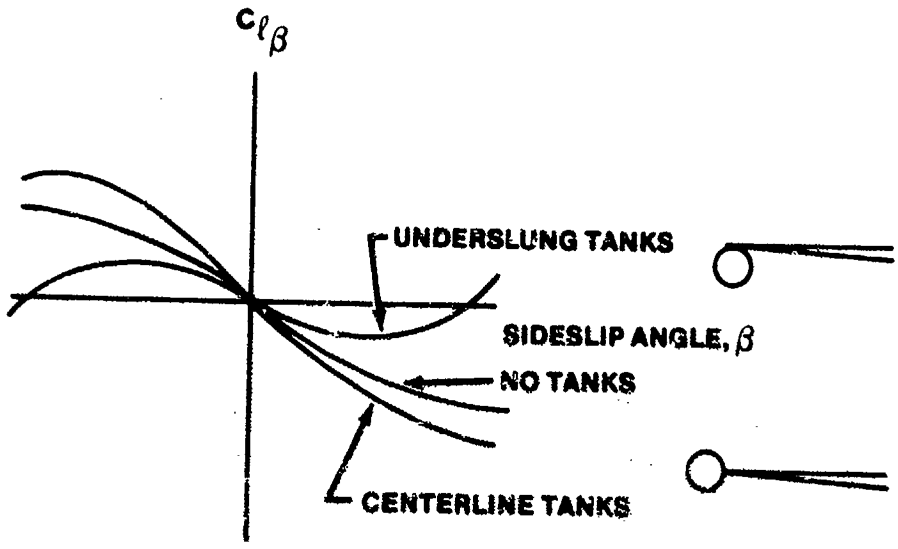


FIGURE 7.37. EFFECT OF TIP TANKS ON DIHEDRAL EFFECT, C_{ℓ_β} OF F-80

7.4.1.5 Tip Tanks. Tip tanks, pylon tanks, or other external stores will generally exert an indirect influence on C_{ℓ_β}. Unfortunately, the effect of a given external store configuration is hard to predict analytically, and it is usually necessary to rely on empirical results. To illustrate the effect of various external store configurations, data for the F-80 are presented in Figure 7.37. The data are for an F-80 in cruise configuration, 230 gallon centerline tip tanks, and 165 gallon underslung tanks. These data show that the centerline tanks increase dihedral effect while the underslung tanks reduce stable dihedral effect considerably.

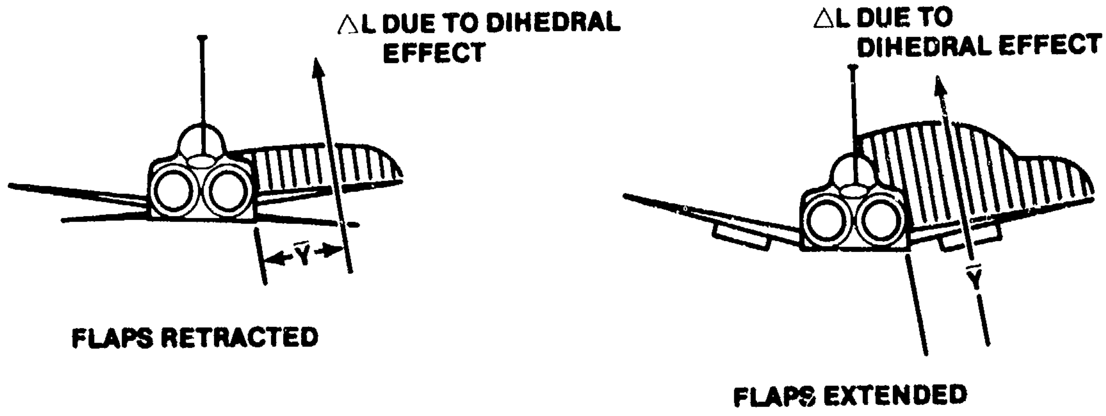


FIGURE 7.38. EFFECT OF FLAPS ON WING LIFT DISTRIBUTION

7.4.1.6 Partial Span Flaps. Partial span flaps indirectly affect static lateral stability by shifting the center of lift of the wing, thus changing the effective moment arm Y. If the partial span flap is on the inboard portion of the wing (as is usually the case), then it will shift the center of lift inboard and reduce the effective moment arm. Therefore, although the values of ΔL remain the same, the rolling moment will decrease. This in turn has a detrimental effect on C_{ℓ_β} (Figure 7.38). The higher the effectiveness of the flaps in increasing the lift coefficient, the greater will be the change in span lift distribution and the more detrimental will be the effect of the inboard flaps. Therefore, the decrease in lateral stability due to flap extension may be large.

Extended flaps may also cause a secondary, and generally small, variation in the effective dihedral. This secondary effect depends upon the planform of the flaps themselves. If the shape of the wing gives a sweepback to the leading edge of the flaps, a slight stabilizing dihedral effect results when the flaps are extended. If the leading edges of the flaps are swept forward, flap extension causes a slight destabilizing dihedral effect. These effects are produced by the same phenomenon that produced a change in C_{ℓ_β} with wing sweep.

7.4.1.7 Wing-Fuselage Interference. For a complete analysis of dihedral effect, account must be taken of the various interference effects between parts of the aircraft. Of these, probably the most important is wing-fuselage interference; more precisely, the change in angle of attack of the wing near the root due to the flow pattern about the fuselage in a sideslip. To illustrate, this consider a cylindrical body yawed with respect to the relative wind.

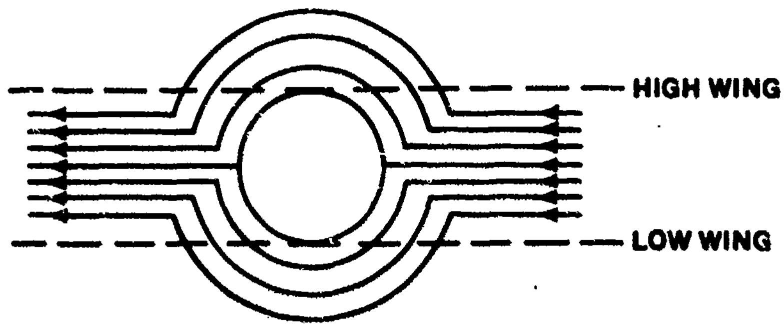


FIGURE 7.39. FLOW PATTERN ABOUT A FUSELAGE IN SIDESLIP

The fuselage induces vertical velocities in a sideslip which, when combined with the mainstream velocity, alters the local angle of attack of the wing. When the wing is located at the top of the fuselage (high-wing), then the angle of attack will be increased at the wing root, and a positive sideslip will produce a negative rolling moment; i.e., the dihedral effect will be enhanced. Conversely, when the aircraft has a low wing, the angle of attack at the root will be decreased, and the dihedral effect will be diminished. Generally, this explains why high-wing airplanes often have little or no geometric dihedral, whereas low-wing aircraft may have a great deal of geometric dihedral.

The magnitude of this effect is dependent upon the fuselage length ahead of the wing, its cross-sectional shape, and the planform and location of the wing.

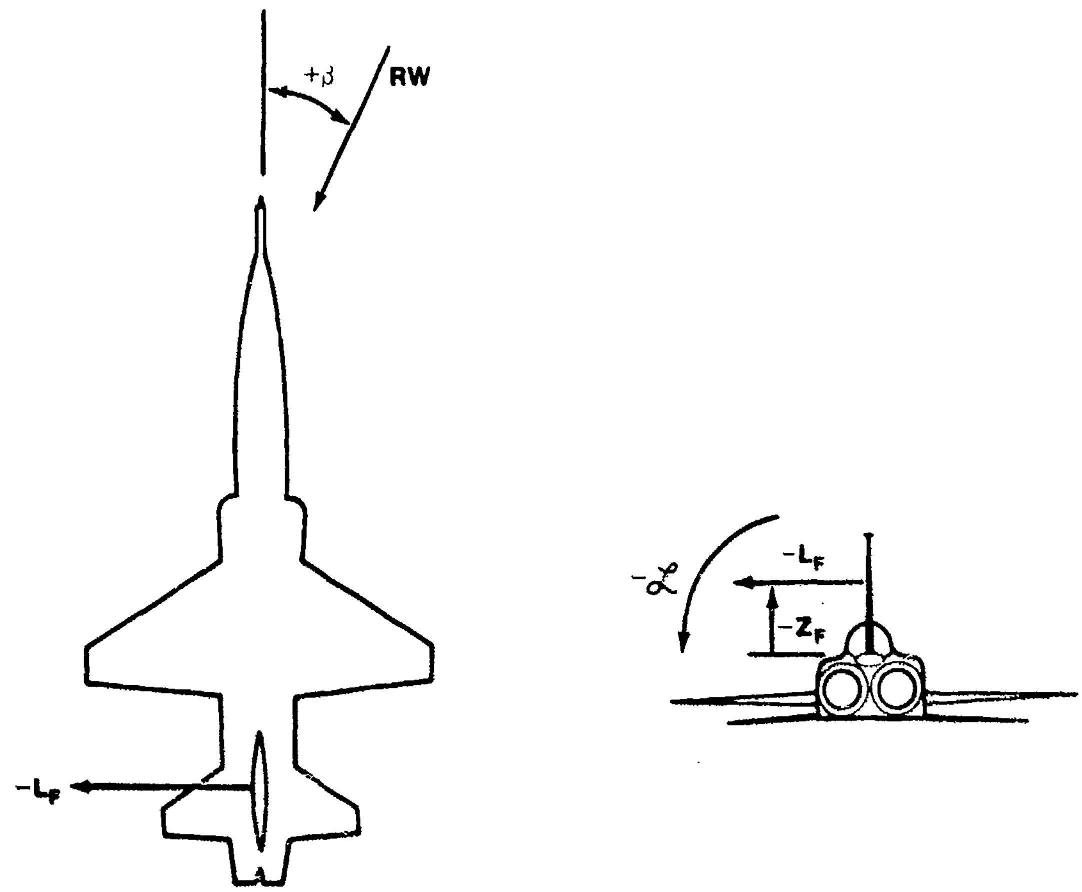


FIGURE 7.40. ROLLING MOMENT CREATED BY VERTICAL TAIL AT A POSITIVE ANGLE OF SIDESLIP

7.4.1.8 Vertical Tail. As we have already discovered in our $C_{\beta_{Tail}}$ discussion when the sideslip angle is changed, the angle of attack of the vertical tail is changed. This change in angle of attack produces a lift force on the vertical tail. If the center of pressure of the vertical tail is above the aircraft cg, this lift force will produce a rolling moment.

In the situation depicted in Figure 7.40, the negative rolling moment was created by a positive sideslip angle, thus, the vertical tail contributes a stable increment to dihedral effect. This contribution can be quite large. In fact, it can be the major contribution to C_{ℓ_β} on aircraft with large vertical tails such as the T-38. This effect can be calculated in the same manner yawing moments were calculated in the directional case.

Assuming a positive sideslip angle,

$$\mathcal{L}_F = -Z_F L_F \tag{7.65}$$

Since

$$C_{\ell_F} = \frac{\mathcal{L}}{q_w S_w b_w}$$

then

$$C_{\ell_F} = \frac{-Z_F L_F}{q_w S_w b_w} \tag{7.66}$$

but

$$L_F = C_{L_F} q_F S_F$$

Therefore,

$$C_{\ell_F} = \frac{-Z_F \, C_{L_F} \, q_F \, S_F}{q_w S_w b_w} \tag{7.67}$$

Define V_F as

$$V_F = \frac{S_F Z_F}{S_w b_w} \tag{7.68}$$

Assume that for a jet aircraft

$$q_F = q_w \tag{7.69}$$

And Equation 7.67 becomes

$$C_{\ell_F} = -C_{L_F} V_F = -a_F \alpha_F V_F \tag{7.70}$$

Knowing

$$\alpha_F = (\beta - \sigma)$$

$$C_{\ell_F} = -a_F V_F (\beta - \sigma) \tag{7.71}$$

$$C_{\ell_{\beta_{\text{Vertical tail}}}} = \frac{\partial C_{\ell_F}}{\partial \beta} = -\, a_F \, V_F \left(1 - \frac{\partial \sigma}{\partial \beta}\right) \qquad (7.72)$$

$(-)(-) \quad (+) \quad = (-)$ Tail on top

$(-)(+) \quad (+) \quad = (+)$ Tail on bottom

Equation 7.72 reveals that a vertical tail contributes a stable increment to C_{ℓ_β}, whereas a ventral fin $[V_F = (+)]$ would contribute an unstable increment to C_{ℓ_β}. Also, if the lift curve slope of the vertical tail is increased, by end plating for example, the stable dihedral effect would be greatly increased. For example, the C-5 has a high horizontal stabilizer that acts as an end plate on the vertical tail, and this increases the stable dihedral effect. In fact, the increase is so large that it is necessary to add negative geometric dihedral to the wings and a ventral fin to maintain a reasonable value of stable dihedral effect.

7.4.2 $C_{\ell_{\delta_a}}$ LATERAL CONTROL POWER

Lateral control is normally achieved by altering the lift distribution so that the total lift on the two wings differs, thereby creating a rolling moment. This is done by destroying lift on one wing by a spoiler, or by altering the lift on both wings with ailerons (Figure 7.41).

Many modern aircraft designs use differential deflections of the horizontal stabilizers for roll control. When the pilot makes a roll input, the horizontal stabilizer on one side will deflect trailing edge down, while the stabilizer on the other side deflects trailing edge up. The difference in lift on the two sides of the stabilizer results in a rolling moment.

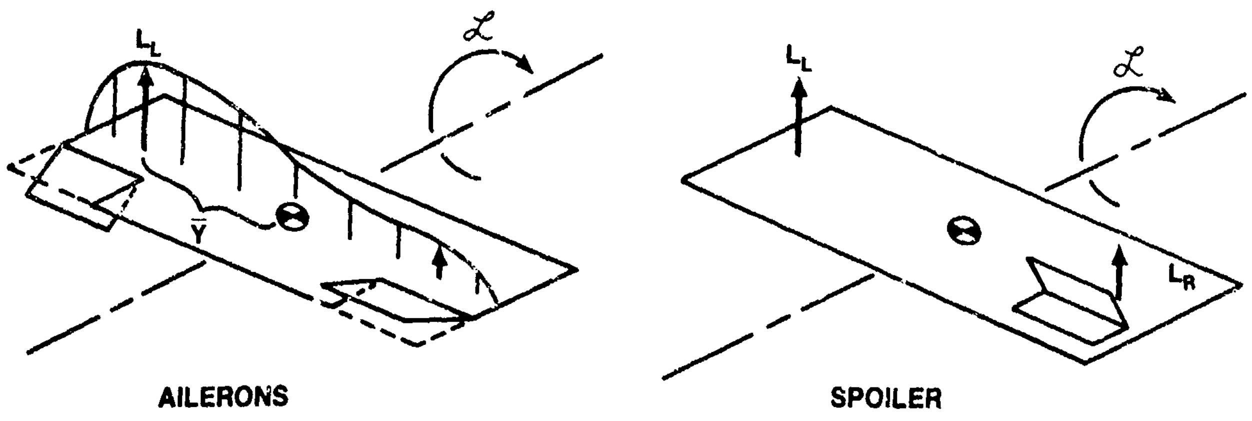


FIGURE 7.41. LATERAL CONTROL

This discussion will be limited to the use of ailerons as the means of lateral control. A measure of aileron power is the rolling moment created by a given aileron deflection. A positive deflection of either aileron, $+\delta_a$, is defined as one which produces a positive rolling moment, (right wing down). $C_{\ell_{\delta_a}}$ is positive by definition. Total aileron deflection is defined as the sum of the two individual aileron deflections. Thus,

$$\delta_{a_{Total}} = \delta_{a_{Left}} + \delta_{a_{Right}} \tag{7.73}$$

The assumption will be made that the wing cp shift due to aileron deflection will not alter the value of C_{ℓ_β}. The distance from the x-axis to the cp of the wing will be labeled Y. When the ailerons are deflected, they produce a change in lift on both wings. This total change in lift, ΔL, produces a rolling moment, $\mathcal{L}$.

$$= L_L Y - L_R Y = (L_L - L_R) Y = \Delta L \; Y \tag{7.74}$$

Since

$$L = C_L q S$$

then

$$\Delta L = \frac{\partial C_L}{\partial \alpha} \Delta\alpha \, q \, S \tag{7.75}$$

therefore

$$\mathcal{L} = \frac{\partial C_{L_a}}{\partial \alpha_a} \Delta\alpha_a \, q_a \, S_a \, Y \qquad (7.76)$$

where the "a" subscripts refer to "aileron" values.

But

$$\frac{\partial C_{L_a}}{\partial \alpha_a} = a_a \qquad (7.77)$$

therefore

$$\mathcal{L} = a_a \, \Delta\alpha_a \, q_a \, S_a \, Y \qquad (7.78)$$

Recalling

$$C_\ell = \frac{\mathcal{L}}{q_w \, S_w \, b_w}$$

then

$$C_\ell = \frac{a_a \Delta\alpha_a \, S_a \, Y}{S_w \, b_w} \left(\cancel{\frac{q_a}{q_w}}\right)^1 \qquad (7.79)$$

If we let $\qquad \Delta\alpha_a = \delta_{a_{Left}} + \delta_{a_{Right}} = \delta_{a_{Total}}$

then

$$C_\ell = \frac{a_a \, \delta_{a_{Total}} \, S_a \, Y}{S_w \, b_w} \qquad (7.80)$$

and

$$\frac{\partial C_\ell}{\partial \delta_a} = C_{\ell_{\delta_a}} = a_a \left(\frac{S_a}{S_w}\right)\left(\frac{Y}{b}\right) \qquad (7.81)$$

Thus, from Equation 7.81 the lateral control power is a function of the aileron airfoil section (a_a), the area of the aileron in relation to the area of the wing S_a/S_w, and the location of the wing cp (Y/b).

7.4.3 C_{ℓ_p} Roll Damping

The forces generated when an airplane is rolling about its x-axis, at some roll rate, p, produce rolling moments which tend to oppose the motion. Thus the algebraic sign of C_{ℓ_p} is usually negative.

The primary contributors to roll damping are the wings and the tail. The wing contribution to C_{ℓ_p} arises from the change in wing angle of attack that results from the rolling velocity. It has already been shown that the downgoing wing in a rolling maneuver experiences an increase in angle of attack. This increased α tends to develop a rolling moment that opposes the original rolling moment. However, when the wing is near the aerodynamic stall, a rolling motion may cause the downgoing wing to exceed the stall angle of attack. In this case, the local lift curve slope may fall to zero or even reverse sign. The algebraic sign of the wing contribution to C_{ℓ_p} may then become positive. This is what occurs when a wing "autorotates," as in spinning (Figure 7.42).

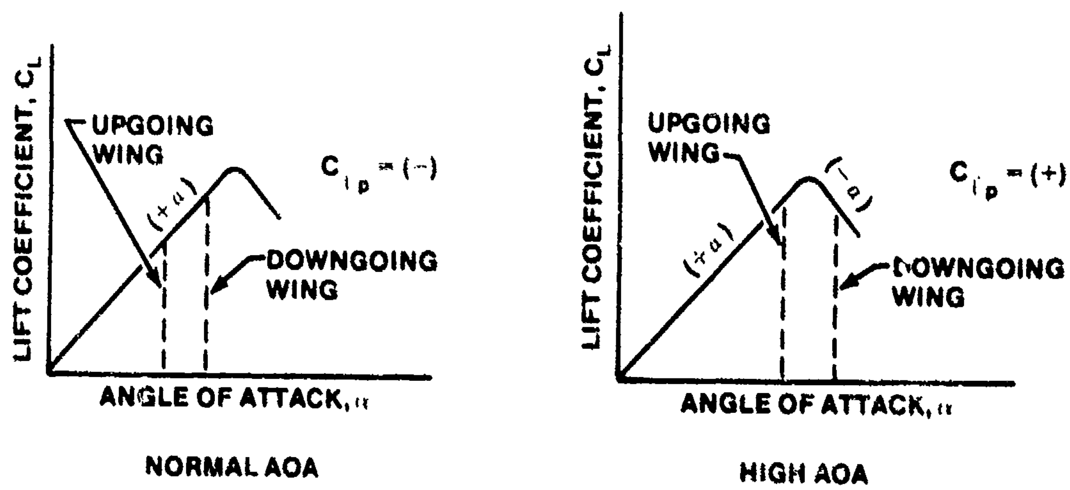


FIGURE 7.42. HIGH AOA EFFECTS ON C_{ℓ_p}

The vertical tail contribution to C_{ℓ_p} arises from the fact that when the aircraft is rolled, the angle of attack on the vertical tail is changed. This change in angle of attack develops a lift force which opposes the original rolling moment. This contribution to a negative C_{ℓ_p} is the same regardless of whether the tail is above (conventional tail) or below (ventral fin) the aircraft roll axis.

7.4.4 C_{ℓ_r} Rolling Moment Due to Yaw Rate

The primary contributions to C_{ℓ_r} come from two sources, the wings and the vertical tail (Figure 7.43).

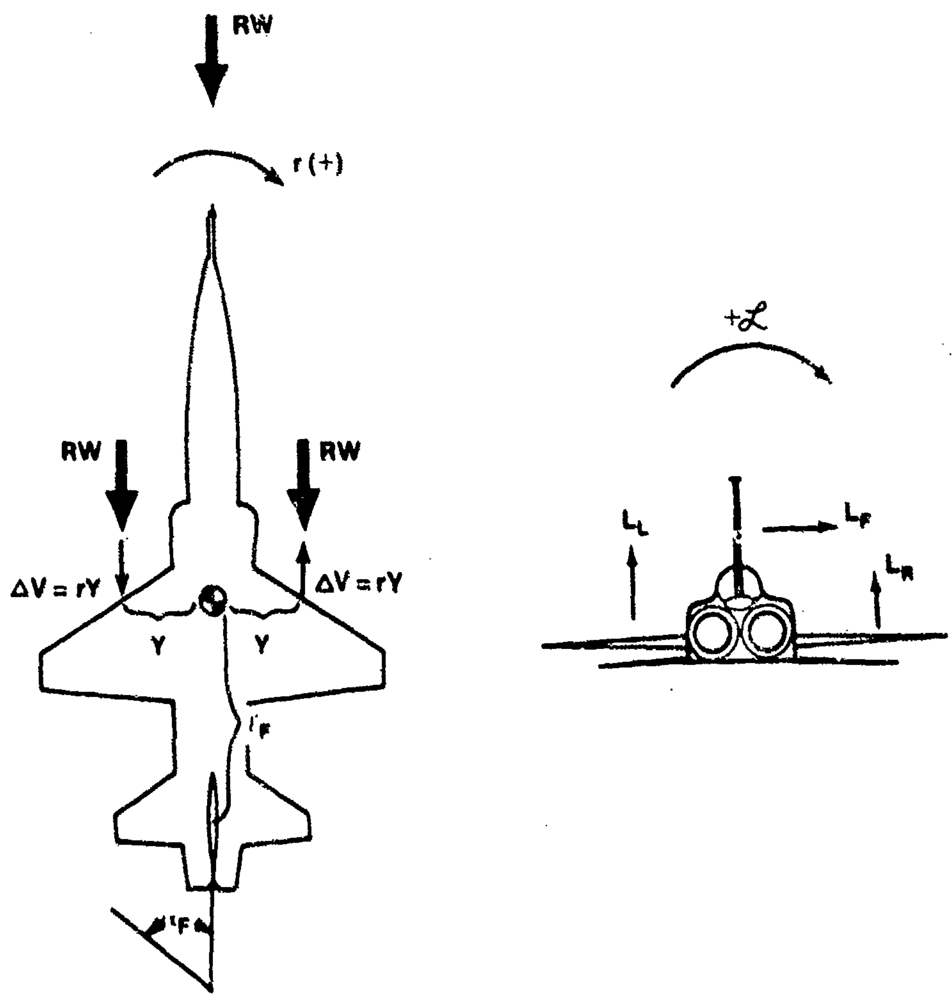


FIGURE 7.43. C_{ℓ_r} CONTRIBUTORS

As the aircraft yaws, the velocity of the relative wind is increased on the advancing wing to produce more lift and thus produces a rolling moment. A right yaw would produce more lift on the left wing and thus a rolling moment to the right. Thus, the algebraic sign of the wing contribution to C_{ℓ_r} is positive.

The tail contribution to C_{ℓ_r} arises from the fact that as the aircraft is yawed, the angle of attack on the vertical tail is changed. The lift force thus produced, L_F, will create a rolling moment if the vertical tail cp is above or below the cg. For a conventional vertical tail, the sign of C_{ℓ_r} will be positive, while for a ventral fin the sign will be negative.

7.4.5 $C_{\ell_{\delta_r}}$ Rolling Moment Due to Rudder Deflection

When a rudder is deflected it creates a lift force on the vertical tail. If the cp of the vertical tail is above or below the aircraft cg a rolling moment will result. Refer to Figure 7.44.

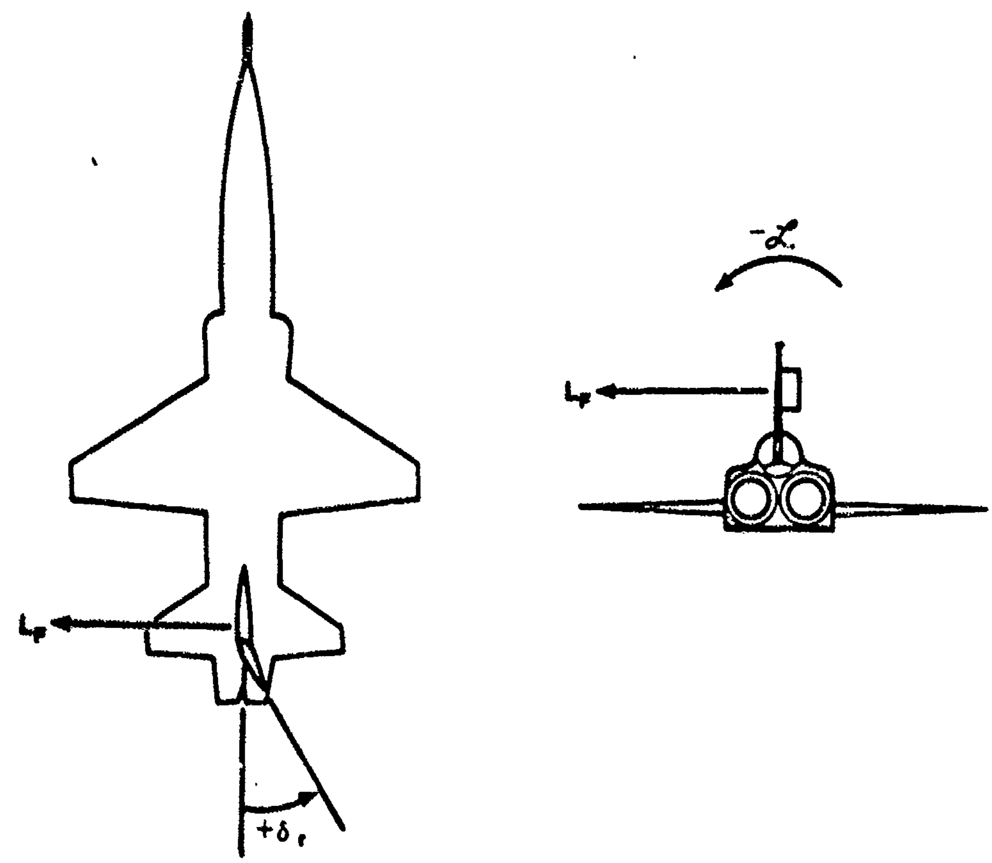


FIGURE 7.44. LIFT FORCE DEVELOPED AS A RESULT OF δ_r

It can be seen that if the cp of the vertical tail is above the cg, as with a conventional vertical tail, the sign of $C_{\ell_{\delta_r}}$ will be negative. However, with a ventral fin, the sign would be positive.

The effects of $C_{\ell_{\delta_r}}$ and C_{ℓ_β} are opposite in nature. When the rudder is deflected to the right, initially, a rolling moment to the left is created due to $C_{\ell_{\delta_r}}$. However, as sideslip develops due to the rudder deflection, dihedral effect, C_{ℓ_β}, comes into play and causes a resulting rolling moment to the right. Therefore, when a pilot applies right rudder to pick up a left wing, he initially creates a rolling moment to the left and, finally, to the right (Figure 7.45).

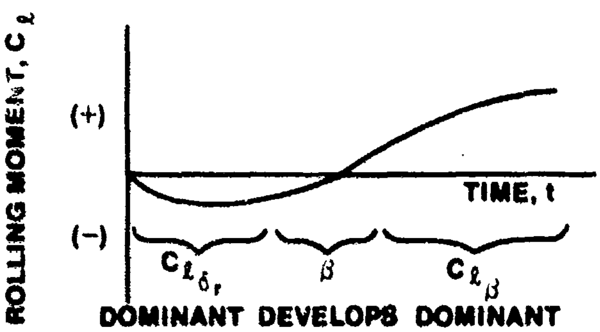


FIGURE 7.45. TIME EFFECTS ON ROLLING MOMENT DUE TO $C_{\ell_{\delta_r}}$ and C_{ℓ_β} CAUSED BY + δ_r

7.4.6 $C_{\ell_{\dot\beta}}$ Rolling Moments Due to Lag Effects in Sidewash

In the discussion of $C_{n_{\dot\beta}}$, it was pointed out that during an increase in β, the angle of attack of the vertical tail will be less than it will finally be in steady state condition. If the cp of the vertical tail is displaced from the aircraft cg, this change in α_F due to lag effects will alter the rolling moment created during the β build up period. Because of lag effects, C_{ℓ_F} will be less during the β build up period than at steady state. Thus, for a conventional vertical tail, the algebraic sign of $C_{\ell_{\dot\beta}}$ is positive.

Again, it should be pointed out that there is widespread disagreement over the interpretation of data concerning lag effects in sidewash and that the foregoing is only one basic approach to a complex problem.

7.4.7 High Speed Consideration of Static Lateral Stability

Most of the contributions to C_ℓ are due to L_{wing}, ΔL_{wing} or $L_{vertical\ tail}$. As airspeed affects these parameters, it also affects static lateral stability.

7.4.7.1 C_{ℓ_β}. Generally, C_{ℓ_β} is not greatly affected by Mach. However, in the transonic region the increase in the lift curve slope of the vertical tail increases this contribution to C_{ℓ_β} and usually results in an overall increase in C_{ℓ_β} in the transonic region.

7.4.7.2 $C_{\ell_{\delta_a}}$. Because of the decrease in the lift curve slope of all aerodynamic surfaces in supersonic flight, lateral control power decreases as Mach increases supersonically.

Aeroelasticity problems have been quite predominant in the lateral control system, since in flight at very high dynamic pressures the wing torsional deflections which occur with aileron use are considerable and cause noticeable changes in aileron effectiveness (Figure 7.46). At high dynamic pressures, dependent upon the given wing structural integrity, the twisting deformation might be great enough to nullify the effect of aileron deflection and the aileron effectiveness will be reduced to zero. Since at speeds above the point where this phenomenon occurs, rolling moments are created which are opposite in direction to the control deflection, this speed is termed "aileron reversal speed."

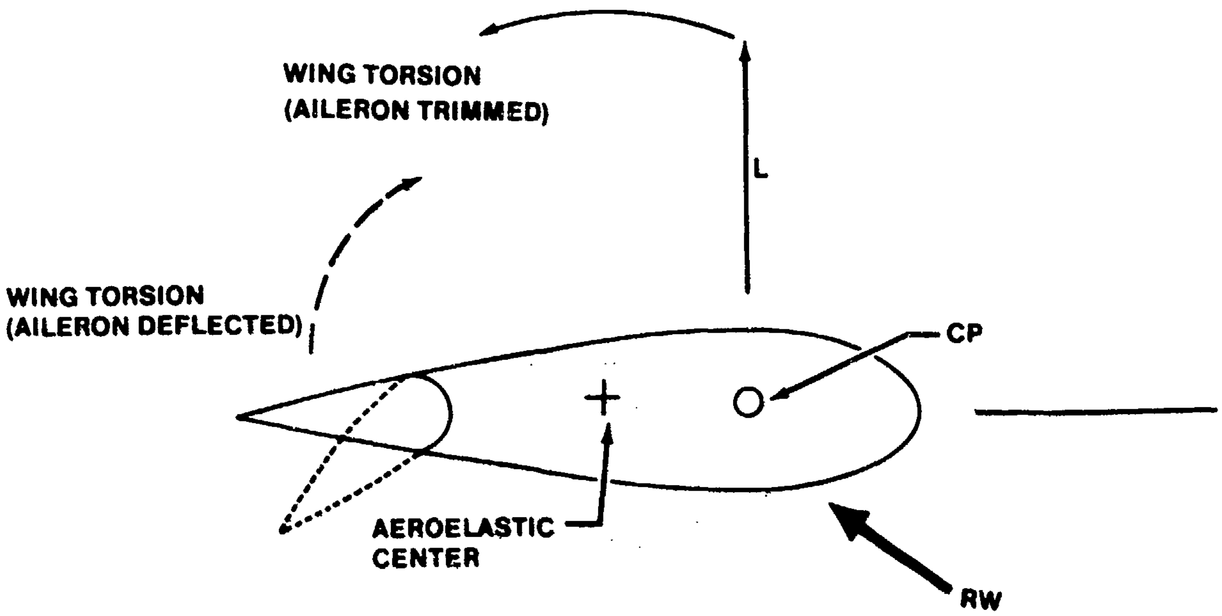


FIGURE 7.46. AEROELASTIC EFFECTS

In order to alleviate this characteristic, the wing must have a high torsional stiffness which presents a significant design problem in sweptwing aircraft. For an aircraft design of the B-47 type, it is easy to visualize how aeroelastic distortion might result in a considerable reduction in lateral control capability at high speeds. In addition, lateral control effectiveness at transonic Mach may be reduced seriously by flow separation effects as a result of shock formation. However, modern high-speed fighter designs have been so successful in introducing sufficient rigidity into wing structures and employing such design modifications as split ailerons, inboard ailerons, spoiler systems etc., that the resulting high control power coupled with the low C_{ℓ_p} of low aspect ratio planforms, has resulted in the lateral control becoming an accelerating device rather than a rate control. That is to say, a steady state rolling velocity is normally not reached prior to attaining the desired bank angle. Consequently, many high speed aircraft have a type of differential aileron system to provide the pilot with much more control surface during approach and landings and to restrict the degree of control in other areas of flight.

Spoiler controls are quite effective in reducing aeroelastic distortions since the pitching moment changes due to spoilers are generally smaller than those for a flap type control surface. However, a problem associated with

spoilers is their tendency to reverse the roll direction for small stick inputs during transonic flight. This occurs as a result of re-energizing the boundary layer by a vortex generator effect for very small deflections of the spoiler, which can reduce the magnitude of the shock induced separation and actually increase the lift on the wing. This difficulty can be eliminated by proper design.

7.4.7.3 C_{ℓ_p}. Since "damping" requires the development of lift on either the wing or the tail, it depends on the value of the lift curve slope. Thus, as the lift curve slope of the wing and tail decreases supersonically, C_{ℓ_p} decreases. Also, since most supersonic designs make use of low aspect ratio surfaces, C_{ℓ_p} will tend to be less for these designs.

7.4.7.4 C_{ℓ_r} and $C_{\ell_{\delta_r}}$. Both of these derivatives depend on the development of lift and will decrease as the lift curve slope decreases supersonically.

7.4.7.5 C_{ℓ_β}. Data on the supersonic variation of this derivative is sketchy, but it probably will not change significantly with Mach.

Variation of all the C_ℓ component derivatives with Mach is illustrated in Figure 7.47.

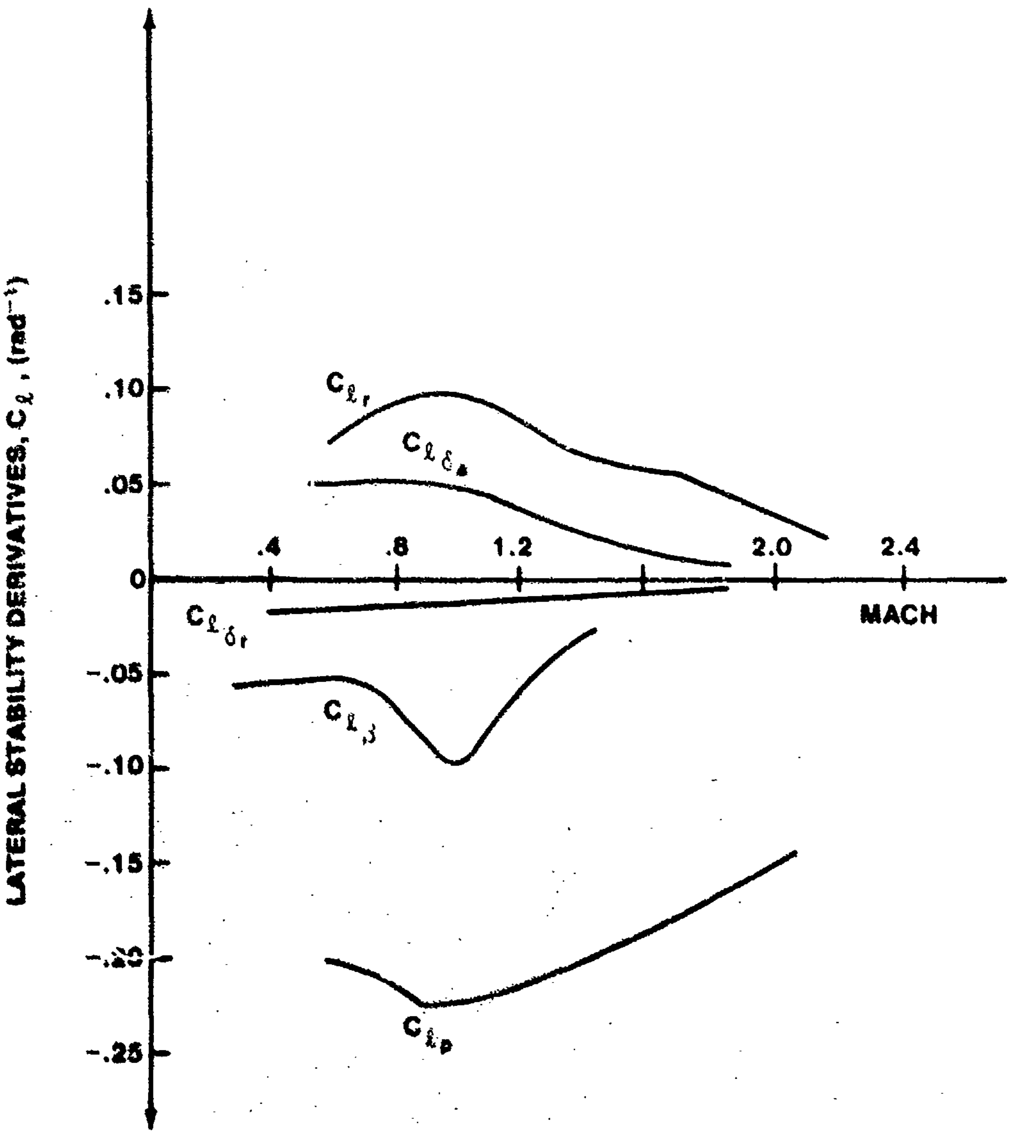


FIGURE 7.47. CHANGES IN LATERAL STABILITY DERIVATIVES WITH MACH (F-4C)

7.4.8 Controls Fixed Static Lateral Stability (Flight Test Relationship)

Having discussed the lateral stability derivatives, we are now ready to develop a parameter which can be measured in flight to determine the static lateral stability of an aircraft. As in the directional stability case, the maneuver that will be flown will be steady straight sideslip (reference Figure 7.21). Recalling the static lateral equation of motion and the fact that in a steady straight sideslip $p = r = \dot{\beta} = \Sigma G = 0$, then

$$C_{\ell_\beta}\beta + \cancelto{0}{C_{\ell_{\dot\beta}}\dot\beta} + \cancelto{0}{C_{\ell_p}\hat p} + \cancelto{0}{C_{\ell_r}\hat r} + C_{\ell_{\delta_r}}\delta_r + C_{\ell_{\delta_a}}\delta_a = \cancelto{0}{C_\ell}$$

Thus

$$C_{\ell_\beta}\beta + C_{\ell_{\delta_a}}\delta_a + C_{\ell_{\delta_r}}\delta_r = 0 \tag{7.82}$$

Solving for δ_a

$$\delta_a = -\frac{C_{\ell_\beta}}{C_{\ell_{\delta_a}}}\beta - \frac{C_{\ell_{\delta_r}}}{C_{\ell_{\delta_a}}}\delta_r \tag{7.83}$$

and differentiating with respect to β

$$\frac{\partial\delta_a}{\partial\beta} = -\frac{C_{\ell_{\beta\,(\text{Fixed})}}}{C_{\ell_{\delta_a}}} - \frac{C_{\ell_{\delta_r}}}{C_{\ell_{\delta_a}}}\frac{\partial\delta_r}{\partial\beta}$$

Disregarding the term that is usually the smallest contributor to the expression, $C_{\ell_{\delta_r}}$, we arrive at the following flight test relationship:

$$\frac{\partial\delta_a}{\partial\beta} = f\left(-\frac{C_{\ell_\beta}}{C_{\ell_{\delta_a}}}\right) \tag{7.84}$$

Since $C_{\ell_{\delta_a}} = a_a\,(S_a/S_w)\,(Y/b)$, all of which are known and fixed by design, then the only dominant variable remaining is C_{ℓ_β}. Therefore $\partial\delta_a/\partial\beta$ can be taken as a direct measure of the static lateral stability of an aircraft, controls fixed.

Since C_{ℓ_β} has to be negative in order to have lateral stability and $C_{\ell_{\delta_a}}$ is positive by definition, then $\partial\delta_a/\partial\beta$ should have a positive slope as shown in Figure 7.48.

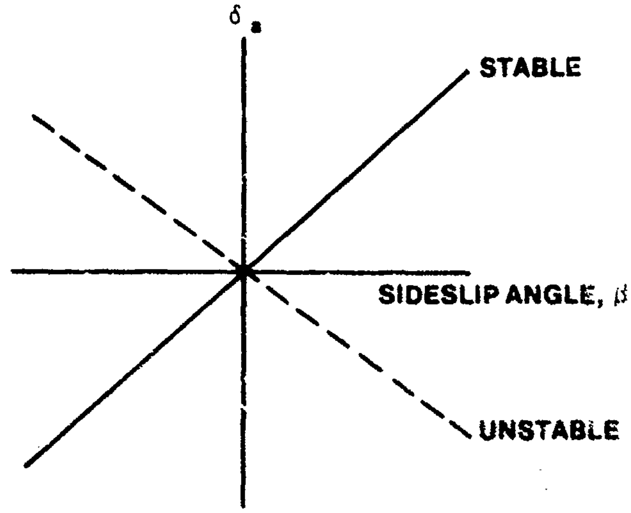


FIGURE 7.48. AILERON DEFLECTION δ_a VERSUS SIDESLIP ANGLE

7.4.9 Controls Free Static Lateral Stability (Flight Test Relationship)

On aircraft with reversible control systems, the ailerons are free to float in response to their hinge moments. Using the same approach as in the directional case, it is possible to derive an expression that will relate the "aileron free" static lateral stability to parameters that can be easily measured in flight. For the discussion of aileron hinge moments, a change in angle of attack on a wing will be defined as positive if it causes a positive rolling moment. This may be contrary to the sign convention used in the longitudinal case.

In a steady straight sideslip, $\Sigma \mathcal{L} = 0$ which implies that $\Sigma \mathcal{L}_{\text{Hinge Pin}} = 0$. Now if moments are summed about the aileron hinge pin, then a pilot must apply aileron forces to oppose the aerodynamic hinge moment in order to keep the ailerons deflected the required amount to maintain $\Sigma \mathcal{L}_{\text{Hinge Pin}} = 0$. This aileron force, F_a, acts through a moment arm and gearing mechanism, both accounted for by some constant K.

Thus in steady straight flight

$$\Sigma \mathcal{L}_{\text{Hinge Pin}} = F_a K + H_a = 0 \tag{7.85}$$

where H_a = the aileron hinge moment. Or

$$F_a = -GH_a \tag{7.86}$$

where G = 1/K (definition).

Recalling coefficient format,

$$C_h = \frac{H_a}{\underbrace{(1/2\ \rho\ V_T^2)}_{q_a} S_a\ c_a} \tag{7.87}$$

Thus

$$H_a = C_h \underbrace{(1/2\ \rho\ V_T^2)}_{q_a} S_a\ c_a \tag{7.88}$$

But we have already shown from Equation 7.23 that

$$C_h = C_{h_{\alpha_a}} \alpha_a + C_{h_{\delta_a}} \delta_a \tag{7.89}$$

Therefore,

$$H_a = \underbrace{(1/2\ \rho\ V_T^2)}_{q_a} S_a\ c_a \left(C_{h_{\alpha_a}} \alpha_a + C_{h_{\delta_a}} \delta_a \right) \tag{7.90}$$

Thus Equation 7.86 becomes

$$F_a = \underbrace{-G\ (1/2\ \rho\ V_T^2)}_{q_a}\ S_a\ c_a \left(C_{h_{\alpha_a}} \alpha_a + C_{h_{\delta_a}} \delta_a \right) \tag{7.91}$$

Recalling that for a floating control surface

$$C_{h_{\alpha_a}} \alpha_a = - C_{h_{\delta_a}} \delta_{a_{(Float)}} \tag{7.92}$$

Therefore

$$F_a = - \underbrace{G\ (1/2\ \rho\ V_T^2)}_{q_a} S_a\ c_a\ C_{h_{\delta_a}} \left(\delta_a - \delta_{a_{(Float)}} \right) \tag{7.93}$$

The difference between where the pilot pushes the aileron, δ_a, and the amount it floats, $\delta_{a_{(Float)}}$, is the free position of the aileron, $\delta_{a_{(Free)}}$. Therefore,

$$F_a = - \underbrace{G\ (1/2\ \rho\ V_T^2)}_{q_a} S_a\ c_a\ C_{h_{\delta_a}} \delta_{a_{(Free)}} \tag{7.94}$$

Differentiating with respect to β

$$\frac{\partial F_a}{\partial \beta} = - \underbrace{G\ (1/2\ \rho\ V_T^2)}_{q_a} S_a\ c_a\ C_{h_{\delta_a}} \frac{\partial \delta_{a_{(Free)}}}{\partial \beta} \tag{7.95}$$

From Equation 7.84, it can be shown that

$$\frac{\partial \delta_{a_{(Free)}}}{\partial \beta} = - \frac{C_{\ell_{\beta_{(Free)}}}}{C_{\ell_{\delta_a}}} \tag{7.96}$$

Thus

$$\frac{F_a}{\partial \beta} = \overset{(+)}{G} \underbrace{\left(1/2\ \overset{(+)}{\rho}\ V_T^2\right)}_{q_a} \overset{(+)}{S_a}\ \overset{(+)}{c_a}\ \overset{(-)}{\frac{C_{h_{\delta_a}}}{\underset{(+)}{C_{\ell_{\delta_a}}}}}\ \overset{(-)}{C_{\ell_{\beta(\mathrm{Free})}}} = (+) \text{ for stability} \qquad (7.97)$$

This equation shows that the parameter $\partial F_a/\partial \beta$ can be taken as an indication of the aileron free static lateral stability of an aircraft since all terms are either constant or set by design, except C_{ℓ_β}. More importantly, $\partial F_a/\partial \beta$ can be readily measured in flight.

An analysis of Equation 7.97 reveals that for stable dihedral effect, a plot of $F_a/\partial \beta$ would have a positive slope (Figure 7.49).

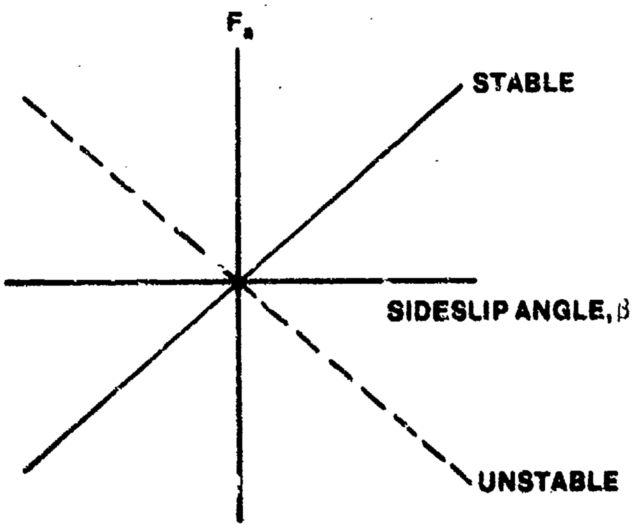


FIGURE 7.49. AILERON FORCE F_a VERSUS SIDESLIP ANGLE

7.5 ROLLING PERFORMANCE

Now that we have shown how aileron force and deflection can be used as a measure of the stable dihedral effect of an aircraft, it is necessary to consider how these parameters affect the rolling capability of the aircraft. For example, full aileron deflection may produce excellent rolling characteristics on certain aircraft; however, because of the large aileron forces required, the pilot may not be able to fully deflect the ailerons, thus making the overall rolling performance unsatisfactory. Thus, it is necessary to evaluate the rolling performance of the aircraft.

The rolling qualities of an aircraft can be evaluated by examining the parameters F_a, δ_a, p and $(pb/2U_0)$. Although the importance of the first three parameters is readily apparent, the parameter $(pb/2U_0)$ needs some additional explanation.

Mathematically $pb/2U_0$ is a nondimensional parameter where p = roll rate (rad/sec); b = wing span (ft); and U_0 = velocity (ft/sec).

Physically $pb/2U_0$ may be described as the helix angle that the wing tip of a rolling aircraft describes (Figure 7.50). In addition, the $pb/2U_0$ that can be produced by full lateral control deflection is a measure of the relative lateral control power available.

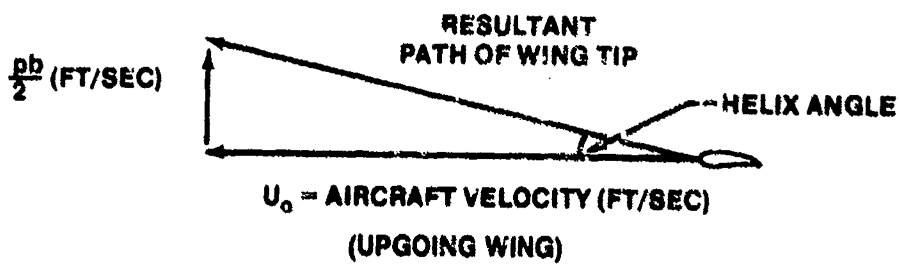


FIGURE 7.50. WING TIP HELIX ANGLE (UPGOING WING)

It can be seen that

$$\tan \text{(Helix Angle)} = \frac{pb}{2U_0} \tag{7.98}$$

Assuming small angles

$$\text{Helix Angle} = \frac{pb}{2U_0} \tag{7.99}$$

This angle also represents the change in angle of attack of a rolling wing (Figure 7.51).

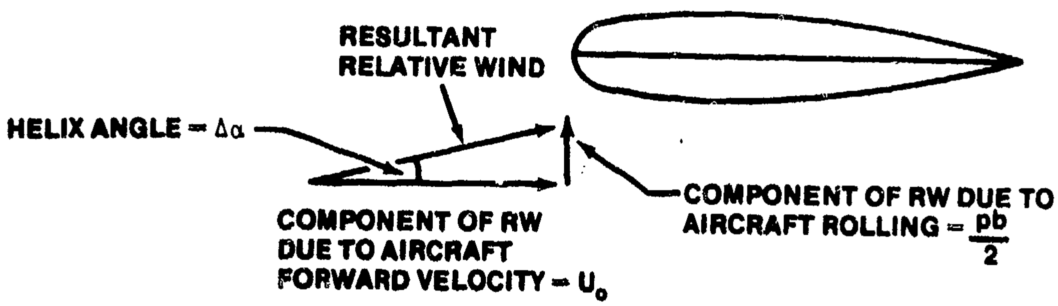


FIGURE 7.51. WIND FORCES ACTING ON A DOWNGOING WING DURING A ROLL

This figure shows that the angle of attack of the downgoing wing is increased due to the roll rate. This implies increased lift opposite the direction of roll on the downgoing wing and, conversely, decreased lift in the direction of roll on the upgoing wing due to decreased α. This is essentially the same effect as C_{ℓ_p}. Thus $pb/2U_0$ represents a damping term.

With the foregoing discussion as background, we are now ready to discuss the effect of F_a, δ_a, p, $pb/2U_0$ on roll performance through the flight envelope of an aircraft.

From Equation 7.94 it can be seen that

$$F_a = f\left(V_T^2, \delta_{a_{(Free)}}\right) \tag{7.98}$$

$$\delta_{a_{(Free)}} = f\,(F_a,\ 1/V_T{}^2) \qquad (7.99)$$

To derive a functional relationship for $(pb/2U_0)$, it is necessary to start with the basic lateral equations of motion,

$$C_\ell = C_{\ell_\beta}\beta + C_{\ell_{\dot\beta}}\hat{\dot\beta} + C_{\ell_p}\hat{p} + C_{\ell_r}\hat{r} + C_{\ell_{\delta_a}}\delta_a + C_{\ell_{\delta_r}}\delta_r \qquad (7.100)$$

and examine the effect of roll terms only, i.e., assume that the roll moment developed is due to the interaction of moments due to δ_a and roll damping only. Therefore, Equation 7.100 becomes

$$C_\ell = C_{\ell_p}\hat{p} + C_{\ell_{\delta_a}}\delta_a \qquad (7.101)$$

Recalling from equations of motion that $\hat{\zeta} = \zeta b/2U_0$, then $\hat{p} = pb/2U_0$. Therefore,

$$C_\ell = C_{\ell_p}\left(\frac{pb}{2U_0}\right) + C_{\ell_{\delta_a}}\delta_a \qquad (7.102)$$

Below Mach or aeroelastic effects, $C_{\ell_{MAX}}$ = constant, so if it is desired to evaluate an aircraft's maximum rolling performance, Equation 7.102 becomes

$$C_{\ell_p}\left(\frac{pb}{2U_0}\right) + C_{\ell_{\delta_a}}\delta_a = \text{constant} \qquad (7.103)$$

$$\frac{pb}{2U_0} = \frac{\text{Constant} - C_{\ell_{\delta_a}}\delta_a}{C_{\ell_p}} \qquad (7.104)$$

$$\left(\frac{pb}{2U_0}\right) = f\ (\delta_a) \tag{7.105}$$

But we have already shown that $\delta_a = f\ (F_a,\ 1/V_T^2)$ therefore,

$$\frac{pb}{2U_0} = f\ (F_a,\ 1/V_T^2) \tag{7.106}$$

A functional relationship for roll rate, p, can be derived from Equation 7.104

$$p = \frac{\text{Constant} - C_{\ell_{\delta_a}} \delta_a}{C_{\ell_p}} \left(\frac{2}{b}\right)\left(U_0\right) \tag{7.107}$$

$$p = f\ (U_0,\ \delta_a) \tag{7.108}$$

and since

$$\delta_a = f\ (F_a,\ 1/V_T^2)$$

then

$$p = f\ (F_a,\ 1/V_T) \tag{7.109}$$

assuming

$$U_0 = V_T \text{ (i.e., no sideslip)}$$

To summarize, the rolling performance of an aircraft can be evaluated by examining the parameters, F_a, δ_a, p, and $(pb/2U_0)$. Functional relationships have been developed in order to look at the variance of these parameters below Mach or aeroelastic effect. These functional relationships are

$$F_a = f\ (V_T^2, \delta_a) \tag{7.110}$$

$$\delta_a = f\ (F_a, 1/V_T^2) \tag{7.111}$$

$$\frac{pb}{2U_0} = f\ (\delta_a) = f\ (F_a, 1/V_T^2) \tag{7.112}$$

$$p = f\ (V_T, \delta_a) = f\ (F_a, 1/V_T) \tag{7.113}$$

These relationships are expressed graphically in Figure 7.52 for a case in which the pilot desires the maximum roll rate at all airspeeds.

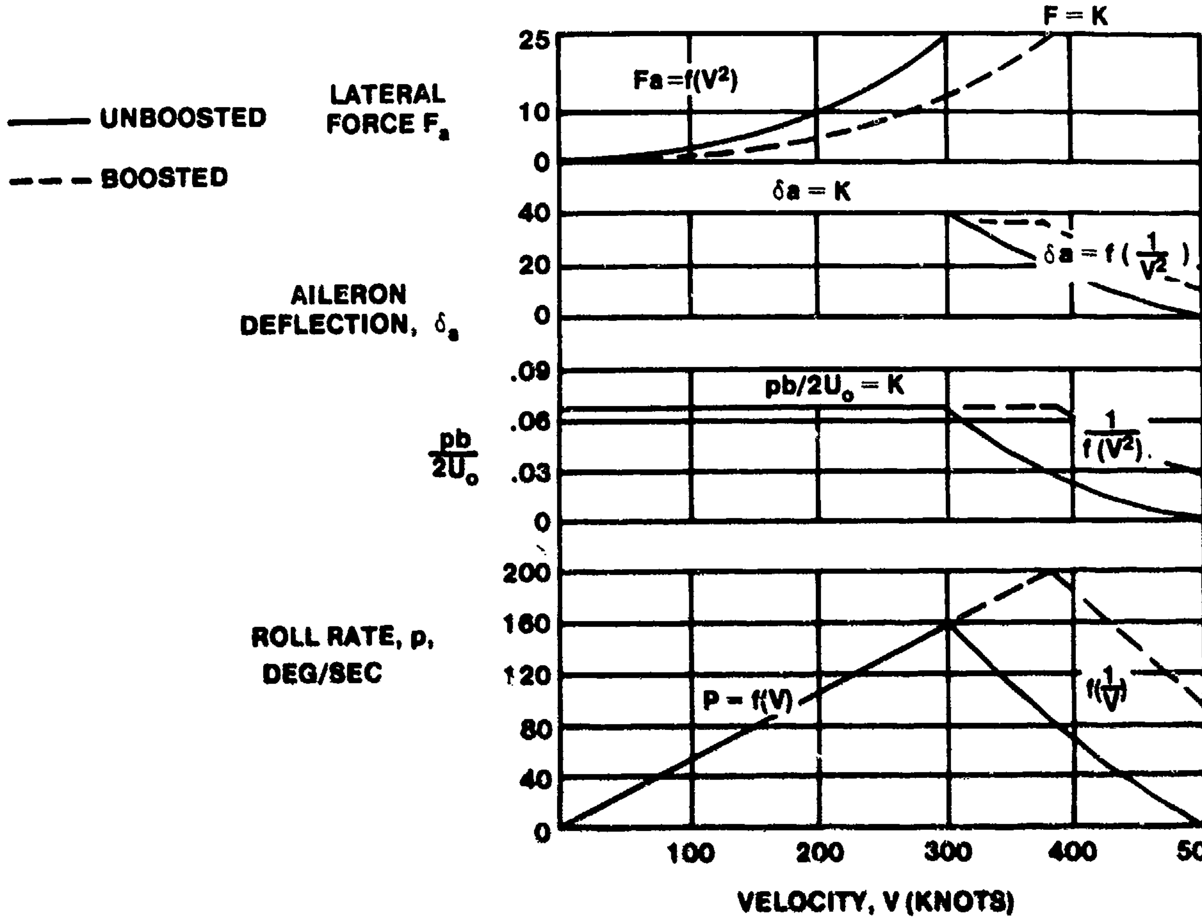


FIGURE 7.52. ROLLING PERFORMANCE

As indicated in Equation 7.110, the force required to hold a constant aileron deflection will vary as the square of the airspeed. The force required by the pilot to hold full aileron deflection will increase in this manner until the aircraft reaches V_{MAX} or until the pilot is unable to apply any more force. In Figure 7.52, it is assumed that the pilot can supply a maximum of 25 pounds force and that this force is reached at 300 knots. If the speed is increased further, the aileron force will remain at this 25 pound maximum value. The curve of aileron deflection versus airspeed shows that the pilot is able to maintain full aileron deflection out to 300 knots. Inspection of Equation 7.111 shows that if aileron force is constant beyond 300 knots, then aileron deflection will be proportional to $(1/V_T^2)$. Equation 7.112 shows that $(pb/2U_0)$ will vary in the same manner as aileron deflection. Inspection of Equation 7.113 shows that the maximum roll rate available will increase linearly as long as the pilot can maintain maximum aileron

deflection, up to 300 knots in this case. Beyond this point, the maximum roll rate will fall off hyperbolically. That is, above 300 knots, p is proportional to $1/V_T$. It follows, then, that at high speeds the maximum roll rate may become unacceptably low. One method of combating this problem is to increase the pilot's mechanical advantage by adding boosted or fully powered ailerons.

By boosting the controls, the pilot can maintain full aileron deflection with less physical effort on his part. Thus, $F_a = 25$ pounds will be delayed to a higher airspeed. The net effect is a shift of the F_a, δ_a and $pb/2U_0$ curves and a resulting increase in p (reference Figure 7.52 dashed lines).

Many modern aircraft have irreversible flight control systems. These systems allow an aircraft to be designed for a specific aileron force at full deflection, regardless of the airspeed. This allows the pilot to hold full deflection at high speeds, resulting in a constant helix angle and increasing roll rate at higher airspeeds. This change in performance is still limited by Mach effects and aeroelasticity.

7.6 LATERAL-DIRECTIONAL STATIC STABILITY FLIGHT TESTS

The lateral-directional characteristics of an aircraft are determined by two different flight tests: the steady straight sideslip test and the aileron roll test. The tests do not measure lateral and directional characteristics independently. Rather, each test yields information concerning both the lateral and the directional characteristics of the aircraft. The requirements of the MIL-F-8785C will be discussed.

7.6.1 Steady Straight Sideslip Flight Test

The steady straight sideslip is a common maneuver which requires the pilot to balance the forces and moments generated on the airplane by a sideslip with appropriate lateral and directional control inputs and bank angle. Since these control forces and positions and bank angles are at least indicative of the sign (if not the magnitude) of the generated forces and moments (and therefore of the associated stability derivatives), the steady straight sideslip is a convenient flight test technique.

All equations relating to the static directional stability of an aircraft were developed under the assumption that the aircraft was in a "steady straight sideslip." This is the maneuver used in the sideslip test. First, trim the aircraft at the desired altitude and airspeed. Apply rudder to develop a sideslip. In order to maintain "straight" flight (constant ground track), bank the aircraft in the direction opposite that of the applied rudder. In Figure 7.53 the aircraft is in a steady sideslip. The moment created by the rudder, $\mathcal{N}_{\delta_r}$, must equal the moment created by the

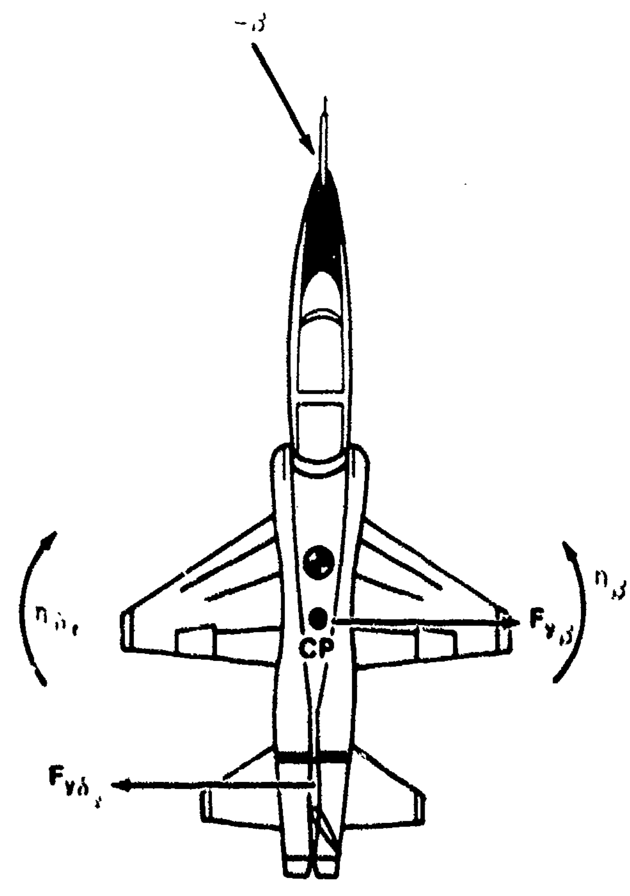


FIGURE 7.53. STEADY SIDESLIP

aerodynamic forces acting on the aircraft , n_β. In this condition the side forces are unbalanced. F_{y_β}, will always be greater the $F_{y_{\delta_r}}$. Thus, in the case depicted, the aircraft will accelerate, or turn, to the right. In order to stop this turn, it is necessary to bank the aircraft, in this case to the left (Figure 7.54). The bank allows a component of aircraft weight, W sin ϕ, to act in the y direction and balance the previously unbalanced side forces. Thus, the pilot establishes a "straight sideslip." By holding this condition constant with respect to time or varying it so slowly in a continuously stabilized condition that rate effects are negligible, he establishes a "straight sideslip" - the condition that was used to derive the flight test relationships in static directional stability theory.

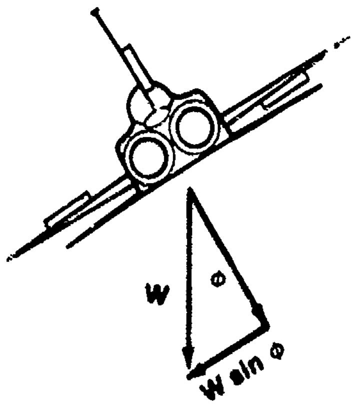


FIGURE 7.54. STEADY STRAIGHT SIDESLIP

MIL-F-7875C, Paragraph 3.3.6 outlines the sideslip tests that must be performed in an aircraft. The specification requires that sideslips be tested to full rudder pedal deflection, 250 pounds of rudder pedal force, or maximum aileron deflection, whichever occurs first. Often sideslips must be discontinued prior to reaching these limits due to controllability or structural problems.

The following MIL-F-8785C paragraphs apply to sideslip tests:
3.2.3.7, 3.3.5, 3.3.6, 3.3.6.1, 3.3.6.2, 3.3.6.3, 3.3.6.3.1, 3.3.6.3.2.

One property of basic importance in the sideslip test is the directional stiffness of an aircraft or its static directional stability. To review, the static directional stability of an aircraft is defined by the initial tendency of the aircraft to return to or depart from its equilibrium angle of sideslip when disturbed from the equilibrium condition. In order to determine if the aircraft possesses static directional stability, it is necessary to determine how the yawing moments change as the sideslip angle is changed. For positive directional stability, a plot of C_{n_β} must have a positive slope (Figure 7.55).

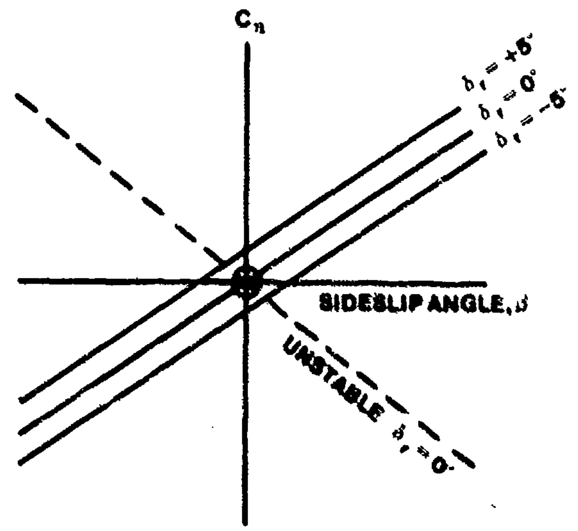


FIGURE 7.55. WIND TUNNEL RESULTS OF YAWING MOMENT COEFFICIENT C_n VERSUS SIDESLIP ANGLE

Plots like those presented in Figure 7.55 are obtained from wind tunnel data. The aircraft model is placed at various angles of sideslip with various angles of rudder deflection, and the unbalanced moments are measured. However it is impossible to determine from flight tests the unbalanced moments at varying angles of sideslip. It was shown in static directional theory, however, that the rudder deflection required to fly in a steady straight sideslip is an indication of the amount of yawing moment tending to return the aircraft to or remove it from its original trimmed angle of sideslip. A plot is made of rudder deflection required versus sideslip angle in order to determine the sign of the rudder fixed static directional stability, C_{n_β}.

The control fixed stability parameter, $\partial\delta_r/\partial\beta$, for a directionally stable aircraft has a negative slope as shown in Figure 7.56. Paragraph 3.3.6.1, requires that right rudder pedal deflection ($+\delta_r$) accompany left sideslips ($-\beta$). Further, for angles of sideslip between $\pm 15^{\circ}$, a plot of $\partial\delta_r/\partial\beta$ should be essentially linear. For larger sideslip angles, an increase in β must require an increase in δ_r. In other words, the slope of $\partial\delta_r/\partial\beta$ cannot go to zero.

Drastic changes occur in the transonic and supersonic speed regions. In the transonic region where the flight controls are most effective, a small δ_r may give a large β and thus $\partial\delta_r/\partial\beta$ may appear less stable. However, as speed increases, control surface effectiveness decreases, and $\partial\delta_r/\partial\beta$ will increase in slope. This apparent change in C_{n_β} is due solely to a change in control surface effectiveness and can give an entirely erroneous indication of the magnitude of the static directional stability if not taken into account.

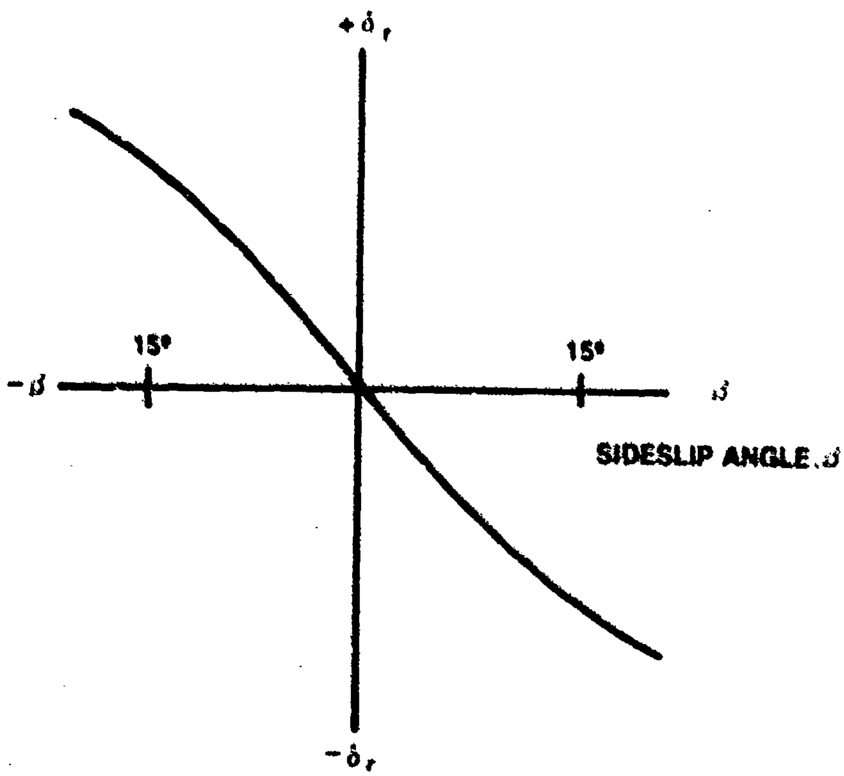


FIGURE 7.56. RUDDER DEFLECTION δ_r VERSUS SIDESLIP

A plot of rudder force required versus sideslip, $\partial F_r/\partial\beta$, is an indication of the rudder-free static directional stability of an aircraft. A plot of $\partial F_r/\partial\beta$ must have a negative slope for positive rudder-free static directional stability. Paragraph 3.3.6.1 requires that a plot of $\partial F_r/\partial\beta$ be essentially linear between $\pm 10^{\circ}$ of β from the trim condition. However, at greater angles of sideslip, the rudder forces may lighten but may never go to zero, or overbalance. These requirements are depicted in Figure 7.57.

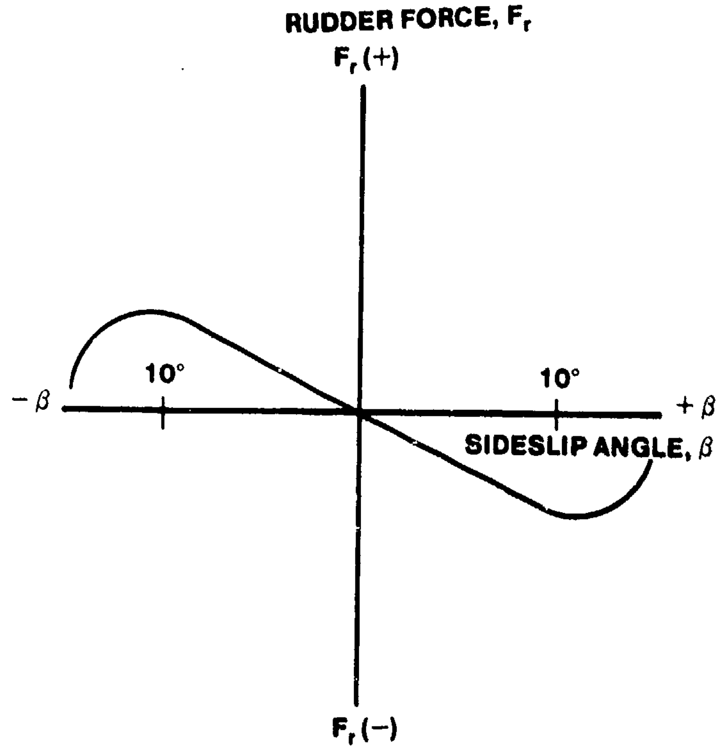


FIGURE 7.57. CONTROL FREE SIDESLIP DATA

The control force information in Figure 7.57 is acceptable as long as the algebraic sign of F_r/β is negative. At very large sideslip angles, the slope F_r/β may be positive. This is acceptable as long as the rudder force required does not go to zero.

Static lateral characteristics are also investigated during the sideslip test. It was shown in the theory of static lateral stability that $\partial\delta_a/\partial\beta$ may be taken as an indication of the control-fixed dihedral effect of an aircraft, $C_{\ell_{\beta\,(\text{Fixed})}}$. For stable dihedral effect, it was shown that a plot of $\partial\delta_a/\partial\beta$

must have a positive slope. Right aileron control deflection shall accompany right sideslips and left aileron control shall accompany left sideslips. A plot of $\partial\delta_a/\partial\beta$ for stable dihedral effect is presented in Figure 7.58.

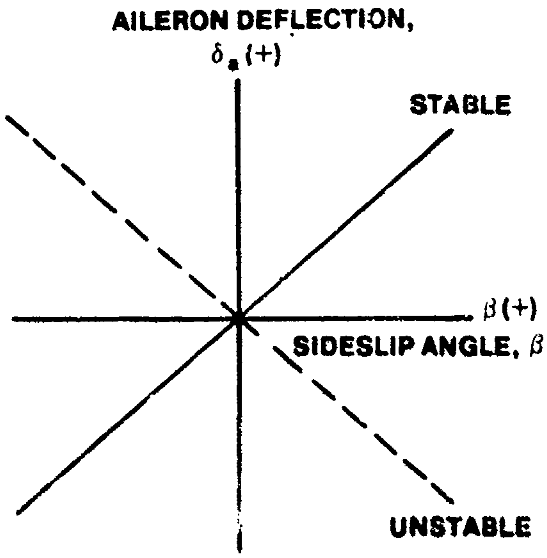


FIGURE 7.58. CONTROL FIXED SIDESLIP DATA

Paragraph 3.3.6.3.2 limits the amount of stable dihedral effect an aircraft will exhibit by specifying that no more than 75% of roll control power available to the pilot, and no more than 10 lbs of roll stick force or 20 lbs of roll wheel force are required for sideslip angles which may be experienced in service employment.

Theoretical discussion of control free dihedral effect revealed that $\partial F_a/\partial\beta$ gives an indication of $C_{\ell_{\beta(\text{Free})}}$, and that for stable dihedral effect $\partial F_a/\partial\beta$ is positive (Figure 7.59). Paragraph 3.3.6.3 states that left aileron force should be required for left sideslips and that a plot of $\partial F_a/\partial\beta$ should be essentially linear for all of the mandatory sideslips tested.

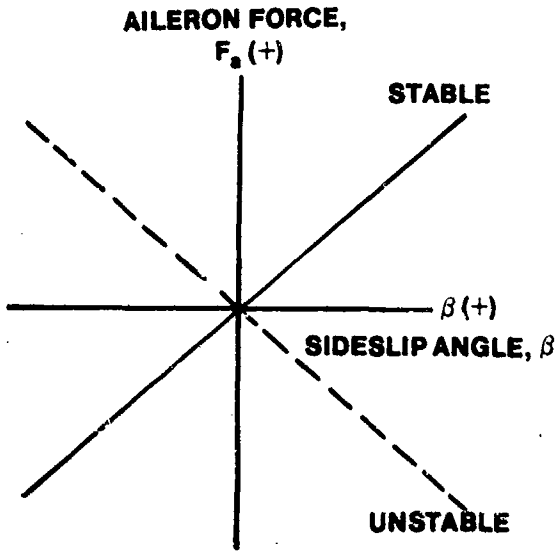


FIGURE 7.59. CONTROL FREE SIDESLIP DATA

Paragraph 3.3.6.3.1 does permit an aircraft to exhibit negative dihedral effect in wave-off conditions as long as no more than 50% of available roll control or 10 lbs of aileron control force is required in the negative dihedral direction.

Paragraph 3.3.6.2 also states that "an increase in right bank angle must accompany an increase in right sideslip."

A longitudinal trim change will most likely occur when the aircraft is sideslipped. Paragraph 3.2.3.7 places definite limits on the allowable magnitude of this trim change. It is preferred that an increasing pull of force accompany an increase in sideslip angle and that the magnitude and direction of the trim change should be similar for both left and right sideslips. The specification also limits the magnitude of the control force accompanying the longitudinal trim change depending on the type of controller in the aircraft (stick or wheel). A plot of elevator force versus sideslip angle that complies with MIL-F-8785C is presented in Figure 7.60.

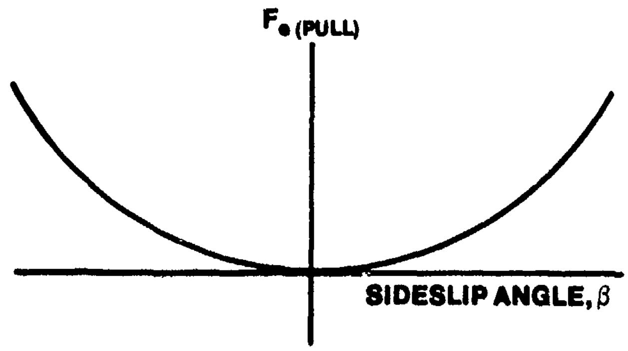


FIGURE 7.60. ELEVATOR FORCE, F_e VERSUS SIDESLIP ANGLE

EXAMPLE DATA

Sample data plots of sideslip test results are presented in Figures 7.61 and 7.62.

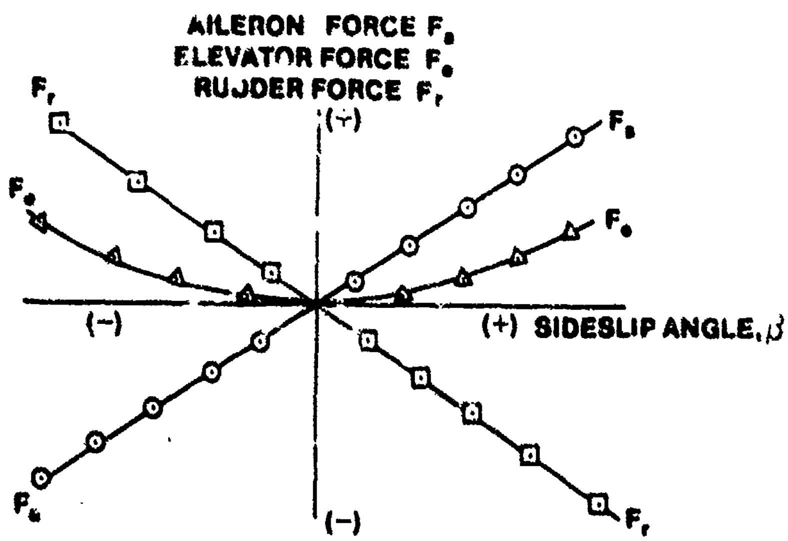


FIGURE 7.61. STEADY STRAIGHT SIDESLIP CHARACTERISTICS
CONTROL FORCES VERSUS SIDESLIP

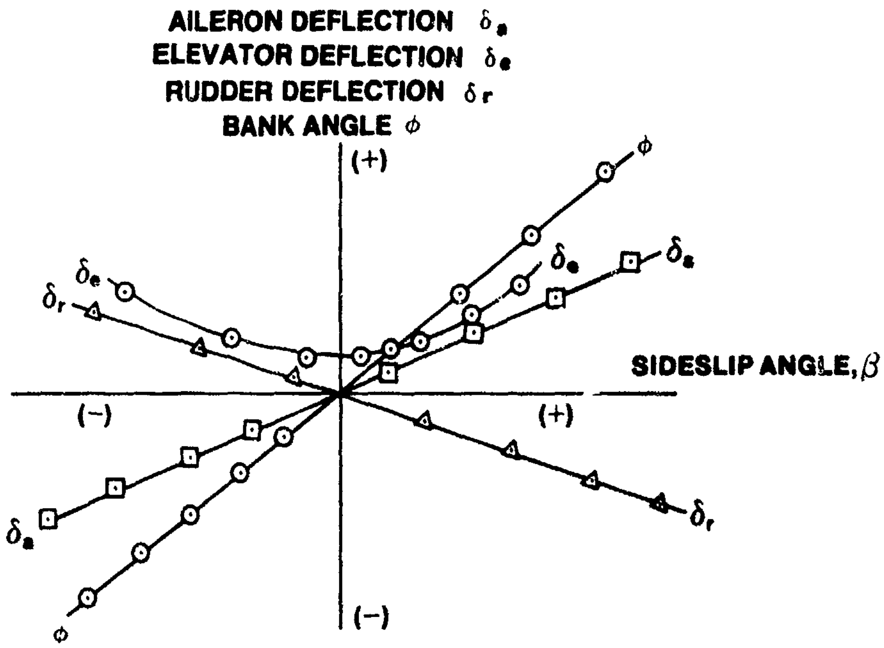


FIGURE 7.62. STEADY STRAIGHT SIDESLIP CHARACTERISTICS CONTROL DEFLECTION AND BANK ANGLE VERSUS SIDESLIP

7.6.2 Aileron Roll Flight Test

The aileron roll flight test technique is used to determine the rolling performance of an aircraft and the yawing moments generated by rolling. Roll coupling is another important aircraft characteristic normally investigated by using the aileron roll flight test technique. The roll coupling aspect of the aileron roll test will not be investigated at the USAF Test Pilot School. However, the theoretical aspects of roll coupling will be covered in Chapter 9.

To accomplish the aileron roll flight test, trim the aircraft at the desired altitude and airspeed. Then, abruptly place the lateral control to a particular control deflection (1/4, 1/2, 3/4, or full) with a step input. Normally, the desired control deflection is obtained by using some mechanical restrictor such as a chain stop. With the lateral control at the desired deflection, roll the aircraft through a specified increment of bank. For control deflections less than a maximum, the aircraft is normally rolled through 90° of bank. Because of the higher roll rates obtained at full control deflection, it is usually desirable to roll the aircraft through 360° of bank. To facilitate aircraft control when rolling through a bank angle change of 90°, start the roll from a 45° bank angle. During the roll, an automatic data recording system may be used to record the following

information: aileron position, aileron force, bank angle, sideslip and roll rate. Aileron rolls are normally conducted in both directions to account for roll variations due to engine gyroscopic effects. Aileron rolls are performed with rudders free, with rudders fixed, and are coordinated with $\beta = 0$ throughout roll.

Exercise caution in testing a fighter type airplane in rolling maneuvers. The stability of the airplane in pitch and yaw is lower while rolling. The incremental angles of attack and sideslip that are attained in rolling can produce accelerations which are disturbing to the pilot and can also cause critical structural loading. The stability of an airplane in a rolling maneuver is a function of Mach, roll rate, dynamic pressure, angle of attack, configuration, and control deflections during the maneuver.

The most important design requirement imposed upon ailerons or other lateral control devices is the ability to provide sufficient rolling moments at low speeds to counteract the effects of vertical asymmetric gusts tending to roll the airplane. This means, in effect, that the ailerons must provide a minimum specified roll rate and a rolling acceleration such that the required rate of roll can be obtained within a specified time, even under loading conditions that result in the maximum rolling moment of inertia (e.g., full tip tanks). The steady roll rate and the minimum time required to reach a particular change in bank angle are the two parameters presently used to indicate rolling capability. Pilot opinion surveys reveal that time to roll a specified number of degrees provides the best overall measure of rolling performance.

The following is a complete list of MIL-F-8785C paragraphs that apply to aileron roll tests: 3.3.2.3; 3.3.2.4; 3.3.2.6; 3.3.4; 3.3.4.1; 3.3.4.1.1; 3.3.4.1.2; 3.3.4.1.3; 3.3.4.2; 3.3.4.3; 3.3.4.4; 3.3.4.5.

The minimum rolling performance required of an aircraft is outlined in MIL-F-8785C, Table IX. This rolling performance is expressed as a function of time to reach a specified bank angle. Table IX is supplemented further by roll performance required of Class IV airplanes in various flight phases. The specific requirements for Class IV airplanes are spelled out in Paragraphs 3.3.4.1.1, .2, .3, and .4. Paragraph 3.3.4.2 and Table X specify the maximum and minimum aileron control forces allowed in meeting the roll requirements of Table IX and the supplemental requirements concerning Class IV aircraft.

Paragraph 3.3.2.5 specifies the maximum rudder force permitted for coordinating the required rolls.

In addition to examining time required to bank a specified number of degrees and aileron forces, F_a, it is necessary to examine the maximum roll rate, p_{max}, to get a complete picture of the aircraft's rolling performance. Therefore, in any investigation of aircraft rolling performance, the maximum roll rate obtained at maximum lateral control displacement is normally plotted versus airspeed.

Paragraph 3.3.4.3 states that there should be no objectionable nonlinearities in roll response to small aileron control deflection or forces. To investigate this area, it is necessary to observe the roll response to aileron deflections less than maximum - such as 1/4 and 1/2 aileron deflections (Figure 7.63).

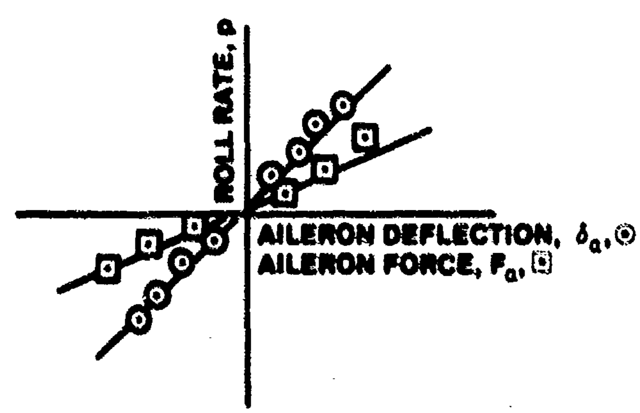


FIGURE 7.63. LINEARITY OF ROLL RESPONSE

Turn coordination requirements are spelled out in Paragraph 3.3.2.6 for steady turning maneuvers.

The other area of prime interest in the aileron roll flight test is the amount of sideslip that is developed in a roll and the phasing of this sideslip with respect to the roll rate. Associated with this characteristic is the roll rate oscillation. These factors influence the pilot's ability to accomplish precise tracking tasks.

7.6.3 Demonstration Flight

To unify all that has been said concerning the sideslip and aileron roll flight test techniques, a complete description of a demonstration mission is presented in the Flying Qualities Phase Planning Guide.

PROBLEMS

7.1. Answer the following questions True (T) or False (F).

T F The primary source of directional instability is the aircraft fuselage.

T F Ailerons usually produce proverse yaw.

T F The tail contribution to C_{n_r} is the dominant damping factor.

T F In a steady straight sideslip $p = 0$.

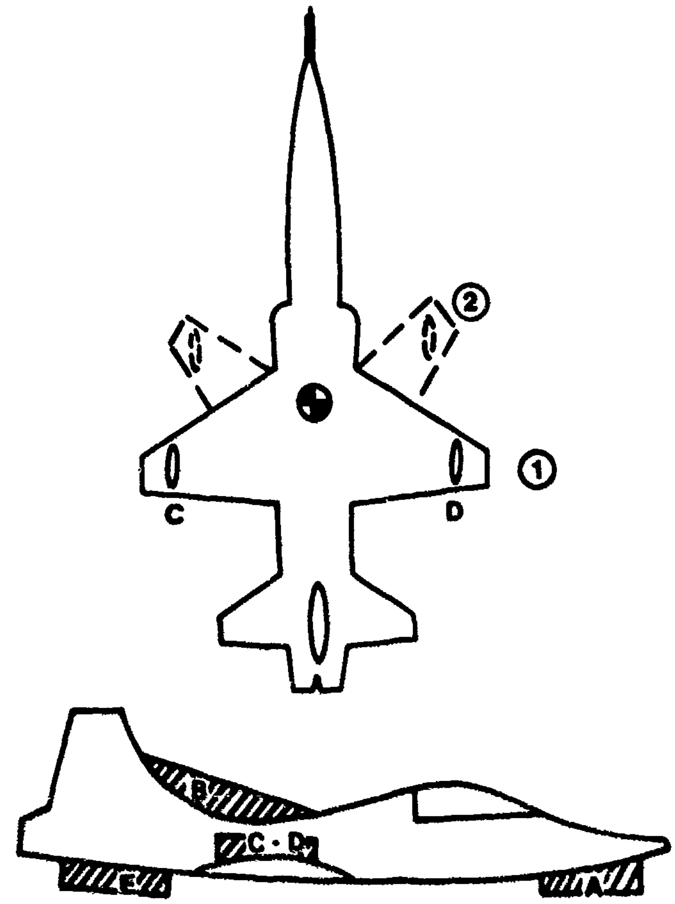
7.2. The aircraft shown in the following diagram is undergoing a design study to improve static directional stability. The Contractor has recommended the addition of surfaces A, B, C, D, and E. However, the System Program Office (SPO) isn't too impressed and wants the following questions answered by the Flight Test Center. With the wings in position 1 or 2 determine if the following contributions to C_{n_β} are stabilizing (+) or destabilizing (-):

	POSITION 1	POSITION 2
a. Vertical Tail		
b. Area E (Ventral)		
c. Canopy Area		
d. Area B (Dorsal)		
e. Area A		
f. Area C		

g. Area D

h. Wing

i. Fuselage



7.3. Lateral-Directional Static Stability is a function of what variables?

7.4. Sketch a curve (C_n versus β) for an aircraft with stable static directional qualities and show the effect on this curve of adding a dorsal.

7.5. Does fuselage sidewash (σ) have a stabilizing or destabilizing effect on C_{n_β}? Why?

7.6. How would you design a flying wing with no protuberances in the Z direction so that it has directional stability?

7.7. What effect do straight wings have on C_{n_β}? Why?

7.8. How does increasing wing sweep (Λ) effect C_{n_β}? Why?

7.9. What effect will increasing AR have on $C_{n_{\beta}FIN}$? Why?

7.10. What is the sign of a left rudder deflection for a tail to the rear aircraft? For a right rudder deflection? Why?

7.11. What would the sign of τ be for a tail to the rear aircraft? Why?

7.12. For a tail to the rear aircraft, draw an airfoil showing the pressure distribution caused by + α_F. What is the sign of H_r?

What is the sign of $\partial C_h/\partial \alpha_F$. Why? Sketch a plot of H_r versus α_F.

7.13. For a tail to rear aircraft, draw an airfoil showing the pressure distribution caused by δ_r. What is the sign of $H_{r_{\delta_r}}$?

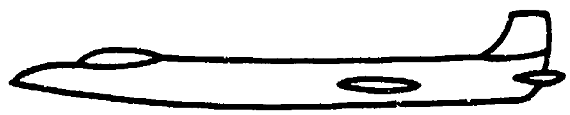
What is the sign of $\partial C_h/\partial \delta_r$? Why? Sketch a plot of H_r versus δ_r.

7.14. Knowing $\delta_{r_{FLOAT}} = - C_{h_{\alpha_F}} \; C_{h_{\delta_r}} \; \alpha_F$ for a tail to the rear aircraft, determine which direction the rudder will float for - αF.

7.15. Knowing $C_{n_{\beta (FREE)}} = V_v \left(a_F \;\; 1 - \frac{\partial \sigma}{\partial \beta} \right) \left(1 - \frac{C_{h_{\alpha_F}}}{C_{h_{\delta_r}}} \right)$,

how does float effect $C_{n_{\beta}}$ for a tail to the rear aircraft? Tail to front?

HINT: You should be able to answer Questions 7.10 - 7.14 for a tail to front aircraft.



7.16. Go-Fast Inc. of Mojave has completed a preliminary design on a new Mach 3.0 fighter. The chief design engineer is concerned that the aircraft may not have sufficient directional stability. List three design changes/additions which would help ensure directional stability.

7.17. You are flying an F-15 Eagle on a sideslip data mission. You establish a steady straight sideslip and record $+5^{\circ}$ of β. You record the following data on your DAS: $\delta_e = -6.25^{\circ}$, $\delta_a = +8.0^{\circ}$, $F_e = +6.8$ lbs, $F_a = +3.7$ lbs, $F_r = -12.3$ lbs.

You had hoped to make a plot of $\partial\delta_r/\partial\beta$, but in true TPS fashion the rudder gage failed to work. The following is wind tunnel data for the F-15:

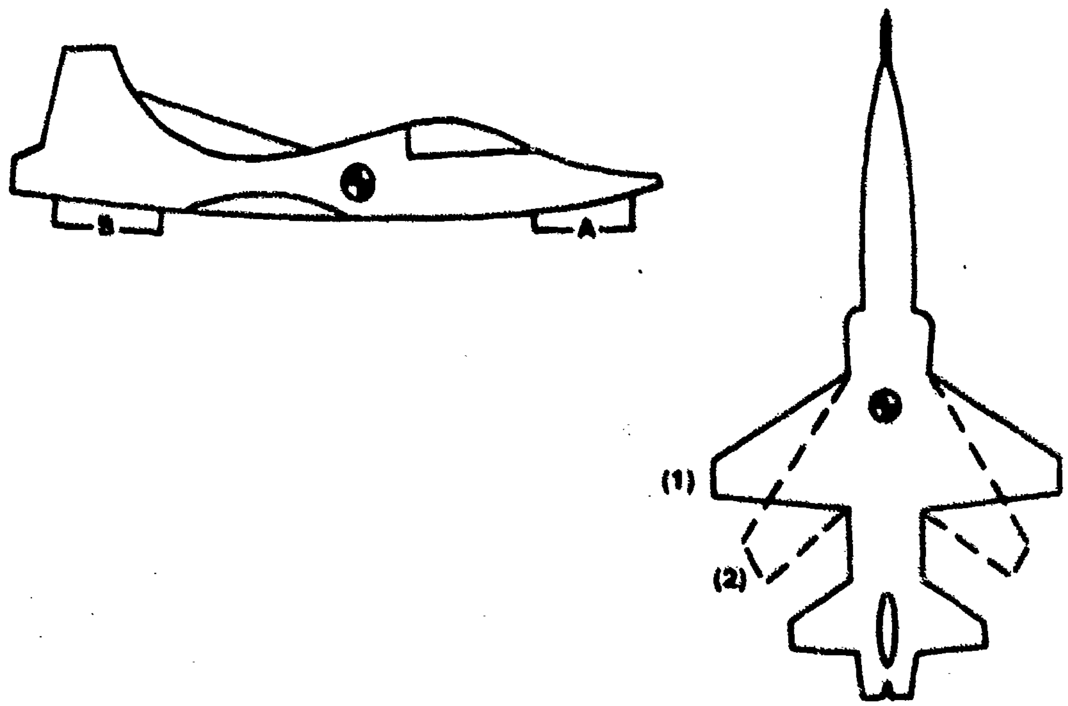
$C_{\ell_\beta} = +0.006$ $\quad C_{n_r} = -0.460$ $\quad C_{n_{\delta_r}} = +0.003$

$C_{\ell_{\delta_a}} = +0.001$ $\quad C_{n_p} = +0.002$ $\quad C_{n_{\dot\beta}} = -0.0006$

a. Determine the value of δ_r at your test point.

b. Assuming that at $\beta = 0$ both δ_r and F_r are $= 0$, does the aircraft exhibit static directional stability rudder Fixed and rudder Free? Sketch plots of δ_r vs β and F_r vs β.

7.18. Given the following swing-wing fighter:



With wings in Position (1), what is the sign of C_{ℓ_β} for the following components?

a. Wing

b. Wing-Fuselage Interference

c. Vertical Tail

d. Area B (Ventral)

e. Area A

f. Canopy

7.19. What is the effect on $C_{\ell_{\beta_{wing}}}$ of sweeping the wings to Position (2)?

7.20. What is the sign of C_{ℓ_r} for the following?

a. Vertical Tail

b. Area B

c. Area A

d. Canopy

7.21. What is the sign of $C_{\ell_{\delta_r}}$?

7.22. What is the sign of C_{ℓ_p} for the following?

a. Vertical Tail

b. Area B

c. Area A

d. Canopy

7.23. For this swing wing fighter

$$C_{\ell_\beta} = -0.0020$$

$$C_{\ell_\beta} = +0.0006$$

$$C_{\ell_p} = -0.0046$$

$$C_{\ell_r} = +0.0018$$

$$C_{\ell_{\delta_r}} = -0.0005$$

$$C_{\ell_{\delta_a}} = +0.0010$$

You run a steady straight sideslip test and measure $\beta = +5^\circ$ and $\delta_r = -10^\circ$. What was your aileron deflection? Does the aircraft exhibit stick-fixed static lateral stability?

7.24. For an aircraft in a right roll, show the pressure distributions that cause C_{h_α} and $C_{h_{\delta_a}}$ on the right wing. Determine the sign of both.

7.25. Assuming an unboosted reversible flight control system, sketch a curve of $(F_a, \delta_a, pb/2U_0, p)$ versus velocity and explain the shape of each for a maximum rate roll. Show the effect of boosting the system.

7.26. Answer each of the following questions True (T) or False (F).

T F High wings make a negative contribution to C_{ℓ_β}.

T F Taper ratio only affects the magnitude of C_{ℓ_β} but does not provide any asymmetric lift distribution.

T F $C_{\ell_{\beta_{fin}}}$ is increased if the fin area (S_F) is decreased.

T F C_{ℓ_r} and $C_{\ell_{\delta_r}}$ are cross derivatives.

T F C_{ℓ_β} is a significant factor in determining aircraft lateral stability.

BIBLIOGRAPHY

7.1 Anon., Military Specification. Flying Qualities of Piloted Airplanes. MIL-F-8785C, 5 Nov 80, UNCLASSIFIED.

CHAPTER 8

DYNAMIC STABILITY

8.1 INTRODUCTION

Dynamics is concerned with the time history of the motion of physical systems. An aircraft is such a system, and its dynamic stability behavior can be predicted through mathematical analysis of the aircraft's equations of motion and verified through flight test.

In the good old days when aircraft were simple, all aircraft exhibited five characteristic dynamic modes of motion, two longitudinal and three lateral-directional modes. The two longitudinal modes are the short period and the phugoid; the three lateral-directional modes are the Dutch roll, spiral, and roll modes.

As aircraft control systems increase in complexity, it is conceivable that one or more of these modes may not exist as a dominant longitudinal or lateral-directional mode. Frequently the higher order effects of complex control systems will quickly die out and leave the basic five dynamic modes of motion. When this is not the case, the development of special procedures may be required to meaningfully describe an aircraft's dynamic motion. For the purposes of this chapter, aircraft will be assumed to possess the five basic modes of motion.

During this study of aircraft dynamics, the solutions to both first order and second order systems will be of interest, and several important descriptive parameters will be used to define the dynamic response of either a first or a second order system.

The quantification of handling qualities, that is, specifying how the magnitude of some of these descriptive parameters can be used to indicate how well an aircraft can be flown, has been an extensive investigation which is by no means complete. Flight tests, simulators, variable stability aircraft, engineering know-how, and pilot opinion surveys have all played major roles in this investigation. The military specification on aircraft handling qualities, MIL-F-8785C, is the current state-of-the-art and ensures that an aircraft will handle well if compliance has been achieved. No attempt will be made to evaluate how satisfactory MIL-F-8785C is for this purpose, but development of the skills necessary to accomplish an analysis of the dynamic behavior of an aircraft will be studied.

8.2 STATIC VS DYNAMIC STABILITY

The static stability of a physical system is concerned with the initial reaction of the system when displaced from an equilibrium condition. The system could exhibit either:

Positive static stability - initial tendency to return
Static instability - initial tendency to diverge
Neutral static stability - remain in displaced position

A physical system's dynamic stability analysis is concerned with the resulting time history motion of the system when displaced from an equilibrium condition.

8.2.1 Dynamically Stable Motions

A particular mode of an aircraft's motion is defined to be "dynamically stable" if the parameters of interest tend toward finite values as time increases without limit. Some examples of dynamically stable time histories and some terms used to describe them are shown in Figures 8.1 and 8.2.

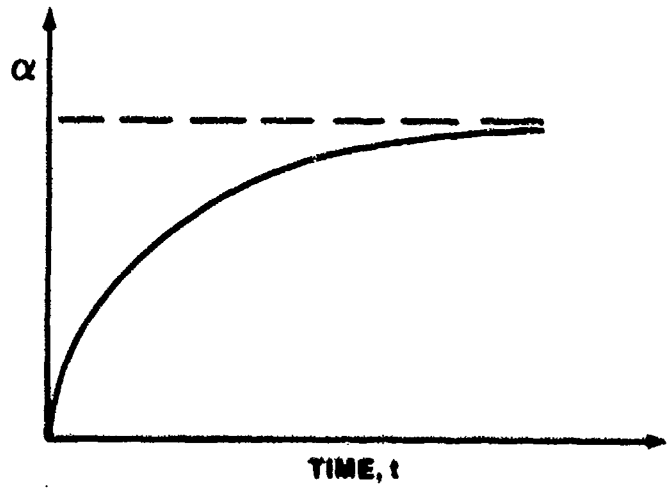


FIGURE 8.1. EXPONENTIALLY DECREASING

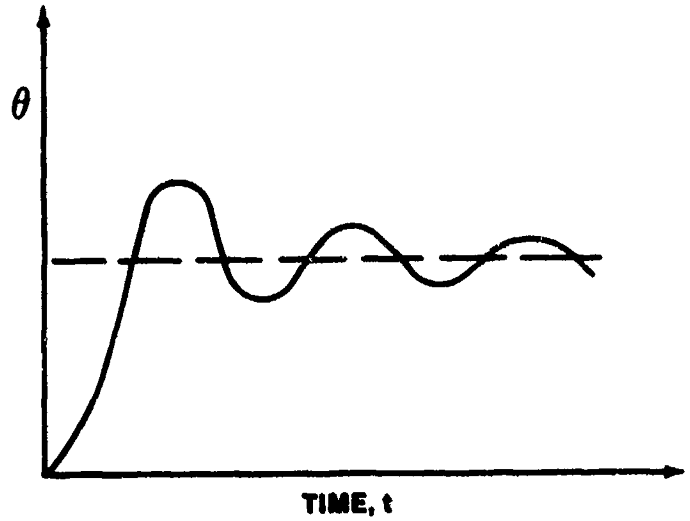


FIGURE 8.2. DAMPED SINUSOIDAL OSCILLATION

8.2.2 Dynamically Unstable Motion

A mode of motion is defined to be "dynamically unstable" if the parameters of interest increase without limit as time increases without limit. Some examples of dynamic instability are shown in Figures 8.3 and 8.4.

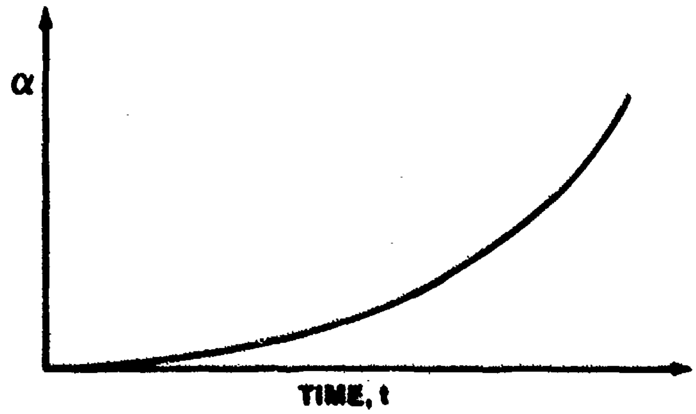


FIGURE 8.3. EXPONENTIALLY INCREASING

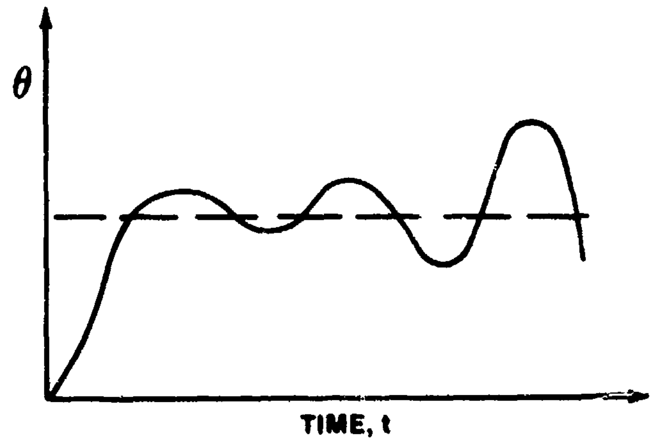


FIGURE 8.4. DIVERGENT SINUSOIDAL OSCILLATION

8.2.3 Dynamically Neutral Motion

A mode of motion is said to have "neutral dynamic stability" if the parameters of interest exhibit an undamped sinusoidal oscillation as time increases without limit. A sketch of such motion is shown in Figure 8.5.

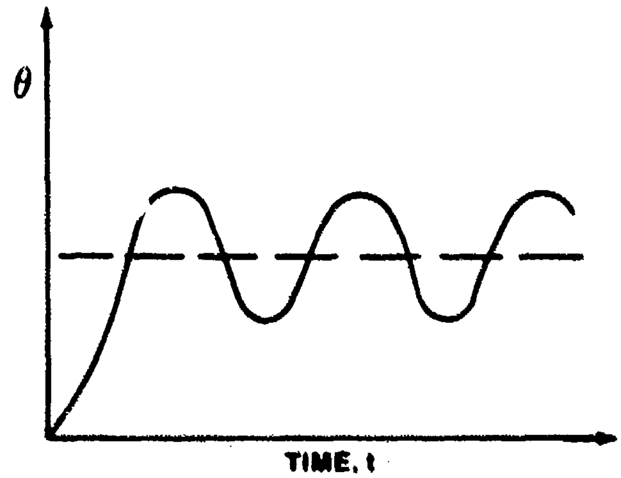


FIGURE 8.5. UNDAMPED OSCILLATION

Example Stability Problem:

A stability analysis can be accomplished to analyze the aircraft shown in Figure 8.6 for longitudinal static stability and dynamic stability. This aircraft is operating at a constant trimmed angle of attack, α_0, in 1g flight.

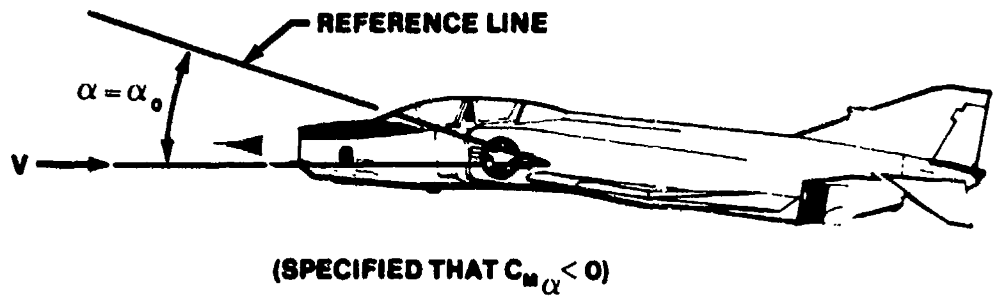


FIGURE 8.6. EXAMPLE STABILITY ANALYSIS

Static Stability Analysis. If the aircraft was displaced from its equilibrium flight conditions by increasing the angle of attack to $\alpha = \alpha_0 + \Delta\alpha$ then the change in pitching moment due to the increase in angle of attack would be nose down because $C_{M_\alpha} < 0$. Thus, the aircraft has positive static longitudinal stability in that its initial tendency is to return to equilibrium.

Dynamic Stability Analysis. The motion of the aircraft as a function of time must be known to describe its dynamic stability. Two methods could be used to find the time history of the motion of the aircraft:

1. Solutions to the aircraft equations of motion could be obtained and analyzed.

2. A flight test could be flown in which the aircraft is perturbed from its equilibrium condition and the resulting motion is recorded and observed.

A sophisticated solution to the aircraft equations of motion with valid aerodynamic inputs can result in good theoretically obtained time histories. However, the fact remains that the only way to discover the aircraft's actual dynamic motion is to flight test and record its motion for analysis.

8.3 EXAMPLES OF FIRST AND SECOND ORDER DYNAMIC SYSTEMS

8.3.1 Second Order System with Positive Damping

The problem of finding the motion of the block shown in Figure 8.7 encompasses many of the methods and ideas that will be used in finding the

time history of an aircraft's motion from its equations of motion.

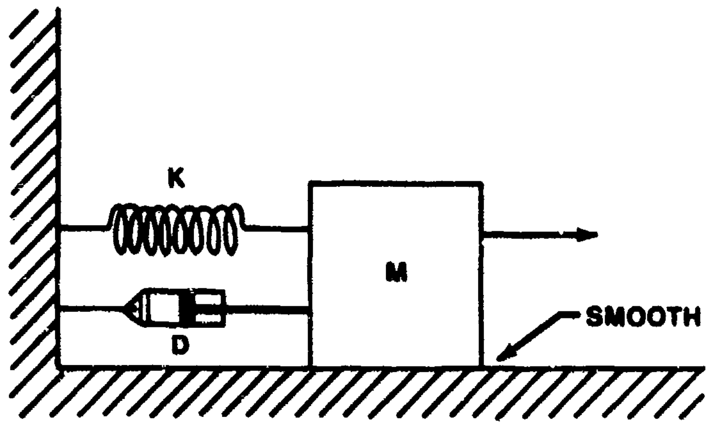


FIGURE 8.7. SECOND ORDER SYSTEM

The differential equation of motion for this physical system is

$$M\ddot{x} + D\dot{x} + Kx = F(t)$$

After Laplace transforming, assuming that the initial conditions are zero, and solving for $X(s)/F(s)$, the transfer function, the result is

$$\frac{X(s)}{F(s)} = \frac{1/M}{s^2 + \frac{D}{M}s + \frac{K}{M}}$$

The denominator of the transfer function which gives the free response of a system will be referred to as its "characteristic equation," and the symbol $\Delta(s)$ will be used to indicate the characteristic equation.

The characteristic equation, $\Delta(s)$, of a second order system will frequently be written in a standard notation.

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \tag{8.1}$$

where

$$\omega_n = \text{natural frequency}$$

$$\zeta = \text{damping ratio.}$$

The two terms, natural frequency and damping ratio, are frequently used to characterize the motion of second order systems.

Also, knowing the location of the roots of $\Delta(s)$ on the complex plane makes it possible to immediately specify and sketch the dynamic motion associated with a system. Continuing to discuss the problem shown in Figure 8.7 and making an identity between the denominator of the transfer function and the characteristic equation

$$\omega_n = \sqrt{\frac{K}{M}}$$

$$\zeta = \frac{D}{2M\sqrt{\frac{K}{M}}}$$

The roots of $\Delta(s)$ can be found by applying the quadratic formula to the characteristic equation

$$s_{1,2} = -\zeta\omega_n \pm i\omega_d$$

Where

$$\omega_d = \omega_n\sqrt{1 - \zeta^2}$$

Note that if $(-1 < \zeta < 1)$, then the roots of $\Delta(s)$ comprise a complex conjugate pair, and for positive ζ would result in root locations as shown in Figure 8.8.

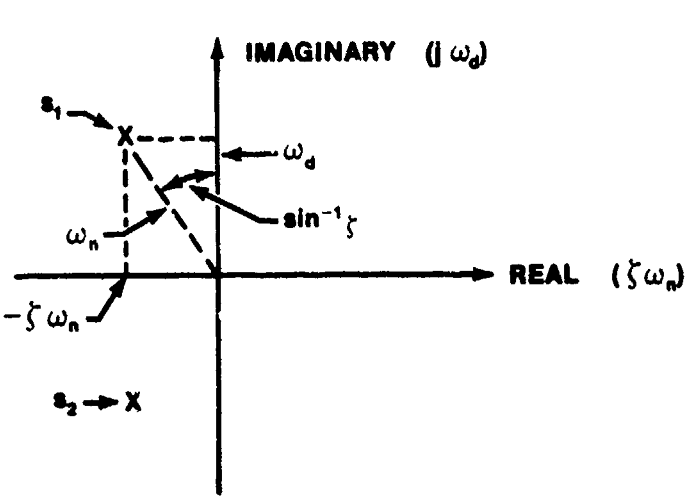


FIGURE 8.8. COMPLEX PLANE

The equation describing the time history of the block's motion can be written by knowing the roots of $\Delta(s)$, $s_{1,2}$, shown above.

$$x(t) = C_1 e^{-\zeta\omega_n t} \cos(\omega_d t + \phi)$$

Where

C_1 and ϕ are constants determined from initial conditions. Knowing either the $\Delta(s)$ root location shown in Figure 8.8 or the equation in $x(t)$ makes it possible to sketch or describe the time history of the motion of the block. The motion of the block shown in Figure 8.7 as a function of time is a sinusoidal oscillation within an exponentially decaying envelope and is dynamically stable.

8.3.2 Second Order System With Negative Damping

A similar procedure to that used in the previous section can be used to find the motion of the block shown in Figure 8.9.

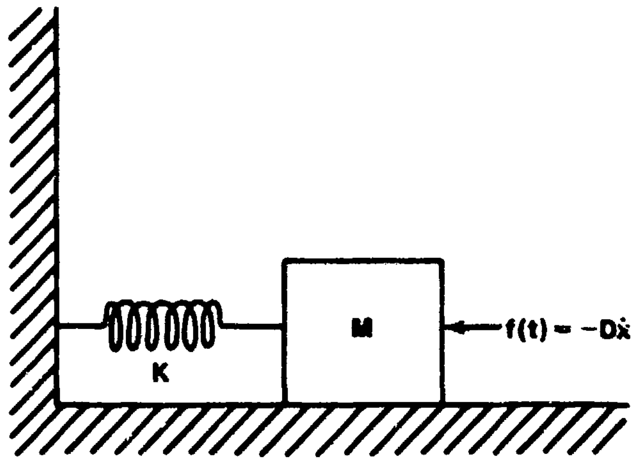


FIGURE 8.9. UNTITLED

The differential equation of motion for this block is

$$M\ddot{x} - D\dot{x} + Kx = f(t)$$

after assuming the inital conditions are zero, the transfer function is

$$\frac{X(s)}{F(s)} = \frac{1/M}{s^2 - \frac{D}{M}s + \frac{K}{M}}$$

By inspection, for this system

$$\omega_n = \sqrt{\frac{K}{M}}$$

$$\zeta = \frac{-D}{2M\sqrt{\frac{K}{M}}}$$

Note that the damping ratio has a negative value. The equation giving the time response of this system is

$$x(t) = C_1 e^{(\text{pos. value})\ t} \cos\left(\omega_d t + \phi\right)$$

where

$$-\zeta\omega_n = \text{pos. value} : \tau = \frac{1}{\zeta\omega_n}$$

For the range $(-1 < \zeta < 0)$, the roots of $\Delta(s)$ for this system could again be plotted on the complex plane from $s_{1,2} = -\zeta\omega_n \pm i\omega_d$ as shown in Figure 8.10.

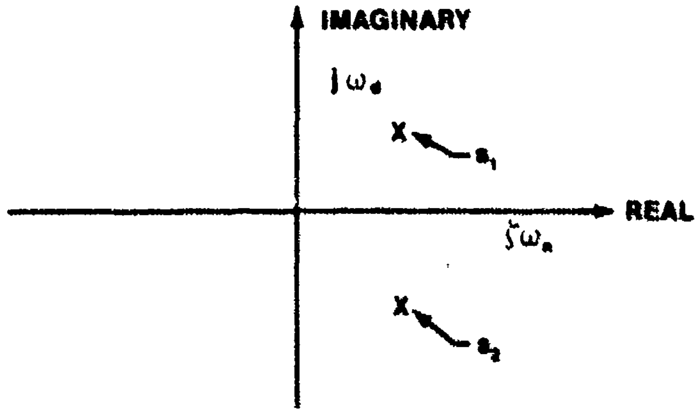


FIGURE 8.10. COMPLEX PLANE

The motion of this system can now be sketched or described. The motion of this system is a sinusoidal oscillation within an exponentially diverging envelope and is dynamically unstable.

8.3.3 Unstable First Order System:

Assume that some physical system has been mathematically modeled and its equation of motion in the s domain is

$$X(s) = \frac{0.5}{0.4s - 0.7} = \frac{1.25}{s - 1.75}$$

For this system the characteristic equation is

$$\Delta(s) = s - 1.75$$

And its root is shown plotted on the complex plane in Figure 8.11.

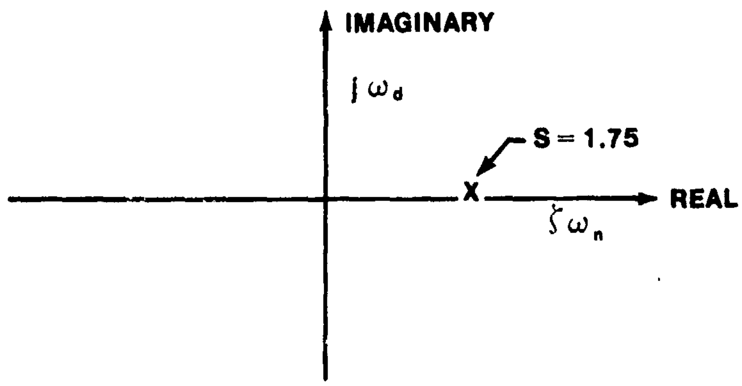


FIGURE 8.11. COMPLEX PLANE

The equation of motion in the time domain becomes

$$\phi(t) = 1.25\ e^{1.75\ t}$$

Note that it is possible to sketch or describe the motion for this system by knowing the location of the root of $\Delta(s)$ or its equation of motion.

For an unstable first order system such as this, one parameter that can be used to characterize its motion is T_2, defined as the time to double amplitude. Without proof,

$$T_2 = \frac{.693}{a} \quad : \quad a = \zeta\omega_n \tag{8.2}$$

For a first order system described by

$$z = C_1\ e^{at} = C\ e^{-\zeta\omega_n t}$$

Note that for a stable first order system, a similar parameter $T_{1/2}$ is the time to half amplitude.

$$T_{1/2} = \frac{-0.693}{a} = \frac{-0.693}{\zeta\omega_n} \qquad (8.3)$$

Where the term, a, must have a negative value for a stable system.

8.3.4 Additional Terms Used To Characterize Dynamic Motion

The time constant, τ, is defined for a stable first order system as the time when the exponent of e in the system equation is -1, or time to reach 63% of final steady state value. For $C\,e^{-\zeta\omega_n t}$

$$\tau = \frac{+1}{a} = \frac{+1}{\zeta\omega_n} \qquad (8.4)$$

The time constant can be thought of as the time required for the parameter of interest to accomplish (1 - 1/e)th of its final steady state. Note that

$$A = 1\,e^{-t/\tau}$$

so for $t = \tau$

$$A = .63 \text{ or } 63\% \text{ of Final Steady State Value.}$$

For $t = 2\tau$

$$A = .86 \text{ or } 86\% \text{ of Final Steady State Value.}$$

Thus the magnitude of the time constant gives a measure of how quickly the dynamic motion of a first order system occurs. Small τ implies a system that, once displaced, returns to equilibrium quickly.

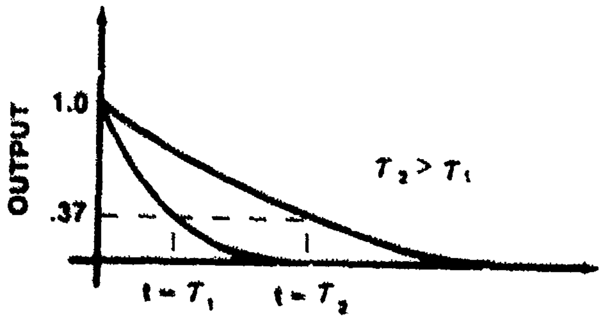


FIGURE 8.12. FIRST ORDER TIME RESPONSE

For a second order system we measure how quickly the envelope of oscillation changes.

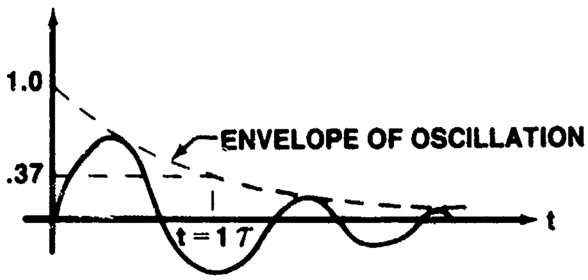


FIGURE 8.12A. SECOND ORDER TIME RESPONSE

Final or steady state value for a given set of equations can be determined using the following expression

$$\text{Final Steady State Value} = \lim_{s \to 0} [s\,F(s)]$$

where F(s) is the Laplace transform of the set of equations with initial conditions equal to zero.

If the Laplace transform of f(t) is F(s), and if $\lim_{t \to \infty} f(t)$ exists, then $\lim_{s \to 0} sF(s) = \lim_{t \to \infty} f(t)$

$$\text{Example: } f(t) = A\theta(t) + B\theta(t) + C\theta(t)$$

$$\text{Input function} = D\,\delta_e(t)$$

where $\delta e(t)$ is a unit step function.

Taking the Laplace transform with the initial conditions equal to zero.

$$F(s) = \frac{\theta(s)}{\delta_e(s)} = \frac{D/A}{s\left(s^2 + \frac{B}{A}s + \frac{C}{A}\right)}$$

Note: $\mathcal{L}$ [Step Function] = 1/s
$\mathcal{L}$ [Impulse Function] = 1

$$\theta_{ss} = \lim_{s \to 0} [s\,F(s)]$$

$$\theta_{ss} = \lim_{s \to 0} \left\{ s \left[\frac{D/A}{s\left(s^2 + \frac{B}{A}s + \frac{C}{A}\right)} \right] \right\}$$

The final steady state value of θ due to a unit step function input is

$$\theta_{ss} = D/C$$

The following list contains some terms commonly used to describe second order system response based on damping ratio values:

Terms	Damping Ratio Value
Overdamped	$1 < \zeta$
Critically damped	$1 = \zeta$
Underdamped	$0 < \zeta < 1$
Undamped	$0 = \zeta$
Negatively damped	$\zeta < 0$

SUMMARY OF DYNAMIC RESPONSE PARAMETERS

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$s_{1,2} = -\zeta\omega_n \pm i\omega_d$$

ζ - Damping Ratio

ω_n - Undamped Natural Frequency

ω_d Damped Frequency

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

τ - Time Constant

$$\tau = \frac{1}{\zeta\omega_n}$$

T - Period

$$T = \frac{2\pi}{\omega_d}$$

$t_{1/2}$ Time to Half Amplitude

$$t_{1/2} = \frac{.69}{\zeta\omega_n}$$

8.4 THE COMPLEX PLANE

It is possible to describe the type response a system will have by knowing the location of the roots of its characteristic equation on the complex plane. A first order response will be associated with each real root, and a complex conjugate pair will have a second order response that is either stable, neutrally stable, or unstable. A complicated system such as an aircraft might have a characteristic equation with several roots, and the total response of such a system will be the sum of the responses associated with each root. A summary of root location and associated response is presented in the following list and in Figures 8.12B and 8.12C.

	Root Location	Associated Response
Case I	On the negative Real axis (1st. Order Response)	Dynamically stable with exponential decay
Case II	In the left half plane off the negative Real axis (2nd. Order Response)	Dynamically stable with sinusoidal oscillation in exponentially decaying envelope
Case III	On the Imaginary axis (2nd. Order Response)	Neutral dynamic stability
Case IV	In the right half plane off the positive Real axis (2nd. Order Response)	Dynamically unstable with sinusoidal oscillation in exponentially increasing envelope
Case V	On the positive Real axis (1st. Order Response)	Dynamically unstable with exponential increase

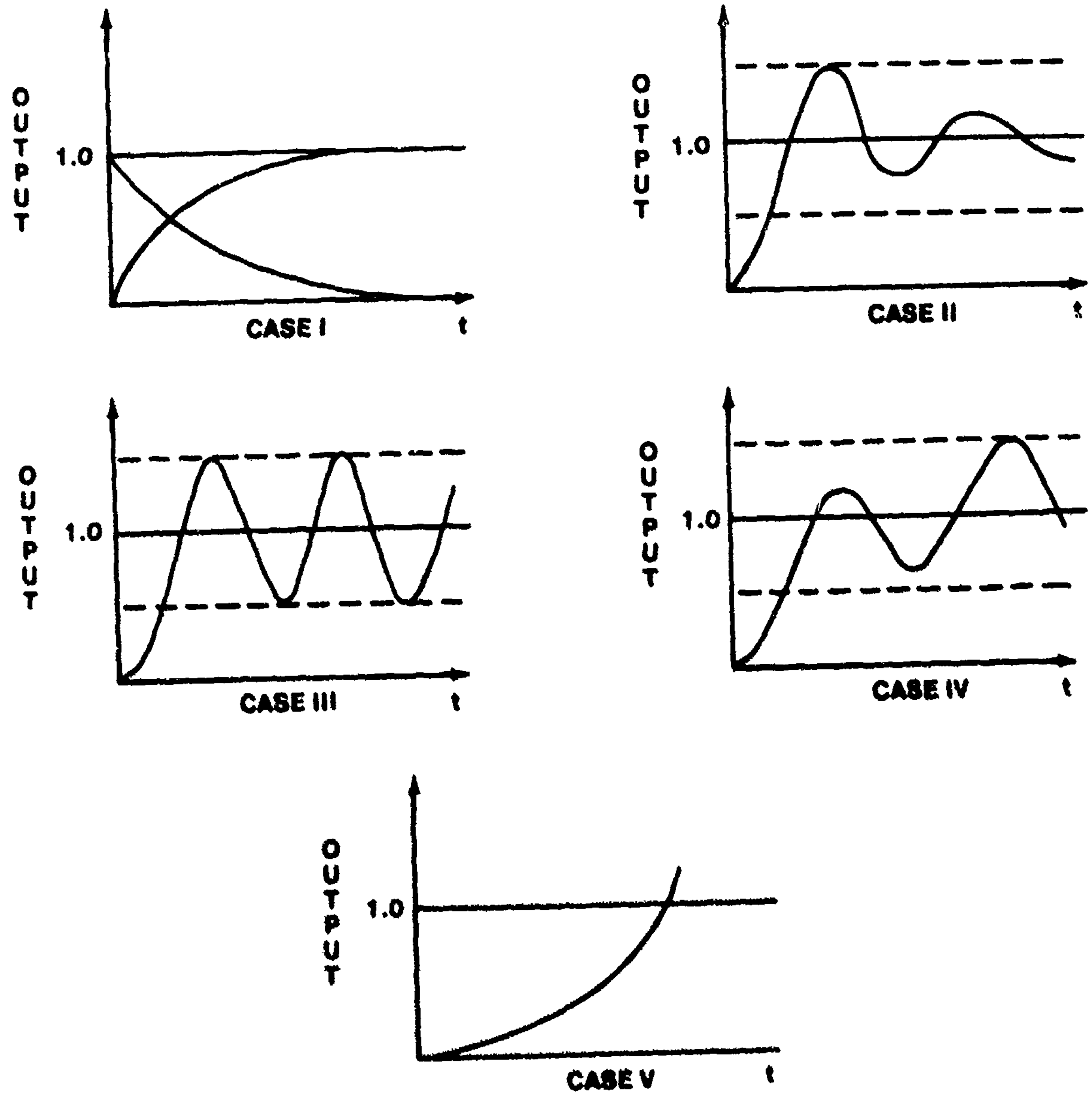


FIGURE 8.12B. POSSIBLE 2ND. ORDER ROOT RESPONSES

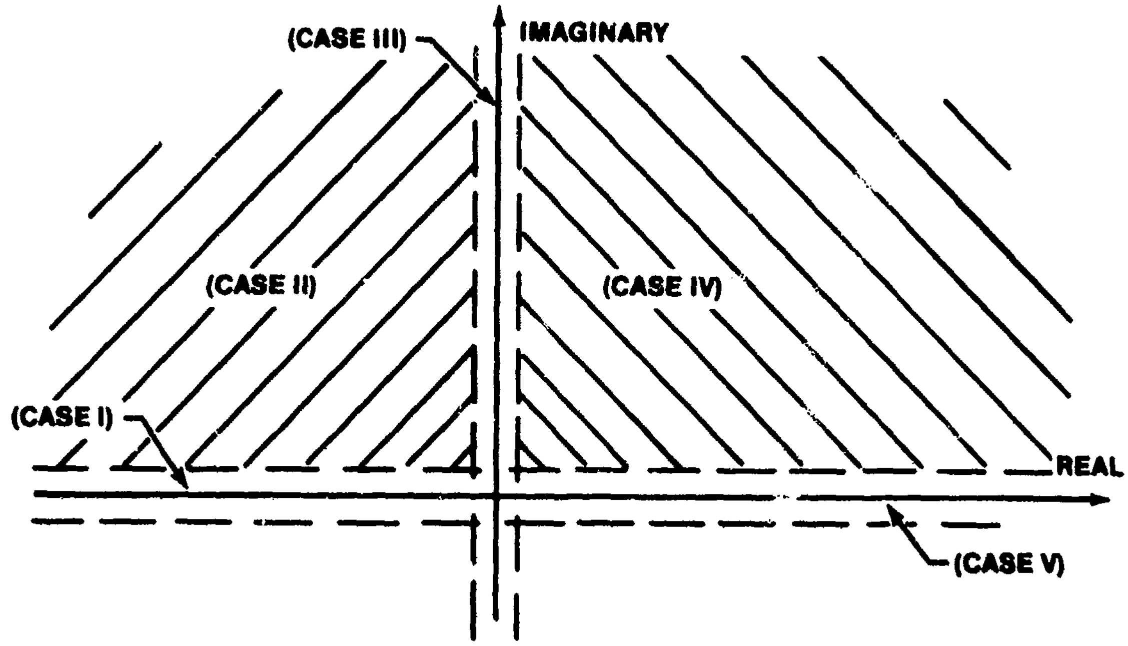


FIGURE 8.12C. ROOT LOCATION IN THE COMPLEX PLANE

8.5 EQUATIONS OF MOTION

Six equations of motion (three translational and three rotational) for a rigid body flight vehicle are required to solve its motion problem. If a rigid body aircraft and constant mass are assumed, then the equations of motion can be derived and expressed in terms of a coordinate system fixed in the body. Solving for the motion of a rigid body in terms of a body fixed coordinate system is particularly convenient in the case of an aircraft when the applied forces are most easily specified in the body axis system.

"Stability axes" can be used as the specified coordinate system. With the vehicle at reference flight conditions, the x axis is aligned into the relative wind; the z axis is 90° from the x axis in the aircraft plane of symmetry, with positive direction down relative to the vehicle; and the y axis completes the orthogonal triad. This xyz coordinate system is then fixed in the vehicle and rotates with it when perturbed from the reference equilibrium conditions. The solid lines in Figure 8.13 depict initial alignment of the stability axes, and the dashed lines show the perturbed coordinate system.

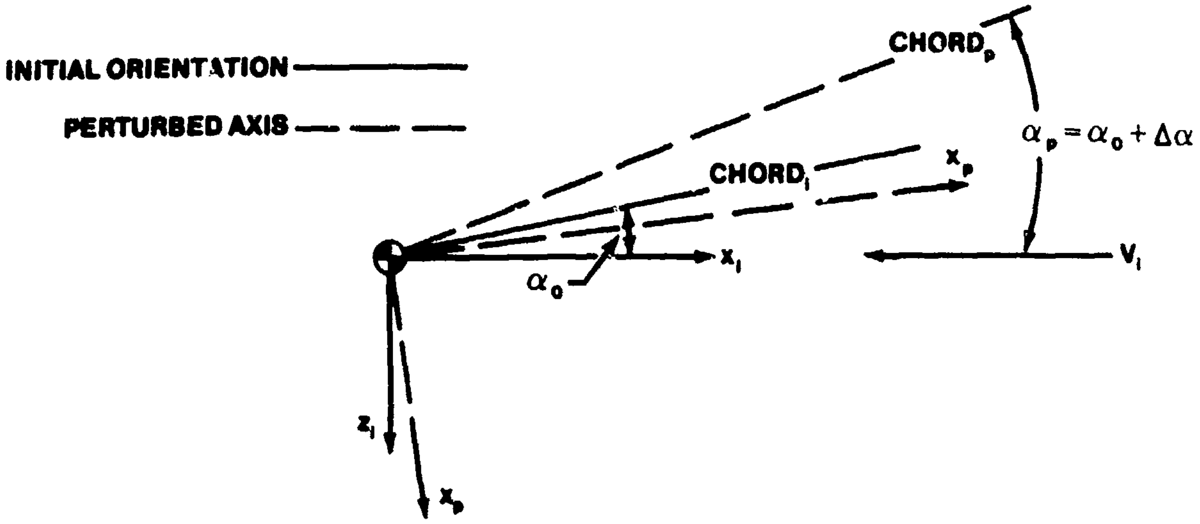


FIGURE 8.13. STABILITY AXIS SYSTEM

Chapter 4, Equations of Motion, Pg. 4, contains the derivation of the complete equations of motion, and the results are listed here.

$$\left.\begin{aligned}
F_x &= m\,(\dot{u} - qw - rv) \\
F_y &= m\,(\dot{v} + rU - pw) \\
F_z &= m\,(\dot{w} + pv - qU) \\
\mathcal{L} &= \dot{p}I_x + qr\,(I_z - I_y) - (\dot{r} + pq)\,I_{xz} \\
\mathcal{M} &= \dot{q}I_y - pr\,(I_z - I_x) + (p^2 - r^2)\,I_{xz} \\
\mathcal{N} &= \dot{r}I_z + pq\,(I_y - I_x) + (qr - \dot{p})\,I_{xz}
\end{aligned}\right\} \quad (8.5)$$

where F_x, F_y, and F_z are forces in the x, y, and z direction, and $\mathcal{L}$ $\mathcal{M}$ and $\mathcal{N}$ are moments about the x, y, and z axes taken at the vehicle center of mass.

Separation of the Equations of Motion:

When all lateral-directional forces, moments, and accelerations are constrained to be zero, the equations which govern pure longitudinal motion result from the six general equations of motion. That is, substituting

$$\left.\begin{aligned} p &= 0 = r \\ \dot{p} &= 0 = \dot{r} \\ \mathcal{L} &= 0 = \mathcal{N} \\ F_y &= 0 \\ v &= 0 \\ \dot{v} &= 0 \end{aligned}\right\} \quad (8.6)$$

into the Equations labeled 8.5 results in the longitudinal equations of motion

$$\left.\begin{aligned} F_x &= m\,(\dot{u} + qw) \\ F_z &= m\,(\dot{w} - qU) \\ \mathcal{M} &= \dot{q}\, I_y \end{aligned}\right\} \quad (8.7)$$

Performing a Taylor series expansion of Equations labeled 8.7 as a function of u, q, and w and assuming small perturbations ($u = u_0 + \Delta u$) results in a linearized set of equations for longitudinal motion. Note that the resulting equations are the longitudinal perturbation equations and that the unknowns are the perturbed values of α, u, and θ from an equilibrium condition. These equations in coefficient form are

$$\left.\begin{aligned} &\frac{2m}{\rho S U_0}\hat{u} + 2C_{D_0}\hat{u} + C_{D_u}\hat{u} + C_{D_\alpha}\alpha + C_{L_0}\theta = C_{D_{\delta_e}}\delta_e \\ &- C_{L_u}\hat{u} - 2C_{L_0}\hat{u} - \frac{2m}{\rho S U_0}\alpha - C_{L_\alpha}\alpha - \frac{c}{2U_0}C_{L_{\dot{\alpha}}}\dot{\alpha} + \frac{2m}{\rho S U_0}\dot{\theta} \\ &- \frac{c}{2U_0}C_{L_q}\dot{\theta} = C_{L_{\delta_e}}\delta_e \\ &C_{m_u}\hat{u} - \frac{c}{2U_0}C_{m_{\dot{\alpha}}}\dot{\alpha} - C_{m_\alpha}\alpha + \frac{2I_y}{\rho S U_0^2 c}\ddot{\theta} - \frac{c}{2U_0}C_{m_q}\dot{\theta} = C_{m_{\delta_e}}\delta_e(s) \end{aligned}\right\} \quad (8.8)$$

Where:

$\hat{u} = \frac{\Delta u}{U_0}$ (A dimensionless velocity parameter has been defined for convenience.)

α, θ are perturbations about their equilibrium values.

C_{D_u}, C_{L_α}, etc., are partial derivatives evaluated at the reference conditions with respect to force coefficients.

Note that these equations are for pure longitudinal motion.

Laplace transforms can be used to facilitate solutions to the longitudinal perturbation equations. For example, taking Laplace transforms of the pitching moment equation and stating that initial perturbation values are zero results in

$$\left[-C_{m_u}\right]\hat{u}(s) + \left[\frac{I_y c^2}{4U_0^2}s^2 - \frac{c}{2U_0}C_{m_q}s\right]\theta(s) - \left[\frac{c}{2U_0}C_{m_{\dot{\alpha}}}s + C_{m_\alpha}\right]\alpha(s) = C_{m_{\delta_e}}\delta_e(s) \tag{8.9}$$

The other two equations could similarly be Laplace transformed to obtain a set of longitudinal perturbation equations in the s domain.

8.5.1 Longitudinal Motion

The Equations 8.8 describe the longitudinal motion of an aircraft about some equilibrium conditions. The theoretical solutions for aircraft motion can be quite good, depending on the accuracy of the various aerodynamic parameters. For example, C_{D_α} is one parameter appearing in the drag force equation, and the goodness of the solution will depend on how accurately the value of C_{D_α} is known. Before an aircraft flies, such values for the various stability derivatives can be extracted from flight test data.

8.5.1.1 Longitudinal Modes of Motion Experience has shown that aircraft exhibit two different types of longitudinal oscillations:

1. One of short period with relatively heavy damping that is called the "short period" mode (sp).

2. Another of long period with very light damping that is called the "phugoid" mode (p).

The periods and damping of these oscillations vary with aircraft configuration and with flight conditions.

The short period is characterized primarily by variations in angle of attack and pitch angle with very little change in forward speed. Relative to the phugoid, the short period has a high frequency and heavy damping.

Typical values for its damped period are in the range of two to five seconds. Generally, the short period motion is the more important longitudinal mode for handling qualities since it contributes to the motion being observed by the pilot when the pilot is in the loop.

The phugoid is characterized mainly by variations in u and θ with α nearly constant. This long period oscillation can be thought of as a constant total energy problem with exchanges between potential and kinetic energy. The aircraft nose drops and airspeed increases as the aircraft descends below its initial altitude. Then the nose rotates up, causing the aircraft to climb above its initial altitude with airspeed decreasing until the nose lazily drops below the horizon at the top of the maneuver.

Because of light damping, many cycles are required for this motion to damp out. However, its long period combined with low damping results in an oscillation that is easily controlled by the pilot, even for a slightly divergent motion. When the pilot is in the loop, he is frequently not aware that the phugoid mode exists as he makes control inputs and obtains aircraft response before the phugoid can be seen. Typical values for its damped period range in the order of 45 to 90 seconds.

Phugoid - Small ω_n
- Large time constant
- Small damping ratio

Short Period - Large ω_n
- Small time constant
- High damping ratio

Example:

Given a T-38 aircraft at M = .8, altitude = 20,000 ft., and at a gross weight of 9,000 lbs, the longitudinal equations in the Laplace domain become (ICs = 0)

$$[1.565 + .0045]\,\hat{u}(s) - .42\alpha(s) + .0605(s)\theta(s) = C_{D_{\delta_e}}\delta_e(s)$$

$$.236\hat{u}(s) + [3.13s + 5.026]\alpha(s) - 3.15s\theta(s) = C_{L_{\delta_e}}\delta_e(s)$$

$$0 + .16\alpha(s) + [.0489s^2 - .039s]\theta(s) = C_{m_{\delta_e}}\delta_e(s)$$

These equations are of the form

$$a\hat{u} + b\alpha + c\theta = d_{\delta_e}$$

$$e\hat{u} + f\alpha + g\theta = h_{\delta_e}$$

$$i\hat{u} + j\alpha + k\theta = \ell_{\delta_e}$$

and using Cramers Rule, this set of equations can be readily solved for any of the variables.

$$\begin{array}{c} (\hat{u} \quad \alpha \quad \theta) \end{array}$$

$$\frac{\alpha(s)}{\delta_e(s)} = \frac{\begin{vmatrix} a & d & c \\ e & h & g \\ i & \ell & k \end{vmatrix}}{\begin{vmatrix} a & b & c \\ e & f & g \\ i & j & k \end{vmatrix}} = \frac{\text{Numerator}(s)}{\text{Denominator}(s)}$$

Recall that the denominator of the above equation in the s domain is the

system characteristic equation and that the location of the roots of $\Delta(s)$ will indicate the type of dynamic response. Solving for the determinant of the denominator yields:

$$\Delta(s) = .239s^4 + .577s^3 + 1.0996s^2 + .00355s + .0028 = 0$$

Factoring of this equation into two quadratics

$$\Delta(s) = (s^2 + .00216s + .00208)\ (s^2 + 2.408s + 4.595) = 0$$

standard format

$$\left(s^2 + 2\zeta\omega_{n_p}s + \omega_{n_p}^{2}\right)\left(s^2 + 2\zeta\omega_{n_{SP}}s + \omega_{n_{SP}}^{2}\right) \qquad (8.10)$$

Each of the quadratics listed in the equation prior to Equation 8.10 will have a natural frequency and damping ratio associated with it, and the values can be computed by comparing the particular quadratic to the standard notation second order characteristic Equation 8.10.

LONGITUDINAL MODES OF MOTION
T-38 EXAMPLE

	Phugoid	Short Period
ζ	.0236	.562
ω_N	.0456 rad/sec	2.143 rad/sec
τ	925.9 sec	.83 sec
T	137.5 sec	3.54 sec

Roots of $\Delta(s)$ for longitudinal motion:

phugoid roots: $s_{1,2} = -.00108 \pm j\ .0456$

Short Period Roots: $s_{3,4} = -1.204 \pm j\, 1.733$

These roots can then be plotted on the s-plane:

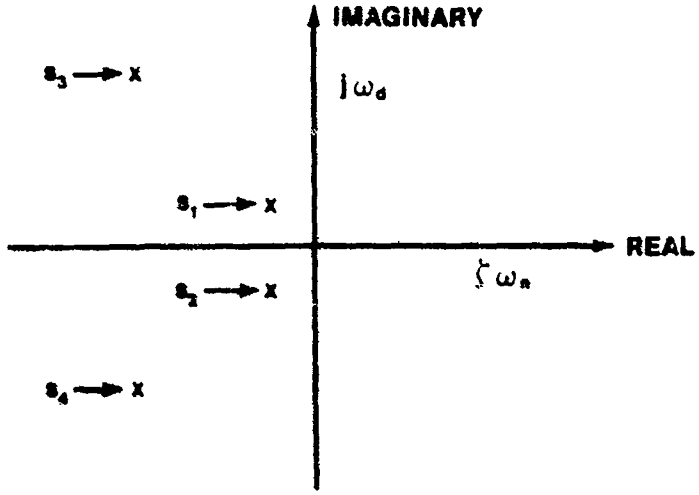


FIGURE 8.14. LONGITUDINAL MOTION COMPLEX PLANE

8.5.1.2 Short Period Mode Approximation. For a one degree of freedom first approximation, the short period is observed to be primarily a pitching motion (Figure 8.15). In addition, the short period motion occurs at nearly constant airspeed, $\Delta u = 0$; and since there is no vertical motion, changes in angle of attack are equal to changes in pitch angle, $\Delta\alpha = \Delta\theta$. With these assumptions applied to the pitching moment equation (Equation 8.8), the results become:

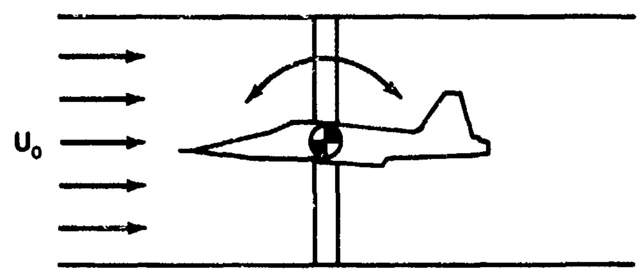


FIGURE 8.15. 1 DEGREE OF FREEDOM MODEL

$$\left[\frac{2I_y}{\rho U_0^{\,2} Sc}\right]\ddot{\alpha} - \left[\frac{c}{2U_0} C_{m_q}\right]\dot{\alpha} - C_{m_\alpha}\alpha = C_{m_{\delta_e}} \delta_e$$

Where C_{m_u} and C_{m_α} have been assumed to be negligible.

Applying the Laplace transform to Equation 8.11 and forming the transfer function $\alpha(s)/\delta_e(s)$ results in an approximate form of the characteristic equation.

$$\Delta(s) = \left[\frac{I_y}{\frac{1}{2}\rho U_0^2 Sc}\right] s^2 - \frac{c}{2U_0} C_{m_q} s - C_{m_\alpha} = 0 \tag{8.12}$$

Comparison of Equation 8.12 with the standard form of the characteristic equation, (Equation 8.1), results in approximations for the short period frequency and damping ratio.

$$\zeta_{SP} = -\frac{C_{m_q}}{\sqrt{-C_{m_\alpha}}}\left[\frac{c}{4U_0}\right]\sqrt{\frac{Sc}{I_y}}\sqrt{\frac{1}{2}\rho U_0^{\,2}} \tag{8.13}$$

$$\omega_{n_{SP}} = \sqrt{-C_{m_\alpha}}\sqrt{\frac{Sc}{I_y}}\sqrt{\frac{1}{2}\rho U_0^{\,2}} \tag{8.14}$$

C_{m_α} - Coefficient of pitching moment due to a change in angle of attack. Proportional to the angular displacement from equilibrium (spring constant)

C_{m_q} - Coefficient of pitching moment due to a change in pitch rate. Proportional to the angular rate (viscous damper)

Both Equations 8.13 and 8.14 can be used to predict trends expected in the short period damping ratio and natural frequency as flight conditions and aircraft configurations change. In addition, these equations show the predominant stability derivatives which affect the short period damping ratio and natural frequency.

8.5.1.3 Equation for Ratio of Load Factor to Angle of Attack Change. The requirements of MIL-F-8785C for the short period natural frequency are stated as a function of n/α and ω_n.

An expression for the slope of the lift curve is

$$C_{L_\alpha} = \frac{\Delta C_L}{\Delta\alpha} \text{ and } \Delta C_L = \frac{\Delta n W}{\frac{1}{2}\rho U_0^2 S} \tag{8.15}$$

$$\frac{\Delta n}{\Delta\alpha} = \frac{1/2\ \rho\ U_0^2\ S\left(C_{L_\alpha}\right)}{W} \tag{8.16}$$

8.5.1.4 Phugoid Mode Approximation Equations. An approach similar to that used when obtaining the short period approximation will be used to obtain a set of equations to approximate the phugoid oscillation. Recalling that the phugoid motion occurs at nearly constant angle of attack, it is logical to substitute $\alpha = 0$ into the longitudinal motion equations. This results in a set of three equations with only two unknowns. Reasoning that the phugoid motion is characterized primarily by altitude excursions and changes in aircraft speed, implies that the lift force and drag force equations are the two equations which should be used. The resulting set of two equations for the phugoid approximation in the Laplace domain is

$$\left[\frac{2m}{\rho S U_0} s + 2C_{D_0}\right] \hat{u}(s) + \left[C_{L_0}\right] \theta(s) = C_{D_{\delta_e}} \delta_e(s) \tag{8.17}$$

$$\left[-2\, C_{L_0}\right] \hat{u}(s) + \left[\frac{2m}{\rho S U_0} s\right] \theta(s) = C_{L_{\delta_e}} \delta_e(s) \tag{8.18}$$

Where C_{D_u}, C_{D_α}, C_{L_u}, C_{L_α}, and C_{L_q} have been assumed to be negligibly small.

The characteristic equation for the phugoid approximation can now be found using the above equation.

$$\Delta(s) = \left[\frac{2m}{\rho S U_0}\right]^2 s^2 + \left[\frac{4m}{\rho S U_0} C_{D_0}\right] s + 2\left[C_{L_0}\right]^2 = 0 \tag{8.19}$$

Note that lift and weight are not equal during phugoid motion, but also realize that the net difference between lift and weight is quite small. If the approximation is made that

$$L = W$$

and then the substitution that

$$W = mg$$

it can be written that

$$\left[\frac{2mg}{\rho S U_0^2}\right] = C_L$$

The phugoid characteristic equation can thus be rewritten as

$$\left[\frac{C_{L_0}^2 \; U_0^2}{g^2}\right] s^2 + 2\left[\frac{C_{L_0} \; U_0}{g} C_{D_0}\right] s + 2\left[C_{L_0}\right]^2 = 0 \qquad (8.20)$$

$$s^2 + 2\frac{g}{U_0}\frac{C_{D_0}}{C_{L_0}} s + \frac{2\,g^2}{U_0^{\,2}} = 0 \qquad (8.21)$$

Comparison of Equation 8.21 with the standard form of Equation 8.1 results in a simplified approximate expression for phugoid natural frequency and is given by

$$\omega_{n_p} = \frac{45.5}{U_0} \qquad (8.22)$$

Where U_0 is true velocity in feet per second.

A simplified approximate expression for the phugoid damping ratio can also be obtained and is given by

$$\zeta_p = \frac{1}{\sqrt{2}}\left[\frac{C_D}{C_L}\right] \qquad (8.23)$$

Equations 8.22 and 8.23 can be used to understand some major contributors to the natural frequency and damping ratio of the phugoid motion.

8.5.2 Lateral Directional Motion Mode

There are three typical asymmetric modes of motion exhibited by aircraft. These modes are the roll, spiral, and Dutch Roll.

8.5.2.1 Roll Mode The roll mode is considered to be a first order response which describes the aircraft roll rate response to an aileron input. Figure 8.16 depicts an idealized roll rate time history to a step aileron input. The time constant is normally from one to three seconds to reach steady state roll rate.

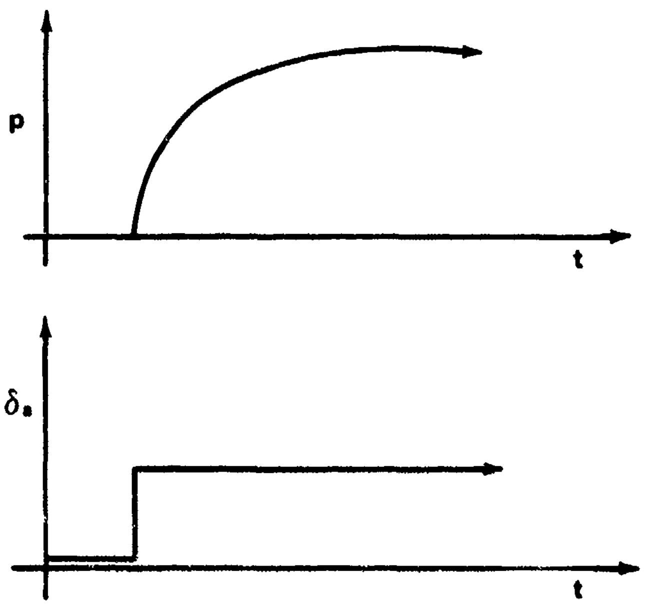


FIGURE 8.16. TYPICAL ROLL MODE

8.5.2.2 Spiral Mode The spiral mode is considered to be a first order response which describes the aircraft bank angle time history as it tends to increase or decrease from a small, nonzero bank angle. After a wings level trim shot, the spiral mode can be observed by releasing the aircraft from bank angles as great as 20° and allowing the spiral mode to occur without control inputs. If this mode is divergent, the aircraft nose continues to drop as the bank angle continues to increase, resulting in the name "spiral mode."

8.5.2.3 Dutch Roll Mode The Dutch roll mode is a coupled yawing and rolling motion slowly dampened, moderately low frequency oscillation. Typically, as the aircraft nose yaws to the right, a right roll due to the yawing motion is generated. This causes increased lift and induced drag on the left wing, and the nose yaws to the left. The combination of restoring forces and moments, damping, and aircraft inertia is generally such that after the motion peaks out to the right, a nose left yawing motion begins accompanied by a roll to the left.

One of the pertinent Dutch roll parameters is ϕ/β, the ratio of bank angle to sideslip angle which may be represented by

$$\phi/\beta = f\left(\frac{C_{\ell_\beta}}{C_{n_\beta}}\right) \qquad (8.24)$$

A very low value for ϕ/β implies little bank change during Dutch roll. In the limit when ϕ/β is zero, the Dutch roll motion consists of a pure yawing motion that most pilots consider less objectionable than the Dutch roll mode with a high value of ϕ/β.

A rudder doublet is frequently used to excite the Dutch roll; Figure 8.17 shows a typical Dutch roll time history.

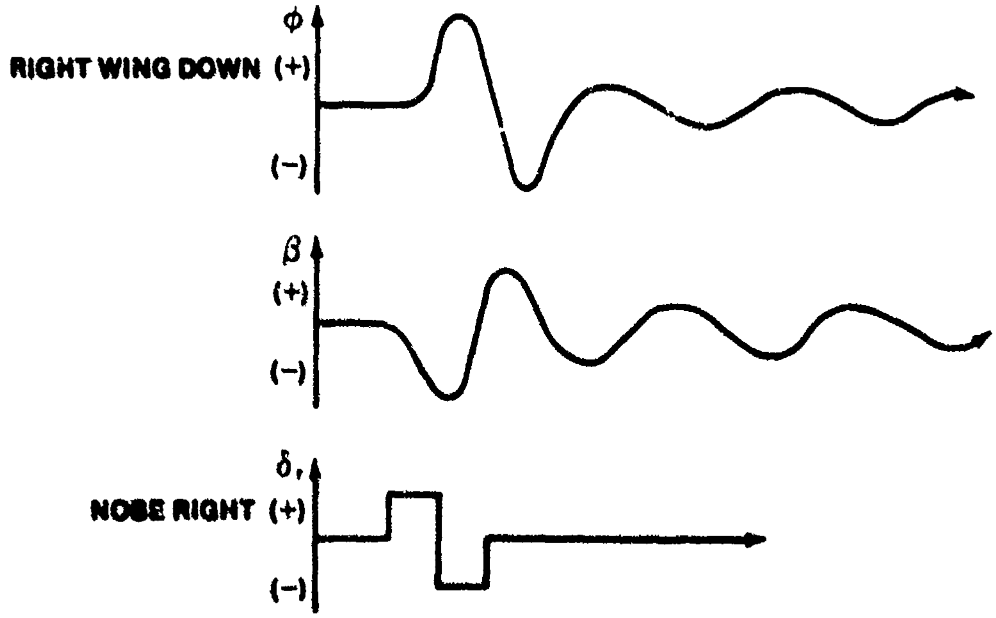


FIGURE 8.17. TYPICAL DUTCH ROLL MODE

8.5.3 Asymmetric Equations of Motion

Similar to the separation of the longitudinal equations, the set of equations which describes lateral-directional motion can be separated from the six general equations of motion. Starting with equilibrium conditions and specifying that only asymmetric forcing functions, velocities, and accelerations exist, results in the lateral directional equations of motion. Assuming small perturbations and using a linear Taylor Series approximation for the forcing functions result in the linear, lateral directional perturbation equations of motion.

$$\frac{-b}{2U_0} C_{y_p} \dot{\phi} - C_{L_0} \phi + \left[\frac{2m}{\rho S U_0} - \frac{b}{2U_0} C_{y_r} \right] \dot{\psi} + \frac{2m}{\rho SU} \dot{\beta} - C_{y_\beta} \beta = C_{y_{\delta_r}} \delta_r + C_{y_{\delta_a}} \delta_a$$

$$\frac{2I_x}{\rho S b U_0^2} \ddot{\phi} - \frac{b}{2U_0} \dot{\phi} + \frac{2I_{xz}}{\rho S b U_0^2} \ddot{\psi} - \frac{b}{2U_0} C_{\ell_r} \dot{\psi} - C_{\ell_\beta} \beta = C_{\ell_{\delta_r}} \delta_r + C_{\ell_{\delta_a}} \delta_a \qquad (8.25)$$

$$-\frac{2I_{xz}}{\rho S b U_0^2} \ddot{\phi} - \frac{b}{2U_0} C_{n_p} \dot{\phi} + \frac{2I_z}{\rho S b U_0^2} \ddot{\psi} - \frac{b}{2U_0} C_{n_r} \dot{\psi} - C_{n_\beta} \beta = C_{n_{\delta_r}} \delta_r + C_{n_{\delta_\alpha}} \delta_a$$

The lateral-directional equations of motion have been non dimensionalized by span, b, as opposed to chord. Also, that the stability derivative C_{ℓ_p} is not a lift referenced stability derivative but that the script ℓ refers to rolling moment. If the perturbation products of are not small, then the lateral-directional motion will couple directly into longitudinal motion as seen from the pitching moment equation (Equation 8.5). Our analysis will assume that conditions are such that coupling does not exist.

8.5.3.1 Roots Of $\Delta(s)$ For Asymmetric Motion The roots of the lateral-directional characteristic equation typically are comprised of a relatively large negative real root, a small root that is either positive or negative, and a complex conjugate pair of roots.

The large real root is the one associated with the roll mode of motion. Note that a large negative value for this root implies a fast time constant.

The small real root that might be either positive or negative is associated with the spiral mode. A slowly changing time response results from this small root, and the motion is either stable for a negative root or divergent for a positive root.

The complex conjugate pair of roots corresponds to the Dutch roll mode, which frequently exhibits high frequency and light damping conditions. This second order motion is of great interest in handling qualities investigations.

For the T-38 Example (Pg. 8.22), the Lateral Directional characteristic equation is: $(s^2 + .87\ s + 18.4)_{1,2}\ (s + 6.822)_3\ (s + .00955)_4 = 0$

Lateral-Directional Modes of Motion - T-38 Example

	SPIRAL_4	ROLL_3	$\text{DUTCH ROLL}_{1,2}$
ω_n	———	———	4.29 rad/sec
ζ	———	———	.102
τ	105 sec	.1465 sec	2.31 sec

8.5.3.2 Approximate Roll Mode Equation. This approximation results from the hypothesis that only rolling motion exists and use of the rolling moment equation results in the roll mode approximation equation.

$$\left[\left(\frac{2I_x}{\rho S b U_0^2}\right) s - \left(\frac{b}{2U_0} C_{\ell_p}\right)\right] \dot{\phi}(s) = C_{\ell_{\delta_a}} \delta_a \tag{8.26}$$

The roll mode characteristic equation root is

$$s_R = \left[\frac{b^2\ S\ \rho\ U_0}{4\ I_x}\right] C_{\ell_p} \tag{8.27}$$

$$\text{NOTE:}\quad \tau_R = \frac{1}{s_R} = \left[\frac{2I_x\ U_0}{b^2}\right] \left[\frac{1}{\frac{1}{2}\rho U_0^2}\right] \left[\frac{1}{C_{\ell_p}}\right]$$

Note that C_{ℓ_p} less than zero implies stability for the roll mode and that a larger negative value of s_R implies an aircraft that approaches its steady

state roll rate quickly. A functional analysis can be made using Equation 8.27 to predict trends in τ_R, the roll mode time constant, as flight conditions change.

8.5.3.3 Spiral Mode Stability. After the Dutch roll is damped, the long time period spiral mode begins. If the Dutch roll damps and leaves the aircraft at some small β, then the effect is to induce a rolling moment. If the bank angle gradually increases and the aircraft enters a spiral dive, the mode is unstable. Since the motion is slow (long time period), we may say

$$s = \frac{\left(C_{n_r} C_{\ell_\beta} - C_{n_\beta} C_{\ell_r}\right)\left(\rho U_0 S b^2\right)}{4 I_z C_{\ell_\beta}}$$

The spiral mode will be stable when the sign of the above root is negative. Since C_{ℓ_β} is negative, the root will be negative as long as

$$C_{\ell_\beta} C_{n_r} > C_{n_\beta} C_{\ell_r} \tag{8.28}$$

Note that C_{n_r} and C_{ℓ_β} are both negative while C_{n_β} and C_{ℓ_r} are both positive. Thus, for spiral stability, we must increase the dihedral effect $\left(C_{\ell_\beta}\right)$ and decrease the weathercock effect $\left(C_{n_\beta}\right)$.

8.5.3.4 Dutch Roll Mode Approximate Equations. For airplanes with relatively small dihedral effect, C_{ℓ_β}, the Dutch roll mode consists primarily of sideslipping and yawing. An approximation to the Dutch roll mode of motion can be obtained from Equation 8.25 by specifying that pure sideslip occurs ($\beta = -\psi$) and eliminating the rolling degree of freedom. The resulting approximations to Dutch roll damping and natural frequency are:

$$\zeta_{DR} = -\frac{bC_{n_r}}{2U_0}\sqrt{\frac{S\rho u_0^{\,2}}{2C_{n_\beta} I_z}}$$

$$\omega_{nDR} = \sqrt{\frac{C_{n_0}\ Sb\rho u_0^{\,2}}{2I_z}} \tag{8.29}$$

In practice, the Dutch roll mode natural frequency is well predicted, but because of the usually large values of C_{ℓ_β} in addition to significant values of I_{xz}, the Dutch roll damping is not well predicted.

If we specify that there are no large changes in yawing moments ($\Sigma\Delta = 0$) and the Dutch roll mode consists primarily of rolling motion, the resulting approximations to the Dutch roll damping and natural frequency are

$$\zeta_{DR} = -C_{y_\beta}\frac{\rho U_0 S}{4m}\sqrt{\frac{C_{\ell_p} b}{C_{\ell_\beta}^{\,2} g}}$$

$$\omega_{n_{DR}} = \sqrt{\frac{2g\, C_{\ell_\beta}}{bC_{\ell_p}}} \tag{8.30}$$

Just as in the case of Equation 8.29, the Dutch roll damping ratio is not well predicted. To predict the Dutch roll damping ratio, a complete evaluation of Equation 8.25 must be made.

These approximations do give a physical insight into the parameters that affect the Dutch roll mode, and the effect that changes in these parameters such as those caused by configuration changes, stores, and fuel leading may have on flying qualities.

8.5.3.5 Coupled Roll Spiral Mode. This mode of lateral-directional motion has rarely been exhibited by aircraft, but the possibility exists that it can indeed happen. If this mode is present, the characteristic equation for asymmetric motion has two pairs of complex conjugate roots instead of the usual one complex conjugate pair along with two real roots. The phenomenon

which occurs is the roll mode root decreases in absolute magnitude while the spiral mode root becomes more negative until they meet and split off the real axis to form a second complex conjugate pair of roots, as depicted in Figure 8.18.

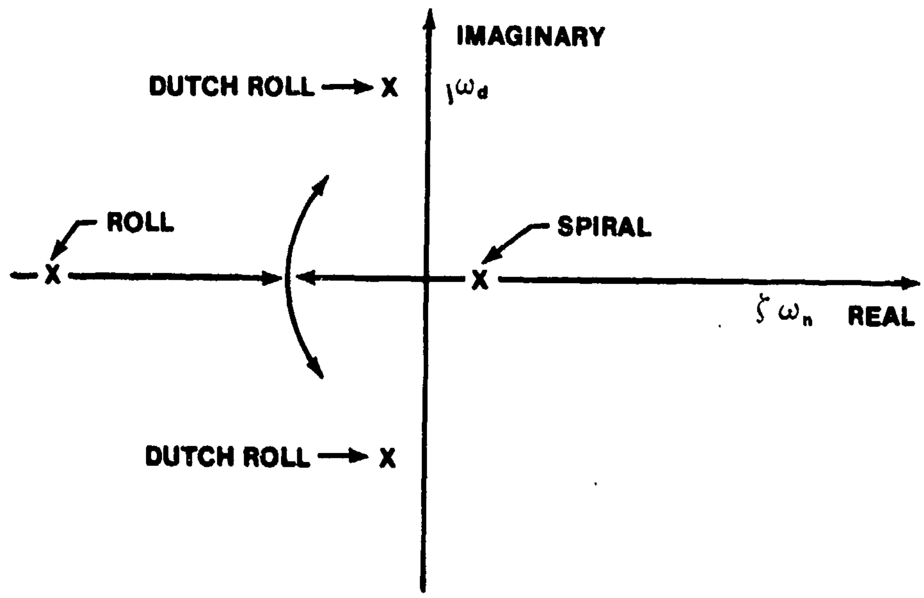


FIGURE 8.18. COUPLED ROLL SPIRAL MODE

At least two in-flight experiences with this mode have been documented and have shown that a coupled roll spiral mode causes significant piloting difficulties. One occurrence involved the M2-F2 lifting body, and a second involved the Flight Dynamics Lab variable-stability NT-33. Some designs of V/STOL aircraft have indicated that these aircraft would exhibit a coupled roll spiral mode in a portion of their flight envelope (Reference 8.3). Some pilot comments from simulator evaluations are "rolly," "requires tightly closed roll control loop," or "will roll on its back if you don't watch it."

A coupled roll spiral mode can result from a high value for C_{ℓ_β} and a low value for C_{ℓ_p}. The M2-F2 lifting body did in fact possess a high dihedral effect and quite a low roll damping. Examination of the equations for the roll mode and spiral mode characteristic equation roots shows how the root locus shown in Figure 8.18 could result as C_{ℓ_p} decreases in absolute magnitude and C_{ℓ_β} increases.

8.6 STABILITY DERIVATIVES

Introduction:

Some of the stability derivatives are particularly pertinent in the study of the dynamic modes of aircraft motion, and the more important ones appearing in the functional equations which characterize the dynamic modes of motion should be understood. C_{M_q}, C_{M_α}, C_{ℓ_p}, C_{ℓ_β}, C_{n_r}, and C_{n_β} are discussed in the following paragraphs.

8.6.1 C_{n_β}

The stability derivative, C_{n_β}, is the change in yawing moment coefficient with variation in sideslip angle. It is usually referred to as the static directional derivative or the "weathercock" derivative. When the airframe sideslips, the relative wind strikes the airframe obliquely, creating a yawing moment, N, about the center of gravity. The major portion of C_{n_β} comes from the vertical tail, which stabilizes the body of the airframe just as the tail feathers of an arrow stabilize the arrow shaft. The C_{n_β} contribution due to the vertical tail is positive, signifying static directional stability, whereas the C_{n_β} due to body is negative, signifying static directional instability. There is also a contribution to C_{n_β} from the wing, the value of which is usually positive but very small compared to the body and vertical tail contributions.

The derivative C_{n_β} is very important in determining the dynamic lateral stability and control characteristics. Most of the references concerning full-scale flight tests and free-flight wind tunnel model tests agree that C_{n_β} should be as high as possible for good flying qualities. A high value of C_{n_β} aids the pilot in effecting coordinated turns and prevents excessive sideslip and yawing motions in extreme flight maneuvers and in rough air. C_{n_β}

primarily determines the natural frequency of the Dutch roll oscillatory mode of the airframe, and it is also a factor in determining the spiral stability characteristics.

8.6.2 C_{n_r}

The stability derivative C_{n_r} is the change in yawing moment coefficient with change in yawing velocity. It is known as the yaw damping derivative. When the airframe is yawing at an angular velocity, r, a yawing moment is produced which opposes the rotation. C_{n_r} is made up of contributions from the wing, the fuselage, and the vertical tail, all of which are negative in sign. The contribution from the vertical tail is by far the largest, usually amounting to about 80% or 90% of the total C_{n_r} of the airframe.

The derivative C_{n_r} is very important in lateral dynamics because it is the main contributor to the damping of the Dutch roll oscillatory mode. It also is important to the spiral mode. For each mode, large negative values of C_{n_r} are desired.

8.6.3 C_{M_α}

This stability derivative is the change in pitching moment coefficient with varying angle of attack and is commonly referred to as the longitudinal static stability derivative. When the angle of attack of the airframe increases from the equilibrium condition, the increased lift on the horizontal tail causes a negative pitching moment about the center of gravity of the airframe. Simultaneously, the increased lift of the wing causes a positive or negative pitching moment, depending on the fore and aft location of the lift vector with respect to the center of gravity. These contributions together with the pitching moment contribution of the fuselage are combined to establish the derivative C_{M_α}. The magnitude and sign of the total C_{M_α} for a particular airframe configuration are thus a function of the center of gravity position, and this fact is very important in longitudinal stability and control. If the center of gravity is ahead of the neutral point, the value of

C_{M_α} is negative, and the airframe is said to possess static longitudinal stability. Conversely, if the center of gravity is aft of the neutral point, the value of C_{M_α} is positive, and the airframe is then statically unstable. C_{M_α} is perhaps the most important derivative as far as longitudinal stability and control are concerned. It primarily establishes the natural frequency of the short period mode and is a major factor in determining the response of the airframe to elevator motions and to gusts. In general, a large negative value of C_{M_α} (i.e., large static stability) is desirable for good flying qualities. However, if it is too large, the required elevator effectiveness for satisfactory control may become unreasonably high. A compromise is thus necessary in selecting a design range for C_{M_α}. Design values of static stability are usually expressed not in terms of C_{M_α} but rather in terms of the derivative $C_{M_{C_L}}$, where the relation is $C_{M_\alpha} = C_{M_{C_L}} C_{L_\alpha}$. It should be pointed out that $C_{M_{C_L}}$ in the above expression is actually a partial derivative for which the forward speed remains constant.

8.6.4 C_{M_q}

The stability derivative C_{M_q} is the change in pitching moment coefficient with varying pitch velocity and is commonly referred to as the pitch damping derivative. As the airframe pitches about its center of gravity, the angle of attack of the horizontal tail changes and lift develops on the horizontal tail, producing a negative pitching moment on the airframe and hence a contribution to the derivative C_{M_q}. There is also a contribution to C_{M_q} because of various "dead weight" aeroelastic effects. Since the airframe is moving in a curved flight path due to its pitching, a centrifugal force is developed on all the components of the airframe. The force can cause the wing to twist as a result of the dead weight moment of overhanging nacelles and can cause the horizontal tail angle of attack to change as a result of fuselage bending due to the weight of the tail section. In low speed flight, C_{M_q} comes mostly from the effect of the curved flight

path on the horizontal tail, and its sign is negative. In high speed flight the sign of C_{M_q} can be positive or negative, depending on the nature of the aeroelastic effects. The derivative C_{M_q} is very important in longitudinal dynamics because it contributes a major portion of the damping of the short period mode for conventional aircraft. As pointed out, this damping effect comes mostly from the horizontal tail. For tailless aircraft, the magnitude of C_{M_q} is consequently small; this is the main reason for the usually poor damping of this type of configuration. C_{M_q} is also involved to a certain extent in phugoid damping. In almost all cases, high negative values of C_{M_q} are desired.

8.6.5 C_{ℓ_β}

This stability derivative is the change in rolling moment coefficient with variation in sideslip angle and is usually referred to as the "effective dihedral derivative." When the airframe sideslips, a rolling moment is developed because of the dihedral effect of the wing and because of the usual high position of the vertical tail relative to the equilibrium x-axis. No general statements can be made concerning the relative magnitude of the contributions to C_{ℓ_β} from the vertical tail and from the wing since these contributions vary considerably from airframe to airframe and for different angles of attack of the same airframe. C_{ℓ_β} is nearly always negative in sign, signifying a negative rolling moment for a positive sideslip.

C_{ℓ_β} is very important in lateral stability and control, and it is therefore usually considered in the preliminary design of an airframe. It is involved in damping both the Dutch roll mode and the spiral mode. It is also involved in the maneuvering characteristics of an airframe, especially with regard to lateral control with the rudder alone near stall.

8.6.6 C_{ℓ_p}

The stability derivative C_{ℓ_p}, is the change in rolling moment coefficient with change in rolling velocity and is usually known as the roll damping derivative. When the airframe rolls at an angular velocity p, a rolling moment is produced as a result of this velocity; this moment opposes the rotation. C_{ℓ_p} is composed of contributions, negative in sign, from the wing and the horizontal and vertical tails. However, unless the size of the tail is unusually large in comparison with the size of the wing, the major portion of the total C_{ℓ_p} comes from the wing.

The derivative C_{ℓ_p} is quite important in lateral dynamics because essentially C_{ℓ_p} alone determines the damping in roll characteristics of the aircraft. Normally, it appears that small negative values of C_{ℓ_p} are more desirable than large ones because the airframe will respond better to a given aileron input and will suffer fewer flight perturbations due to gust inputs.

8.7 HANDLING QUALITIES

Because the "goodness" with which an aircraft flies is often stated as a general appraisal . . . "My F-69 is the best damn fighter ever built, and it can outfly and outshoot any other airplane." "It flies good." "That was really hairy." . . . you probably can understand the difficulty of measuring how well an aircraft handles. The basic question of what parameters to measure and how those parameters relate to good handling qualities has been a difficult one, and the total answer is not yet available. The current best answers for military aircraft are found in MIL-F-8785C, the specification for the "Flying Qualities of Piloted Airplanes."

When an aircraft is designed for performance, the design team has definite goals to work toward . . . a particular takeoff distance, a minimum time to climb, or a specified combat radius. If an aircraft is also to be designed to handle well, it is necessary to have some definite handling quality goals to work toward. Success in attaining these goals can be measured by flight tests for handling qualities when some rather firm

standards are available against which to measure and from which to recommend.

In order to make it possible to specify acceptable handling qualities, it was necessary to evolve some flight test measurable parameters. Flight testing results in data which yield values for the various handling quality parameters, and the military specification gives a range of values that should ensure good handling qualities. Because MIL-F-8785C is not the ultimate answer, the role of the test pilot in making accurate qualitative observations and reports in addition to generating the quantitative data is of great importance in handling qualities testing.

One method that has been extensively used in handling qualities quantification is the use of pilot opinion surveys and variable stability aircraft. For example, a best range of values for the short period damping ratio and natural frequency could be identified by flying a particular aircraft type to accomplish a specific task while allowing the ζ and ω_n to vary. From the opinions of a large number of pilots, a valid best range of values for ζ and ω_n could be obtained, as shown in Figure 8.19.

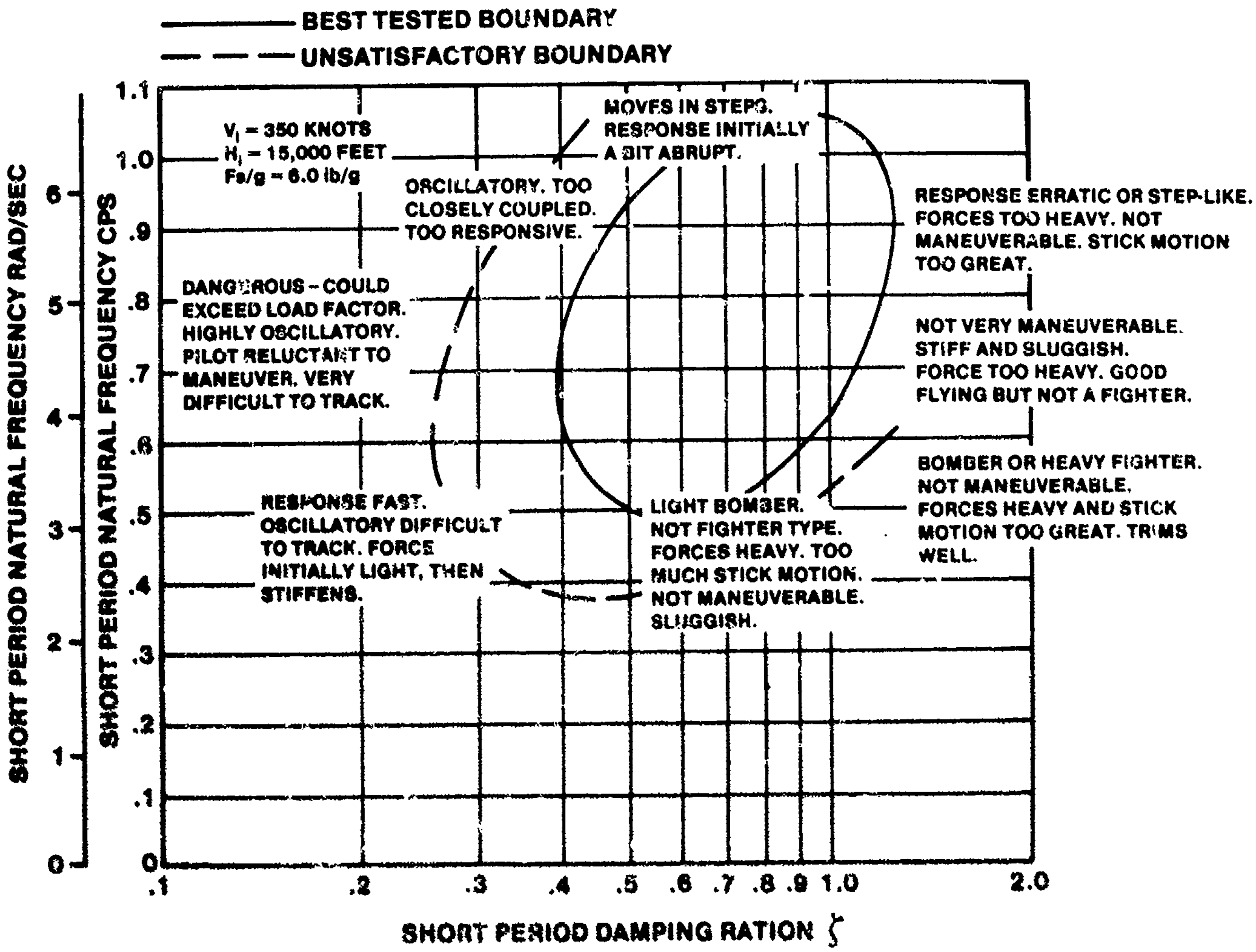


FIGURE 8.19. BEST RANGE FOR ζ AND ω_n FROM PILOT OPINION

8.7.1 Open Loop Vs Closed Loop Response

The ζ and ω_n being discussed here are the aircraft free or open loop response characteristics which describe aircraft motion without pilot inputs. With the pilot in the loop, the free response of the aircraft is hidden as pilot inputs are continually made. The closed loop block diagram shown in Figure 8.20 can be used to understand aircraft closed loop response.

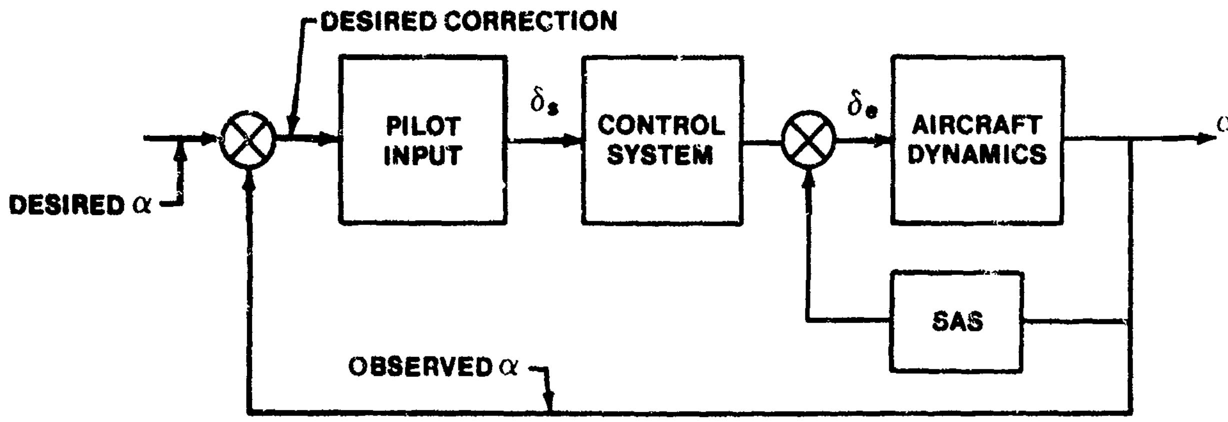


FIGURE 8.20. CLOSED LOOP BLOCK DIAGRAM

The free response of an aircraft does relate directly to how well the aircraft can be flown with a pilot in the loop, and many of the pertinent handling qualities parameters are for the open loop aircraft.

The real test of an aircraft's handling qualities is how well it can be flown closed loop to accomplish a particular mission. Closed loop handling quality evaluations such as air-to-air tracking in a simulated air combat maneuvering mission play an important part of determining how well an aircraft handles.

8.7.2 Pilot In The Loop Dynamic Analysis

Calspan (formerly Cornell Aeronautical Laboratory) has made notable contributions to the understanding of pilot rating scales and pilot opinion surveys. Except for minor variations between pilots, which sometimes prevent a sharp delineation between acceptable and unacceptable flight characteristics, there is very definite consistency and reliability in pilot opinion. In addition, the opinions of well qualified test pilots can be exploited because of their engineering knowledge and experience in many different aircraft types.

The stability and control characteristics of airplanes are generally established by wind tunnel measurement and by technical analysis as part of the airplane design process. The handling qualities of a particular airplane are related to the stability and control characteristics. The relationship is a complex one which involves the combination of the airplane and its pilot in the accomplishment of the intended mission. It is important that the effects of specific stability and control characteristics be evaluated in

terms of their ultimate effects on the suitability of the pilot-vehicle combination for the mission. On the basis of this information, intelligent decisions can be made during the airplane design phase which will lead to the desired handling qualities of the final product.

There are three general ways in which the relationship between stability and control parameters and the degree of suitability of the airplane for the mission may be examined:

1. Theoretical analysis
2. Experimental performance measurement
3. Pilot evaluation

Each of the three approaches has an important role in the complete evaluation. One might ask, however, why is the pilot assessment necessary? At present a mathematical representation of the human operator best lends itself to analysis of specific simple tasks. Since the intended use is made up of several tasks and several modes of pilot-vehicle behavior, difficulty is experienced first in accurately describing all modes analytically, and second in integrating the quality of the subordinate parts into a measure of overall quality for the intended use. In spite of these difficulties, theoretical analysis is fundamental for understanding pilot-vehicle difficulties, and pilot evaluation without it remains a purely experimental process.

Attaining satisfactory performance in a designated mission is a fundamental reason for our concern with handling qualities. Why can't the experimental measurement of performance replace pilot evaluation? Why not measure pilot-vehicle performance in the intended use - isn't good performance consonant with good quality? A significant difficulty arises here in that the performance measurement tasks may not demand of the pilot all that the real mission demands. The pilot is an adaptive controller whose goal is to achieve good performance. In a specific task, he is capable of attaining essentially the same performance for a wide range of vehicle characteristics, at the expense of significant reductions in his capacity to assume other duties and planning operations. Significant differences in task performance may not be measured where very real differences in mission suitability do exist.

The questions which arise in using performance measurements may be summarized as follows: (1) For what maneuvers and tasks should measurements be made to define the mission suitability? (2) How do we integrate and weigh the performance in several tasks to give an overall measure of quality if measurable differences do exist? (3) Is it necessary to measure or evaluate pilot workload and attention factors for performance to be meaningful? If so, how are these factors weighed with those in (2)? (4) What disturbances and distractions are necessary to provide a realistic workload for the pilot during the measurement of his performance in the specified task?

Pilot evaluation still remains the only method of assessing the interactions between pilot performance and workload in determining suitability of the airplane for the mission. It is required in order to provide a basic measure of quality and to serve as a standard against which pilot-airplane system theory may be developed, against which performance measurements may be correlated, and with which significant airplane design parameters may be determined and correlated.

The technical content of the pilot evaluation generally falls into two categories: one, the identification of characteristics which interfere with the intended use, and two, the determination of the extent to which these characteristics affect mission accomplishment. The latter judgment may be formalized as a pilot rating.

8.7.3 Pilot Rating Scales

In 1956, the newly formed Society of Experimental Test Pilots accepted responsibility for one program session at the annual meeting of the Institute of Aeronautical Sciences. A paper entitled "Understanding and Interpreting Pilot Opinion" was presented with the intent to create better understanding and use of pilot opinion in aeronautical research and development. The widespread use of rating systems has indicated a general need for some uniform method of assessing aircraft handling qualities through pilot opinion.

Several rating scales were independently developed during the early use of variable stability aircraft. These vehicles, as well as the use of ground simulation, made possible systematic studies of aircraft handling qualities through pilot evaluation and rating of the effects of specific stability and

control parameters.

Figure 8.21 shows the 10-point Cooper-Harper Rating Scale that is widely used today.

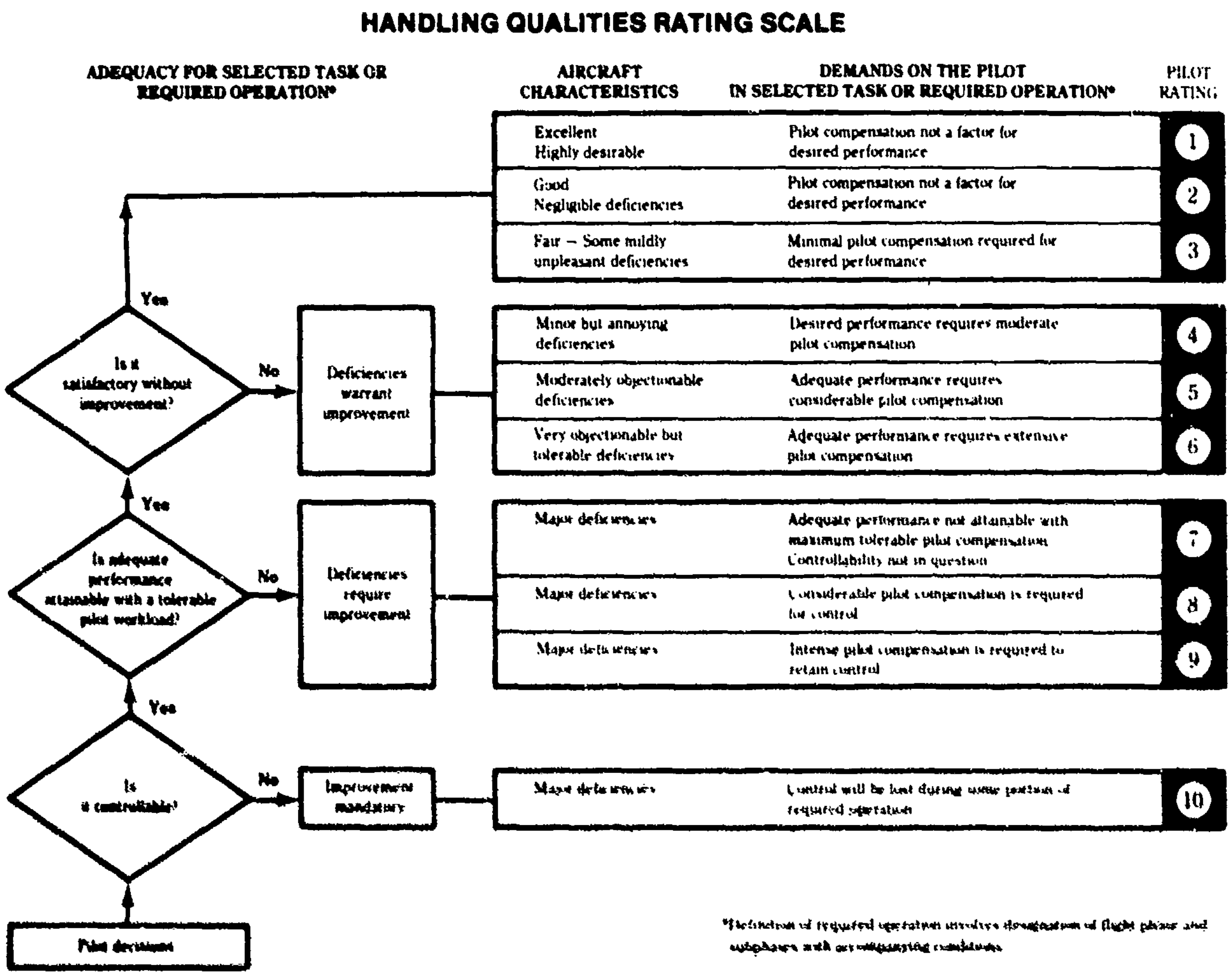


FIGURE 8.21. TEN-POINT COOPER-HARPER PILOT RATING SCALE

A flow chart is shown in Figure 8.22 that traces the series of dichotomous decisions that the pilot makes in arriving at the final rating. As a rule, the first decision may be fairly obvious. Is the configuration controllable or uncontrollable? Subsequent decisions become less obvious as the final rating is approached.

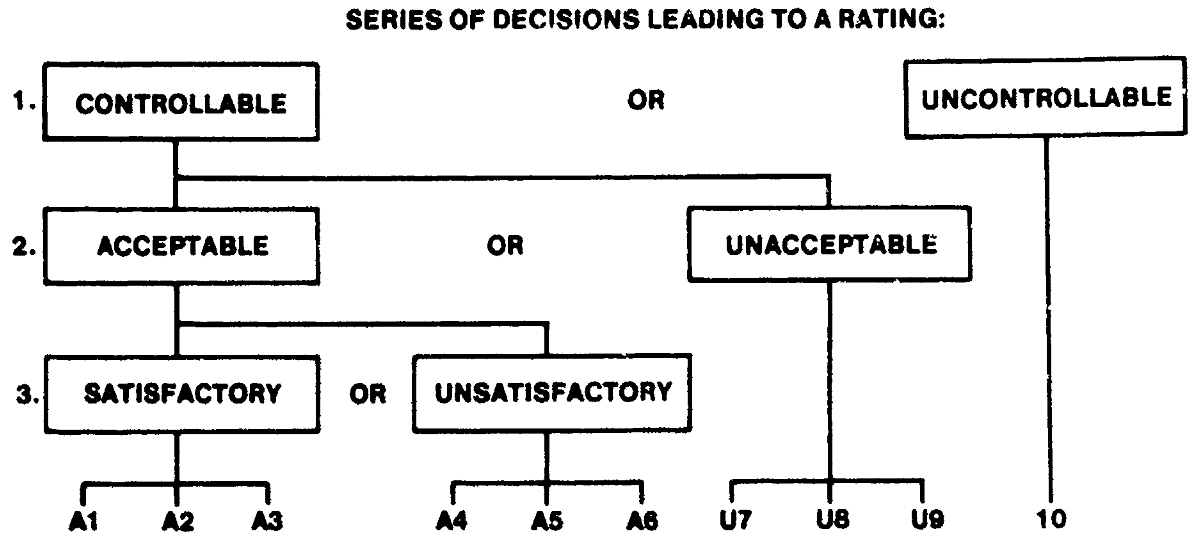


FIGURE 8.22. SEQUENTIAL PILOT RATING DECISIONS

If the airplane is uncontrollable in the mission, it is rated 10. If it is controllable, the second decision examines whether it is acceptable or unacceptable. If unacceptable, the ratings U7, U8, and U9 are considered (rating 10 has been excluded by the "controllable" answer to the first decision). If it is acceptable, the third decision must examine whether it is satisfactory or unsatisfactory. If unsatisfactory, the ratings 4, 5, and 6 are considered; if satisfactory, the ratings 1, 2, and 3 are considered.

The basic categories must be described in carefully selected terms to clarify and standardize the boundaries desired. Following a careful review of dictionary definitions and consideration of the pilot's requirement for clear, concise descriptions, the category definitions shown in Figure 8.23 were selected. When considered in conjunction with the structural outline presented in Figure 8.22 a clearer picture is obtained of the series of decisions which the pilot must make.

CATEGORY	DEFINITION
CONTROLLABLE	CAPABLE OF BEING CONTROLLED OR MANAGED IN CONTEXT OF MISSION, WITH AVAILABLE PILOT ATTENTION.
UNCONTROLLABLE	CONTROL WILL BE LOST DURING SOME PORTION OF MISSION.
ACCEPTABLE	MAY HAVE DEFICIENCIES WHICH WARRANT IMPROVEMENT BUT ADEQUATE FOR MISSION. PILOT COMPENSATION, IF REQUIRED TO ACHIEVE ACCEPTABLE PERFORMANCE, IS FEASIBLE.
UNACCEPTABLE	DEFICIENCIES WHICH REQUIRE MANDATORY IMPROVEMENT. INADEQUATE PERFORMANCE FOR MISSION, EVEN WITH MAXIMUM FEASIBLE PILOT COMPENSATION.
SATISFACTORY	MEETS ALL REQUIREMENTS AND EXPECTATIONS; GOOD ENOUGH WITHOUT IMPROVEMENT. CLEARLY ADEQUATE FOR MISSION.
UNSATISFACTORY	RELUCTANTLY ACCEPTABLE. DEFICIENCIES WHICH WARRANT IMPROVEMENT. PERFORMANCE ADEQUATE FOR MISSION WITH FEASIBLE PILOT COMPENSATION.

FIGURE 8.23. MAJOR CATEGORY DEFINITIONS

8.7.4 Major Category Definitions

To control is to exercise direction of, or to command. Control also means to regulate. The determination as to whether the airplane is controllable or not must be made within the framework of the defined mission or intended use. An example of the considerations of this decision would be the evaluation of fighter handling qualities during which the evaluation pilot encounters a configuration over which he can maintain control only with his complete and undivided attention. The configuration is "controllable" in the sense that the pilot can maintain control by restricting the tasks and maneuvers which he is called upon to perform and by giving the configuration his undivided attention. However, for him to answer "Yes, it is controllable in the mission," he must be able to retain control in the mission tasks with whatever effort and attention are available from the totality of his mission duties. Uncontrollable implies that flight manual limitations may be exceeded during performance of the mission task.

The dictionary shows that "acceptable" means that a thing offered is received with a consenting mind; "unacceptable" means that it is refused or rejected. Acceptable means that the mission can be accomplished;

it means that the evaluation pilot would agree to buy it for the mission to fly, for his son to fly, or for either to ride in as a passenger. "Acceptable" in the rating scale doesn't say how good it is for the mission, but it does say it is good enough. With these characteristics, the mission can be accomplished. It may be accomplished with considerable expenditure of effort and concentration on the part of the pilot, but the levels of effort and concentration required in order to achieve this acceptable performance are feasible in the intended use. By the same token, unacceptable does not necessarily mean that the mission cannot be accomplished; it does mean that the effort, concentration, and workload necessary to accomplish the tasks are of such a magnitude that the evaluation pilot rejects that airplane for the mission.

Consider now a definition of satisfactory. The dictionary defines this as adequate for the purpose. A pilot's definition of satisfactory might be that it isn't necessarily perfect or even good, but it is good enough that he wouldn't ask that it be fixed. It meets a standard, it has sufficient goodness and it can meet all requirements of a mission task. Acceptable but unsatisfactory implies that it is acceptable even though objectionable characteristics should be improved, that it is deficient in a limited sense, or that there is insufficient goodness. Thus, the quality is either:

1. Acceptable (satisfactory) and therefore of the best category,
   or
2. Acceptable (unsatisfactory) and of the next best category,
   or
3. Unacceptable. Not suitable for the mission, but still controllable,
   or
4. Unacceptable for the mission and uncontrollable.

8.7.5 Experimental Use Of Rating Of Handling Qualities

The evaluation of handling qualities has a similarity to other scientific experiments in that the output data are only as good as the care taken in the design and execution of the experiment itself and in the analysis and reporting of the results. There are two basic categories of output data in a handling qualities evaluation: the pilot comment data and the pilot ratings. Both items are important output data. An experiment which ignores one of the two outputs is discarding a substantial part of the output information.

The output data which are most often neglected are the pilot comments, primarily because they are quite difficult to deal with due to their qualitative form and, perhaps their bulk. Ratings, however, without the attendant pilot objections, are only part of the story. Only if the deficient areas can be identified can one expect to devise improvements to eliminate or attenuate the shortcomings. The pilot comments are the means by which the identification can be made.

8.7.6 Mission Definition

Explicit definition of the mission (task) is probably the most important contributor to the objectivity of the pilot evaluation data. The mission (task) is defined here as a use to which the pilot-airplane combination is to be put. The mission must be very carefully examined, and a clear definition and understanding must be reached between the engineer and the evaluation pilot as to their interpretation of this mission. This definition must include:

1. What the pilot is required to accomplish with the airplane
2. The conditions or circumstances under which he must perform the task

For example, the conditions or circumstances might include instrument or visual flight or both, type of displays in the cockpit, input information to assist the pilot in the accomplishment of the task, etc. The environment in which the task is to be accomplished must also be defined and considered in the evaluation, and could include, for example, the presence or absence of turbulence, day versus night, the frequency with which the task has to be repeated, the variability in pilot preparedness for the task and his proficiency level.

8.7.7 Simulation Situation

The pilot evaluation is seldom conducted under the circumstances of the real mission. The evaluation almost inherently involves simulation to some degree because of the absence of the real situation. As an example, the

evaluation of a day fighter is seldom carried out under the circumstances of a combat mission in which the pilot is not only shooting at real targets, but also being shot back at by real guns. Therefore, after the mission has been defined, the relationship of the simulation situation to the real mission must be explicitly stated for both the engineer and the evaluation pilot so that each may clearly understand the limitations of the simulation situation.

The pilot and engineer must both know what is left out of the evaluation program, and also what is included that should not be. The fact that the anxiety and tension of the real situation are missing, and that the airplane is flying in the clear blue of calm daylight air, instead of in the icing, cloudy, turbulent, dark situation of the real mission, will affect results. Regardless of the evaluation tasks selected, the pilot must use his knowledge and experience to provide a rating which includes all considerations which are pertinent to the mission, whether provided in the tasks or not.

8.7.8 Pilot Comment Data

One of the fallacies resulting from the use of a rating scale which is considered for universal handling qualities application is the assumption that the numerical pilot rating can represent the entire qualitative assessment. Extreme care must be taken against this oversimplification because it does not constitute the full data gathering process.

Pilot objections to the handling qualities are important, particularly to the airplane designer who is responsible for the improvement of the handling qualities. But, even more important, the pilot comment data are essential to the engineer who is attempting to understand and use the pilot rating data. If ratings are the only output data, one has no real way of assessing whether the objectives of the experiment were actually realized. Pilot comments supply a means of assessing whether the pilot objections (which lead to his summary rating) were related to the mission or resulted from some extraneous uncontrolled factor in the execution of the experiment, or from individual pilots focusing on and weighing differently various aspects of the mission. Attention to detail is important to ensure that pilot comments are useful.

Pilots must comment in the simplest language. Avoid engineering terms unless they are carefully defined. The pilot should report what he sees and

feels, and describe his difficulties in carrying out that which he is attempting. It is then important for the pilot to relate the difficulties which he is having in executing specific tasks to their effect on the accomplishment of the mission.

The pilot should make specific comments in evaluating each configuration. These comments generally are in response to questions which have been developed in the discussions of the mission and simulation situation. The pilot must be free to comment on difficulties over and above the specific questions asked of him. The test pilot should strive for a balance between a continuous running commentary and occasional comment in the form of an explicit adjective. The former often requires so much editing to find the substance that it is often ignored, while the latter may add nothing to the numerical rating itself.

The pilot comments must be taken during or immediately after each evalution. In-flight comments should be recorded on a tape recorder. Experience has shown that the best free comments are often given during the evaluation. If the comments are left until the conclusion of the evaluation, they are often forgotten. A useful procedure is to permit free comment during the evaluation itself and to require answers to specific questions in the summary comments at the end of the evaluation.

Questionnaires and supplementary pilot comments are most necessary to ensure that: (a) all important or suspected aspects are considered and not overlooked, (b) information is provided relative to why a given rating has been given, (c) an understanding is provided of the tradeoffs with which pilots must continually contend, and (d) supplementary comment that might not be offered otherwise is stimulated. It is recommended that the pilots participate in the preparation of the questionnaires. The questionnaires should be modified if necessary as a result of the pilots' initial evaluations.

8.7.9 Pilot Rating Data

The pilot rating is an overall summation of the pilot observations relating to the mission. The basic question that is asked of the

pilot conditions the answer that he provides. For this reason, it is important that the program objectives are clearly stated and understood by all concerned and that all criteria, whether established or assumed, be clearly defined. In other words, it is extremely important that the basis upon which the evaluation is established be firmly understood by pilots and engineers. Unless a common basis is used, one cannot hope to achieve comparable pilot ratings, and confusing disagreement will often result. Care must also be taken that criteria established at the beginning of the program carry through to the end. If the pilot finds it necessary to modify his tasks, technique or mission definition during the program, he must make it clear just when this change occurred.

A discussion of the specific use of a rating scale tends to indicate some disagreement among pilots as to how they actually arrive at a specific numerical rating. There is general agreement that the numerical rating is only a shorthand for the word definition. Some pilots, however, lean heavily on the specific adjective description and look for that description which best fits their overall assessment. Other pilots prefer to make the decisions sequentially, thereby arriving at a choice between two or three ratings. The decision among the two or three ratings is then based upon the adjective description. In concept, the latter technique is preferred since it emphasizes the relationship of all decisions to the mission.

The actual technique used is somewhere between the two techniques above and not so different among pilots. In the past, the pilot's choice has probably been strongly influenced by the relative usefulness of the descriptions provided for the categories on the one hand, and the numerical ratings on the other. The evaluation pilot is continuously considering the rating decision process during his evaluation. He proceeds through the decisions to the adjective descriptors enough times that his final decision is a blend of both techniques. It is therefore obvious that descriptors should not be contradictory to the mission-oriented framework.

Half ratings are permitted (e.g., rating 4.5) and are generally used by the evaluation pilot to indicate a reluctance to assign either of the adjacent ratings to describe the configuration. Any finer breakdown than half ratings

is prohibited since any number greater than or less than the half rating implies that it belongs in the adjacent group. Any distinction between configurations assigned the same rating must be made in the pilot comments. Use of the 3.5, 6.5, and 9.5 ratings is discouraged as they must be interpreted as evidence that the pilot is unable to make the fundamental decision with respect to category.

As noted previously, the pilot rating and comments must be given on the spot in order to be most meaningful. If the pilot should later want to change his rating, the engineer should record the reasons and the new rating for consideration in the analysis, and should attempt to repeat the configuration later in the evaluation program. If the configuration cannot be repeated, the larger weight (in most circumstances) should be given to the on-the-spot rating since it was given when all the characteristics were fresh in the pilot's mind.

8.7.10 <u>Execution Of Handling Qualities Tests</u>

Probably the most important item is the admonition to execute the test as it was planned. It is valuable for the engineer to monitor the pilot comment data as the test is conducted in order that he becomes aware of evaluation difficulties as soon as they occur. These difficulties may take a variety of forms. The pilot may use words which the engineer needs to have defined. The pilot's word descriptions may not convey a clear, understandable picture of the piloting difficulties. Direct communication between pilot and engineer is most important in clarifying such uncertainties. In fact, communication is probably the most important single element in the evaluation of handling qualities. Pilot and engineer must endeavor to understand one another and cooperate to achieve and retain this understanding. The very nature of the experiment itself makes this somewhat difficult. The engineer is usually not present during the evaluation, and hence he has only the pilot's word description of any piloting difficulty. Often, these described difficulties are contrary to the intuitive judgments of the engineer based on the characteristics of the airplane by itself. Mutual confidence is required. The engineer should be confident that the pilot will give him accurate,

meaningful data; the pilot should be confident that the engineer is vitally interested in what he has to say and trusts the accuracy of his comments.

It is important that the pilot have no foreknowledge of the specific characteristics of the configuration being investigated. This does not exclude information which can be provided to help shorten certain tests (e.g., the parameter variations are lateral-directional, only). But it does exclude foreknowledge of the specific parameters under evaluation. The pilot must be free to examine the configuration without prejudice, learn all he can about it from meeting it as an unknown for the first time, look clearly and accurately at his difficulties in performing the evaluation task, and freely associate these difficulties with their effects on the ultimate success of the mission. A considerable aid to the pilot in this assessment is to present the configuration in a random-appearing fashion.

The amount of time which the pilot should use for the evaluation is difficult to specify a priori. He is normally asked to examine each configuration for as long as is necessary to feel confident that he can give a reliable and repeatable assessment. Sometimes, however, it is necessary to limit the evaluation time to a specific period of time because of circumstances beyond the control of the researcher. If the evaluation time per pilot is limited, a larger sample of pilots, or repeat evaluations will be required for similar accuracy, and the pilot comment data will be of poorer quality.

The evaluation pilot must be confident of the importance of the simulation program and join wholeheartedly into the production of data which will supply answers to the questions. Pilots as a group are strongly motivated toward the production of data to improve the handling qualities of the airplanes they fly. It isn't usually necessary to explicitly motivate the pilot, but it is very important to inspire in him confidence in the structure of the experiment and the usefulness of his rating and comment data. Pilot evaluations are probably one of the most difficult tasks that a pilot undertakes. To produce useful data involves a lot of hard work, tenacity, and careful thought. There is a strong tendency for the pilot to become discouraged about their ultimate usefulness. The pilot needs feedback on the

accuracy and repeatability of his evaluations. The test pilot is the only one who can provide the answers to the questions that are being asked. He must be i ~ured through feedback

u his assessments are good so that he gains confidence in the manner in which he is carrying out the program.

8.8 DYNAMIC STABILITY FLIGHT TESTS

The dynamic response of an aircraft to various pilot control inputs is important in evaluating its handling qualities. The aircraft may be statically stable, yet its dynamic response could be such that a dangerous flight characteristic results. The aircraft must have dynamic qualities that permit the design mission to be accomplished.

The purpose of the dynamic stability flight test is to investigate an aircraft's primary modes of motion. An airplane usually has five major modes of free motion: phugoid, short period, rolling, Dutch roll and spiral. Flight test determines the acceptability of these modes - frequency, damping, and time constant being the characteristics of primary importance.

There are several different forms that the modes of motion may take. Figure 8.24 shows four possibilities for aircraft free motion: a pure divergence, a pure convergence, a damped, or an undamped oscillation. The aircraft being a rather complicated dynamic system, will move in a manner that is a combination of several different modes at the same time. One of the problems of flight testing is to excite each individual mode independently.

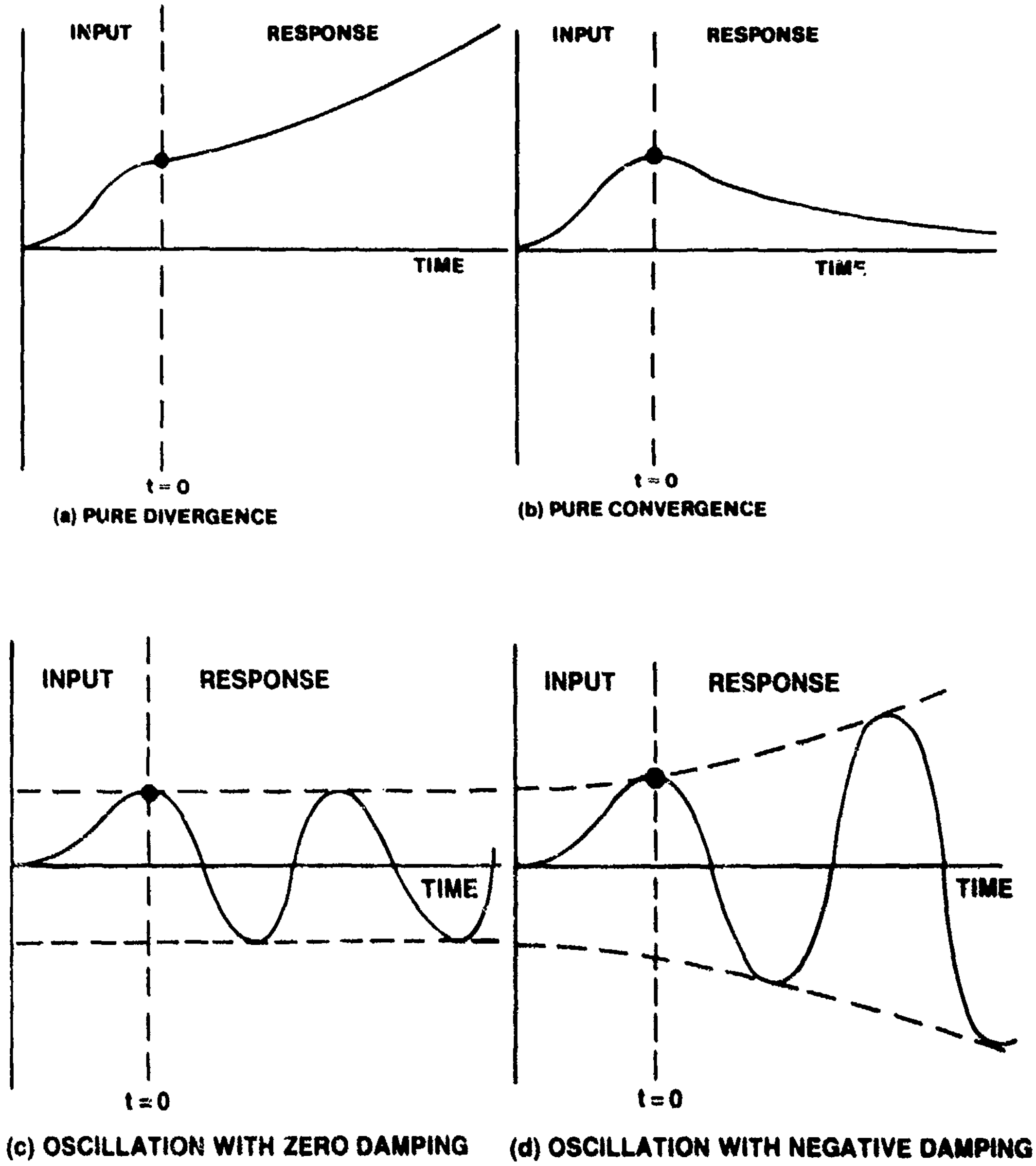


FIGURE 8.24. AIRCRAFT FREE MOTION POSSIBILITIES

8.8.1 Control Inputs

There are several different control inputs that could be used to excite the dynamic modes of motion of an aircraft. To accomplish the task of obtaining the free response of an aircraft, the pilot makes an appropriate control input, removes himself from the loop, and observes the resulting aircraft motion. Three inputs that are frequently used in stability and control investigations will be discussed in this section: the step input, the pulse, and the doublet.

8.8.1.1 Step Input. When a step input is made, the applicable control is rapidly moved to a desired new position and held there. The aircraft motion

resulting from this suddenly applied new control position is recorded for analysis. A mathematical representation of a step input assumes the deflection occurs in zero time and is contrasted to a typical actual control position time history in Figure 8.25. The "unit step" input is frequently used in theoretical analysis and has the magnitude of one radian, which is equivalent to 57.3°. Specifying control inputs in dimensionless radians instead of degrees is convenient for use in the non-dimensional equations of motion.

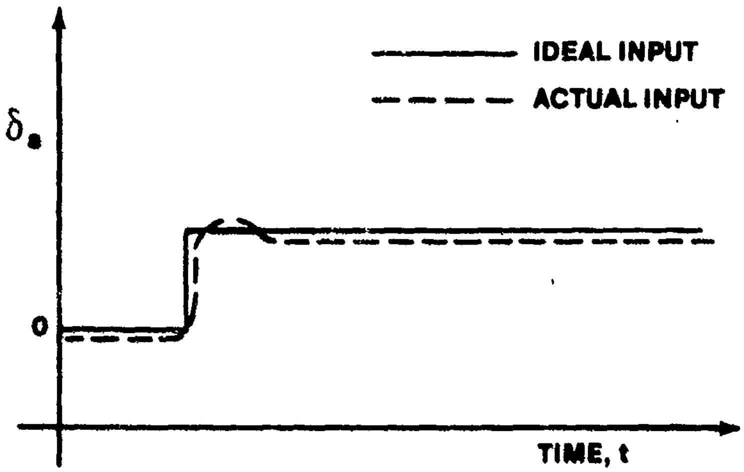


FIGURE 8.25. STEP INPUT

8.8.1.2 Pulse. When a pulse, or singlet is applied, the control is moved to a desired position, held momentarily, and then rapidly returned to its original position. The pilot can then remove himself from the loop and observe the free aircraft response. Again, deflections are theoretically assumed to occur instantaneously. An example of a pulse, or singlet, is shown in Figure 8.26.

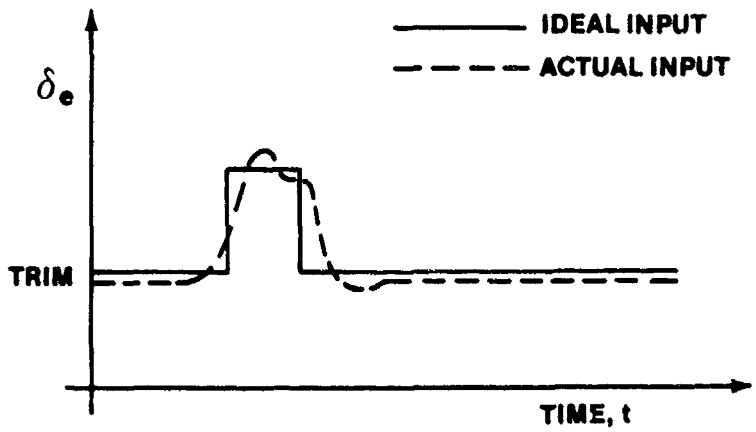


FIGURE 8.26. PULSE INPUT

The "unit impulse" is frequently used in theoretical analysis and is related to the pulse input. The unit impulse is the mathematical result of a limiting process which has an infinitely large magnitude input applied in zero time and an area of unity.

8.8.1.3 Doublet. A doublet is a double pulse which is skew symmetric with time. After exciting a dynamic mode of motion with this input and removing himself from the control loop, the pilot can record the aircraft open loop motion (Figure 8.27).

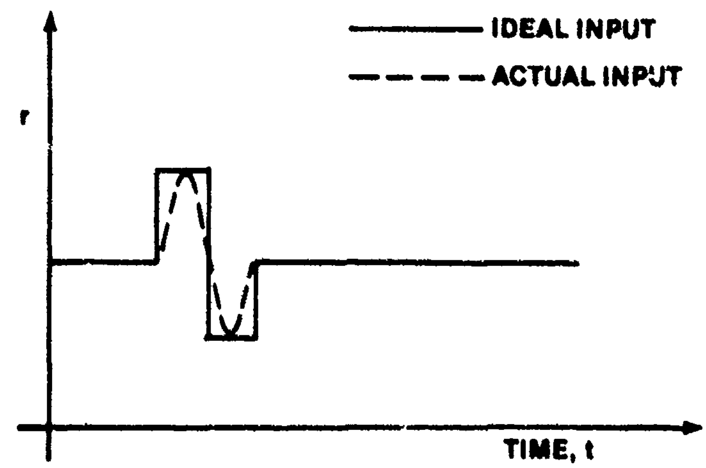


FIGURE 8.27. DOUBLET INPUT

8.8.2 Pilot Estimation Of Second Order Response

Pilot-observed data can be used to obtain approximate values for the damped frequency and damping ratio for second order motion such as the short period or Dutch roll.

To obtain a value for ω_d, the pilot needs merely to observe the number of cycles that occur during a particular increment of time.

Then,

$$f_d = \frac{\text{Number of Cycles}}{\text{Time Increment}} = \text{cycles/sec} \tag{8.31}$$

And

$$\omega_d = \left(f_d \frac{\text{cycles}}{\text{sec}}\right) \left(\frac{2\pi \text{ radians}}{\text{cycle}}\right) = \text{radians/sec} \tag{8.32}$$

The number of cycles can be estimated either by counting overshoots (peaks) or zeroes of the appropriate variable. For short period motion, perturbed θ is easily observed, and if counting overshoots is applied to the motion shown in Figure 8.27, the result is

$$f_d = \frac{1}{2} \frac{(\text{Number of Peaks} - 1)}{(\text{Time Increment})}$$

$$f_d = \frac{\frac{1}{2}(4-1)}{3} = 0.5 \text{ cycles/sec}$$

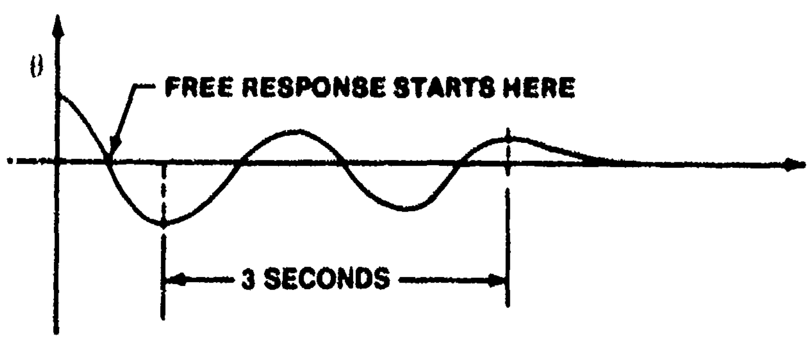


FIGURE 8.28. SECOND ORDER MOTION

If zeroes are counted, then

$$f_d = \frac{\frac{1}{2}\text{ (number of zeroes - 1)}}{\text{(Time Increment)}} \text{ cycles/sec}$$

The pilot can obtain an estimated value for ζ by noting the number of peaks that exist during second order motion and using the approximation

$$\zeta \cong \frac{1}{10}\text{ (7 - Number of Peaks)} \tag{8.33}$$

for $.1 < \zeta < .7$

The motion shown in Figure 8.28 thus has an approximate value

$$\zeta \cong \frac{1}{10}(7 - 4) = .3$$

Note that the peaks which occur during aircraft free response are the ones to be used in Equation 8.33. If zero observable peaks exist during a second order motion, the best estimate for the value of ζ is then "heavily damped, .7 or greater." If seven or more peaks are observed, the best estimate for the value of ζ is "lightly damped, .1 or less."

8.8.3 Short Period Mode

The short period is characterized by pitch angle, pitch rate, and angle of attack change while essentially at constant airspeed and altitude. The short period mode is an important flying quality because its period can approach the limit of pilot reaction time and it is the mode which a pilot uses for longitudinal maneuvers in normal flying. The period and damping may be such that the pilot may induce an unstable oscillation if he attempts to damp the motion with control movements. Hence, heavy damping of this mode is

desirable. Although heavy damping of the short period is desired, investigations have shown that damping alone is insufficient for good flying qualities. In fact, very high damping may result in poor handling qualities. It is the combination of damping and frequency of the motion that is important.

8.8.3.1 Short Period Flight Test Technique. To examine the short period mode, stabilize the airplane at the desired flight condition (altitude, airspeed, normal acceleration). Trim the control forces to zero (for one g normal acceleration) and start recording. Abruptly deflect the longitudinal control to obtain a change in normal acceleration of about one-half g. A suggested technique is to apply a longitudinal control doublet (a small positive displacement followed immediately by a negative displacement of the same magnitude followed by rapidly returning the control to the trimmed position). For stick-fixed stability, return the control to neutral and hold fixed. For stick-free stability release the control after it is returned to neutral.

The abruptness and magnitude of the control input must be approached with due care! Use very small inputs until it is determined that the response is not violent. Start with small magnitudes and gradually work up to the desired excitation. When the aircraft transient motion stops, stop recording data. An input that is too sharp or too large could very easily excite the aircraft structural mode or produce a flutter that might seriously damage the airplane and/or injure the pilot. If the aircraft is equipped with artificial stabilization devices, the test should be conducted with this device off as well as on.

8.8.3.2 Short Period Data Required. The trim conditions of pressure altitude, airspeed, weight, cg position, and configuration should be recorded. The test variables of concern are: airspeed, altitude, angle of attack, normal acceleration, pitch angle, pitch rate, control surface position, and control stick/yoke position.

8.8.3.3 Short Period Data Reduction. Short period mode investigations have shown that frequency as well as damping is important in a consideration of flying qualities. This is so because at a given frequency, damping alters the phase angle of the closed-loop system (which consists of a pilot coupled to the airframe system). Phase angle of the total system governs the dynamic

stability.

The short period frequency, damping, and n/α can be determined from a trace of the time history of aircraft response to a pitch doublet as shown in Figure 8.29.

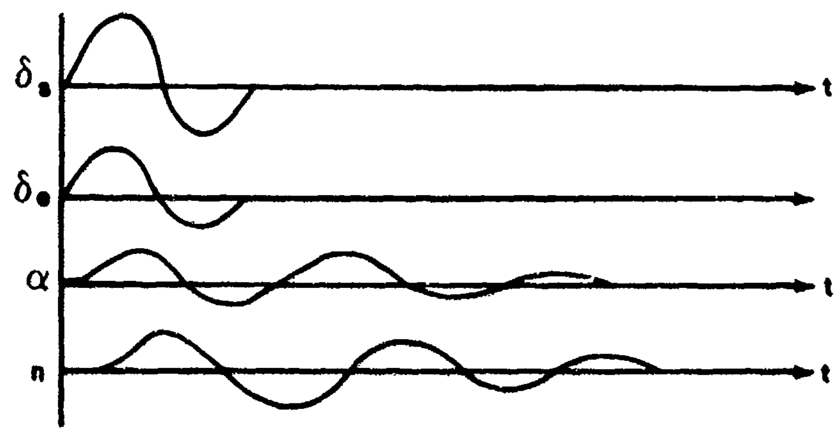


FIGURE 8.29. SHORT PERIOD RESPONSE

The MIL-F-8785C specifies that the angle of attack trace should be used to determine the short period response frequency and damping ratio; however, load factor, pitch angle, and pitch rate are all indicators of the same short period free response characteristics. Either the Log Decrement or Time Ratio data reduction methods can be applied to the short period response trace.

8.8.3.3.1 Log Decrement Method. If the short period response is oscillatory and the damping ratio is 0.5 or less (three or more overshoots), proceed with the log decrement method of data reduction. This method is also called the subsidence ratio or the transient peak-ratio method.

Using the angle of attack trace, draw a mean value line at the steady-state trimmed angle of attack. Measure the values of each peak deviation from trimmed angle of attack for ΔX_1, ΔX_2, ΔX_3, etc., as shown in Figure 8.30.

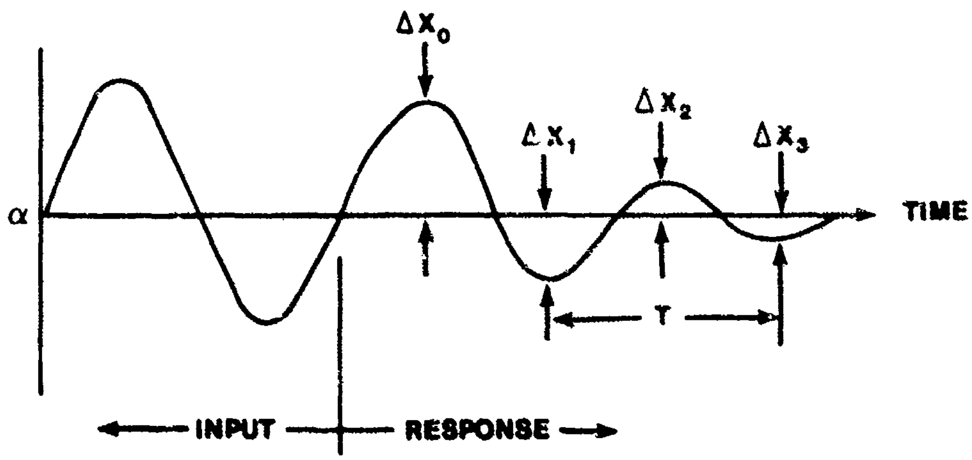


FIGURE 8.30. SUBSIDENCE RATIO ANALYSIS

Determine the transient peak ratios, $\Delta X_1/\Delta X_0$, $\Delta X_2/\Delta X_1$, $\Delta X_3/\Delta X_2$. Enter Figure 8.32 with the transient peak ratio values for $\Delta X_1/\Delta X_0$, $\Delta X_2/\Delta X_1$, $\Delta X_3/\Delta X_2$ and determine corresponding values for damping ratios ζ_1, ζ_2, ζ_3. The average of the damping ratios will yield the value of the overall short period damping ratio. For very lightly damped oscillatory response, Figure 8.33 can be used to determine damping ratio. The m = 1 line is used when comparing peak ratios $\Delta X_1/\Delta X_0$, $\Delta X_2/\Delta X_1$; the m = 2 line is used for peak ratios of $\Delta X_2/\Delta X_0$, $\Delta X_3/\Delta X_1$, etc.

The period, T, of the short period response can be determined by measuring the time between peaks as shown in Figure 8.31. The short period damped frequency is then calculated by $\omega_d = 2\pi/T$ (rad/sec.) The short period natural frequency is computed using $\omega_n = \omega_d/\sqrt{1 - \zeta^2}$.

8.8.3.3.2 Time Ratio Method. If the damping ratio is between .5 and 1.5 (two or less overshoots), then the time ratio method of a data reduction can be used to determine short period response frequency and damping ratio.

Select a peak on the angle of attack trace where the response is free. Divide the amplitude of the peak into the values of 0.736, 0.406, and 0.199. Measure time values t_1, t_2, and t_3 as shown in Figure 8.31. Form the time ratios t_2/t_1, t_3/t_1, and $(t_3 - t_2)/(t_2 - t_1)$.

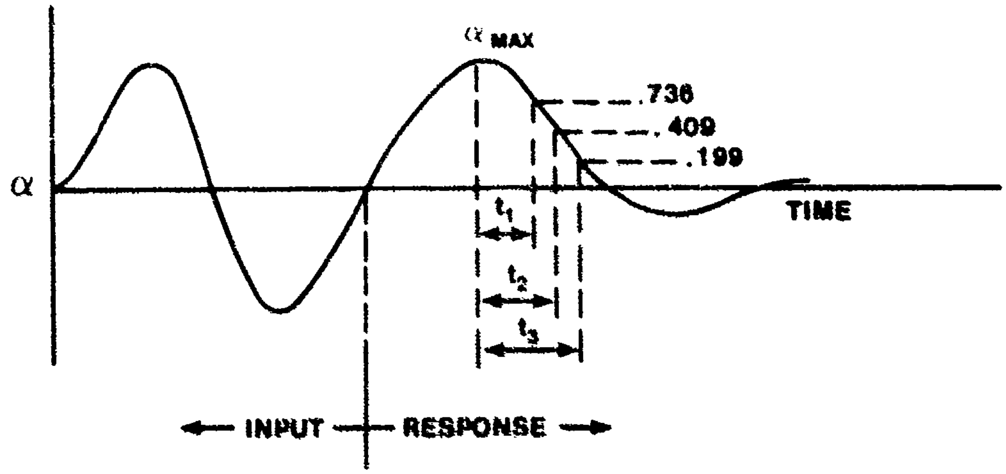


FIGURE 8.31. TIME RATIO ANALYSIS

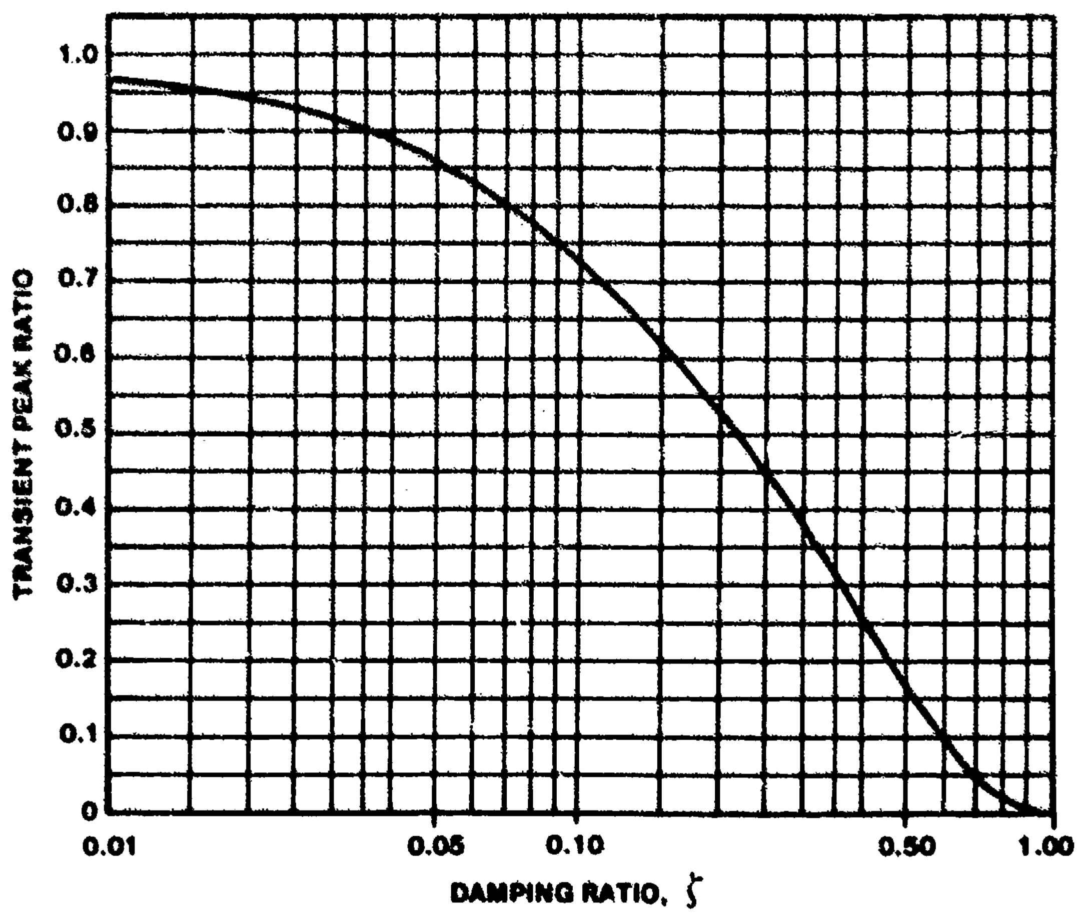


FIGURE 8.32. DETERMINING ζ BY TRANSIENT-PEAK-RATIO METHOD

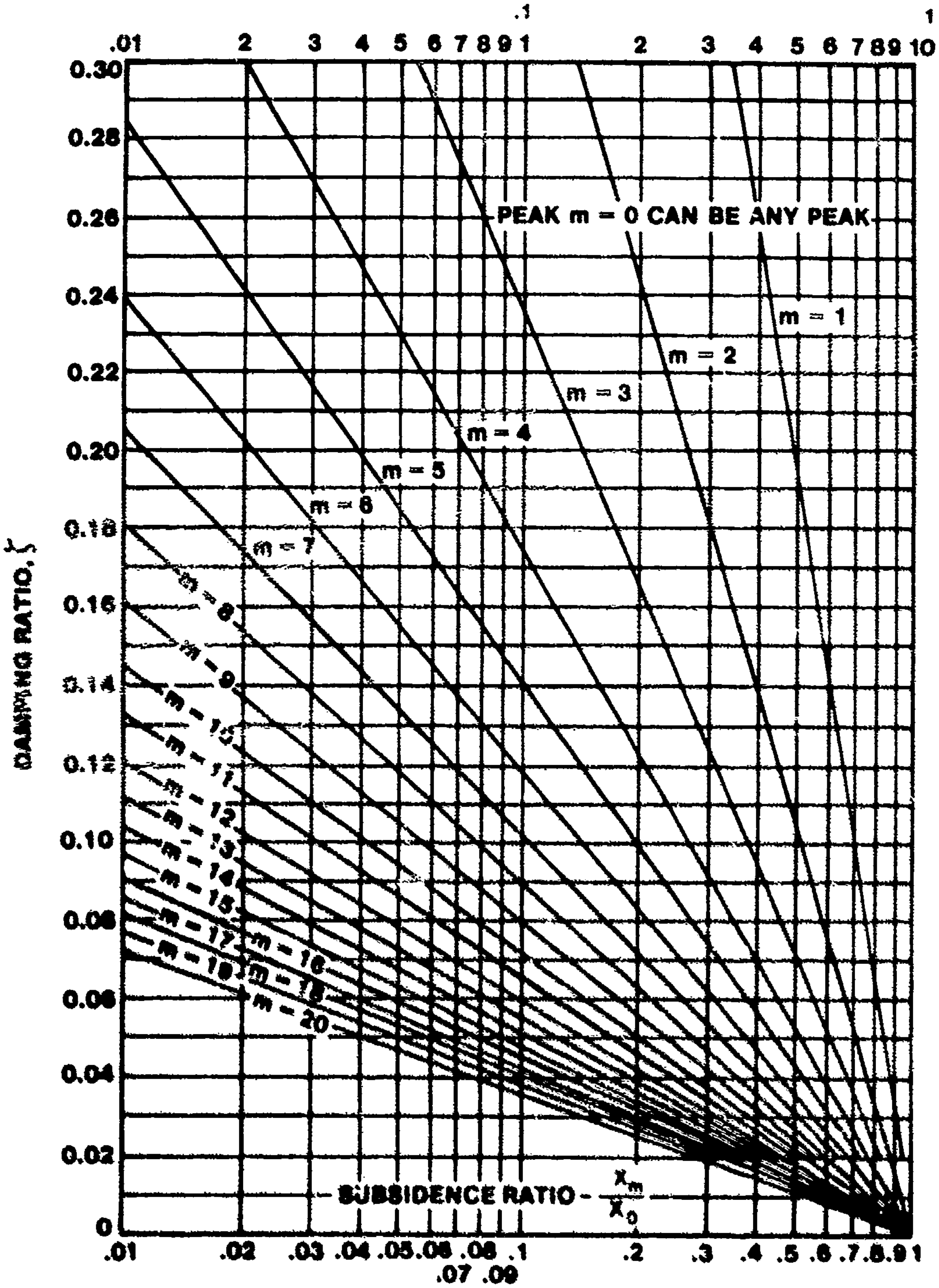


FIGURE 8.33. DAMPING RATIO AS A FUNCTION OF SUBSIDENCE RATIO

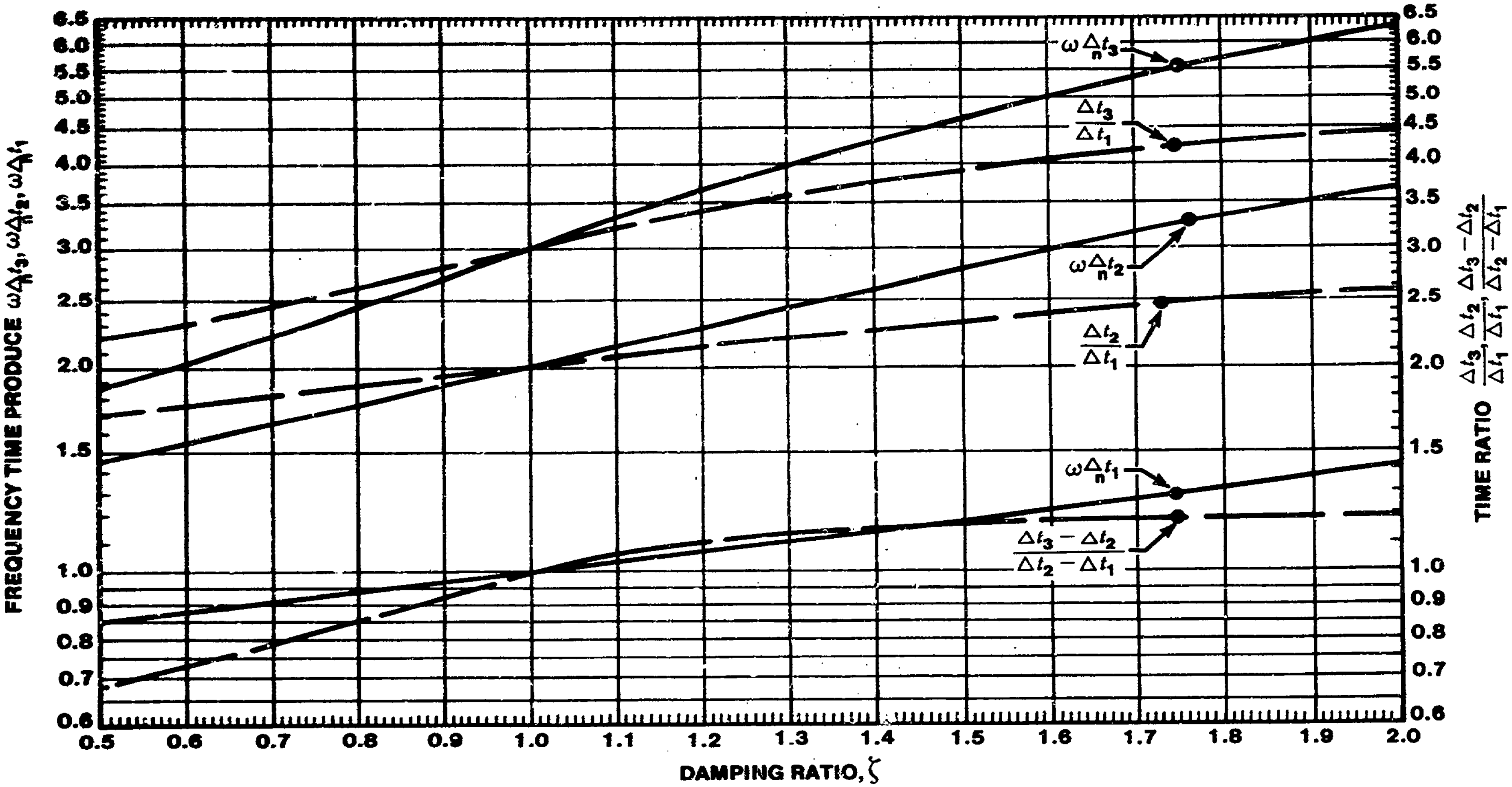


FIGURE 8.34. DETERMINING ζ AND ω_n BY TIME-RATIO METHOD

Enter Figure 8.34 at the Time Ratio side and find the corresponding damping ratio for each time ratio. Average these damping ratios to determine the short period damping ratio, ζ_{sp}.

Re-enter Figure 8.35 with the average short period damping ratio and find the frequency time products $\omega_n t_1$, $\omega_n t_2$, and $\omega_n t_3$. To determine the natural frequency, ω_n, compute:

$$\omega_n = \frac{\omega_n t_1}{t_1} ; \quad \omega_n = \frac{\omega_n t_2}{t_2} ; \quad \omega_n = \frac{\omega_n t_3}{t_3}$$

Average these natural frequencies to determine the overall short period natural frequency.

8.8.3.4 n/α Data Reduction. From the time history traces of load factor and angle of attack free response, determine the peak value of angle of attack that produced the peak load factor, g, as shown in Figure 8.35. Compute the ratio $\Delta n/\Delta\alpha$.

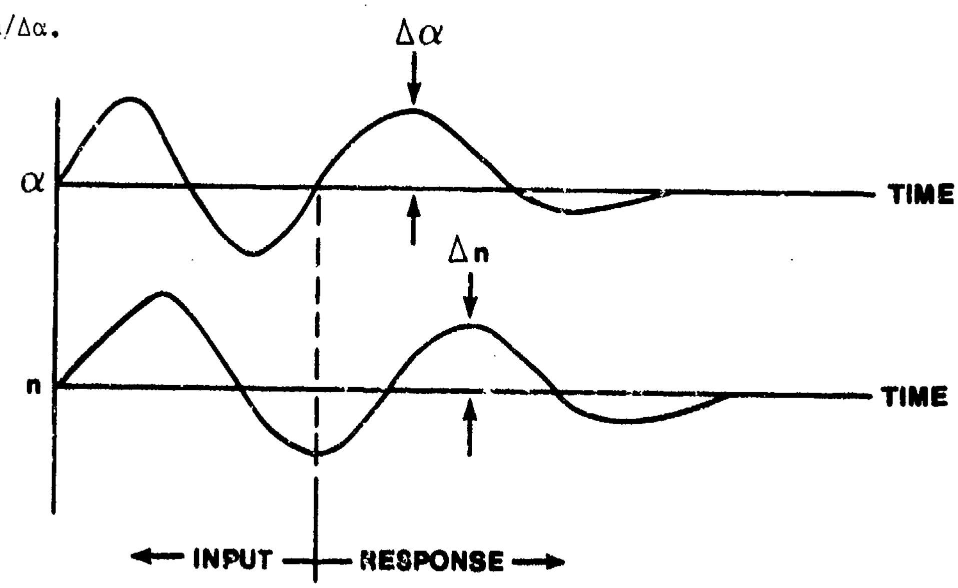


FIGURE 8.35. n/α ANALYSIS

8.8.3.5 Short Period Mil Spec Requirements. MIL-F-8785C specifies that an aircraft's short period response, controls fixed and free, shall meet the requirements of frequency, damping and acceleration sensitivity established in Paragraphs 3.2.2.1 and 3.2.2.1.1. Residual oscillations shall not be greater than 0.05g at the pilot's station nor more than ±3 mils of pitch excursion

pitch excursion for category A Flight Phase tasks.

Tests for short period stability should be conducted from level flight at several altitudes and Mach. Closed loop short period stability tests should also be made at various normal accelerations in maneuvering flight. This stability, when coupled to the pilot, is especially important to tracking and formation flying.

8.8.4 Phugoid Mode

The phugoid mode is generally not considered an important flying quality because its period is usually of sufficient duration that the pilot has little difficulty controlling it. However, under certain conditions it is possible for the damping to degenerate sufficiently so that the phugoid mode becomes important. The phugoid is characterized by airspeed, altitude, pitch angle, and rate variations while at essentially constant angle of attack.

8.8.4.1 Phugoid Flight Test Technique. The phugoid mode may be examined by stabilizing the airplane at the desired flight conditions and trimming the control forces to zero. Smoothly increase the pitch angle until the airspeed reduces 10 to 15 knots below the trim airspeed and return the nose to the trimmed altitude. For stick-fixed stability return the control to neutral and then release it. After the control is released or returned, it may be necessary to maintain wings level by light lateral or slight rudder pressure. Damping and frequency of phugoid motion may be changed appreciably by the presence of small bank angles (5° to 15°). It may be very difficult to return the control to its trimmed position if the aircraft control system has a very large friction band. In such a case, the airspeed increment may be obtained by an increase or decrease in power and by returning it to its trim setting or extending a drag device. In either case the aircraft configuration should be that of the trim condition at the time the data measurements are made.

8.8.4.2 Phugoid Data Required. The trim conditions of pressure altitude, airspeed, weight, cg position and configuration should be recorded.

The damping can be determined by hand recording the maximum and minimum airspeed excursions during at least two cycles of the phugoid free response. In addition, the period can be accurately hand recorded by noting the time between zero vertical velocity points.

8.8.4.3 Phugoid Data Reduction. To determine the phugoid damping ratio (ζ), sketch the damping envelope on the working plot of airspeed versus time. Measure the width of the envelope at the peak values of the oscillation. Form the subsidence ratios (X_m/X_0) . Find the damping ratio for each subsidence ratio from Figure 8.32 or 8.33. Average these damping ratios. If the subsidence ratio is greater than 1.0, then use the inverse of that subsidence ratio. The damping ratio thus determined will be negative, and the mode divergent.

Another method of determining phugoid damping ratio analogous to the above subsidence ratio method is to compute the difference between successive maximum and minimum velocities and assign these magnitudes as ΔX_0, ΔX_1, ΔX_2, etc., as shown in Figure 8.36. Next form the Transient Peak Ratios $\Delta X_1/\Delta X_0$, $\Delta X_2/\Delta X_1$ and find the damping ratio from Figure 8.32 or Figure 8.33.

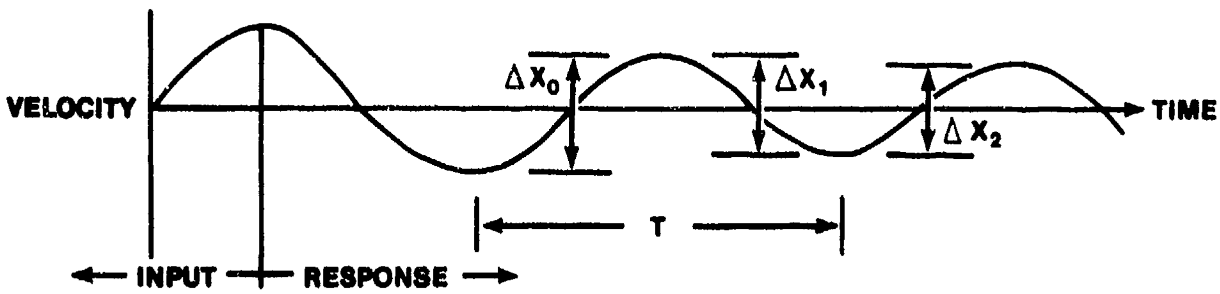


FIGURE 8.36. PHUGOID TRANSIENT PEAK RATIO ANALYSIS

The damped frequency of the phugoid can be determined from the hand recorded period by $\omega_d = 2\pi/T$ (rad/sec). The natural frequency is then computed by

$$\omega_n = \frac{\omega_d}{\sqrt{1 - \zeta^2}}$$

8.8.4.4 Phugoid Mil Spec Requirement. The MIL-F-8785C requirement for phugoid damping is outlined in Paragraph 3.2.1.2.

8.8.5 Dutch Roll Mode

The Dutch Roll lateral-directional oscillations involve roll, yaw, and sideslip. The stability of the Dutch roll mode varies with airplane configuration, angle of attack, Mach, and damper configuration. The

presence of a lightly damped oscillation adversely affects aiming accuracy during bombing runs, firing of guns and rockets, and precise formation work such as in-flight refueling.

Stability of the oscillations is represented by the damping ratio; however, the frequency of an oscillation and the ϕ/β ratio are also important in order to correlate the motion data with the pilot's opinion of handling qualities. If the frequency is higher than pilot reaction time, the pilot cannot control the oscillation and in some cases may reinforce the oscillation to an undesirable amplitude. Since it is the damping frequency combination which influences pilot opinion more than damping alone, some effort should be made to correlate this combination with pilot opinion of the lateral-directional oscillation.

At supersonic speeds, directional stability often decreases with increased Mach and altitude for constant g. An evaluation should proceed cautiously to avoid possible divergent responses that can result from nonlinear aerodynamics.

8.8.5.1 Dutch Roll Flight Test Techniques.

8.8.5.1.1 Rudder Pulse (doublet). Stabilize the airplane in level flight at test flight conditions and trim. Rapidly depress the rudder in each direction and neutralize. Hold at neutral for control-fixed or release rudder for control free response. For aircraft which require excessive rudder force in some flight conditions, the rudder pulse may be applied through the augmented directional flight control system.

8.8.5.1.2 Release from Steady Sideslip. Stabilize the airplane in level flight at test flight conditions and trim forces to zero. Establish a steady straight-path sideslip angle. Rapidly neutralize controls. Either hold controls for control-fixed or release controls for control-free response. Start with small sideslip in case the aircraft diverges.

8.8.5.1.3 Aileron Pulse. Stabilize the airplane in level flight at test flight conditions and trim. Hold aircraft in a steady turn of 10° to 30° of bank. Roll level at a maximum rate reducing the roll rate to zero at level flight. CAUTION. . . Such a test procedure must be monitored by an engineer who is thoroughly familiar with the inertial coupling of that aircraft and its effect upon structural loads and nonlinear stability.

Nonlinearities in the aircraft response may hinder the extraction of the

necessary parameters. These can be induced by large input conditions. Small inputs balanced with instrument sensitivity give the best result.

8.8.5.2 Dutch Roll Data Required. For trim condition, pressure altitude, airspeed, weight, cg position, and aircraft configuration should be recorded. The test variables of concern are bank angle, sideslip angle, yaw rate, roll rate, control positions, and control surface positions.

Flight test data will be obtained as time histories. When determining the damping ratio, the roll rate parameter usually presents the best trace. In addition, the bank angle and sideslip angle time histories will be required to determine the ϕ/β ratio.

8.8.5.3 Dutch Roll Data Reduction. The Dutch roll frequency and damping ratio can be determined from either a bank angle, sideslip angle, or roll rate response time history trace. The roll rate generally gives the best trace for data reduction purposes.

The methods for determining Dutch roll frequency and damping ratio are the same as used for short period data reduction. If the damping ratio is between .5 and 1.5 (2 or less overshoots), then the time ratio method can be employed. For damping ratio of .5 or less (3 or more overshoots), then subsidence ratio methods are applicable for determining Dutch roll frequencies and damping ratio.

The ϕ/β ratio at the test condition can be determined from the ratio of magnitudes of roll angle envelope to sideslip angle envelope at any specified instant of time during the free response motion as shown in Figure 8.37.

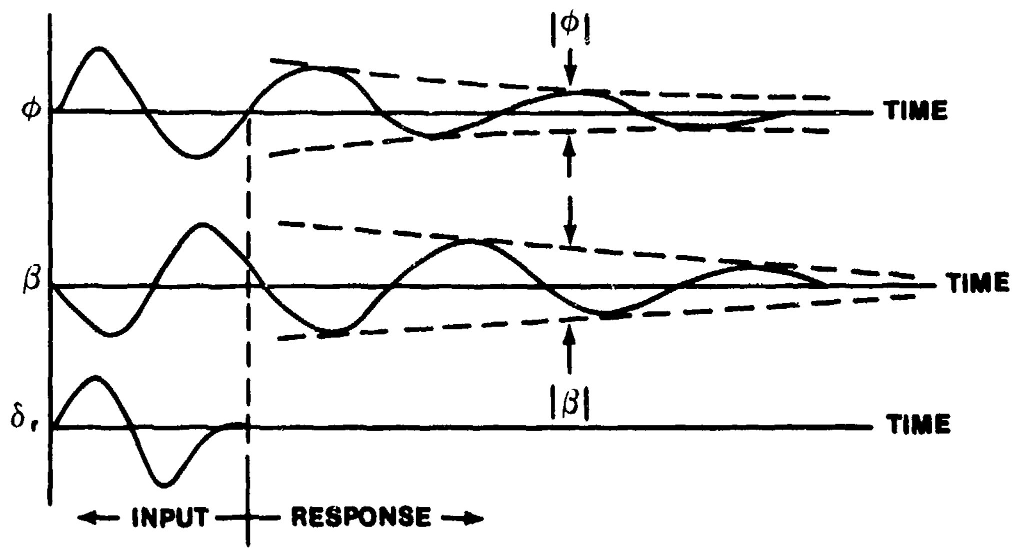


FIGURE 8.37. DETERMINATION OF $|\theta|/|\beta|$ ANALYSIS

8.8.5.4 Dutch Roll Mil Spec Requirements

MIL-F-8785C requirements for Dutch roll frequency and damping ratio are specified in Paragraph 3.3.1.1.

8.8.6 Spiral Mode

The spiral mode is relatively unimportant as a flying quality. However, a combination of spiral instability and lack of precise lateral trimmability may be annoying to the pilot. This problem will be evaluated as a whole due to the difficulty in separating the effects.

The divergent motion is non-oscillatory and is most noticeable in the bank and yaw responses. If an airplane is spirally divergent, it will, when disturbed and not checked, go into a tightening spiral dive. This divergence can be easily controlled by the pilot if the divergence is not too fast.

Excitation of the spiral mode only is difficult because of its relatively large time constant. Any practical input using control surfaces would usually excite other modes as well. If a deficiency in lateral trim control exists, it is often difficult to determine what portion of the resultant motion following a disturbance is caused by the spiral mode. This flight test is used to determine if a combined problem of lateral trim and spiral stability exists. If test results show a definite divergence in hands-off flight, the problem exists.

Spiral divergence is of little importance as a flying quality because it is well within the control capability of the pilot. The ability to hold lateral trim in hands-off flight for 10 to 20 seconds is important.

8.8.6.1 Spiral Mode Flight Test Technique. Trim the aircraft for hands-off flight, ensuring that particular attention is given to lateral control and the ball being centered. Roll into a 20° bank in one direction, release the controls and measure the bank angle after 20 seconds. Repeat the maneuver in a bank to the opposite side.

8.8.6.2 Spiral Mode Data Required. Record aircraft configuration, weight, cg position, altitude and airspeed. The test variables are bank angle, sideslip angle, control position, and control surface position.

8.8.6.3 Spiral Mode Data Reduction. Average the time to double amplitude for right and left banks at each test condition. Figure 8.38 illustrates bank angle data for spiral mode analysis.

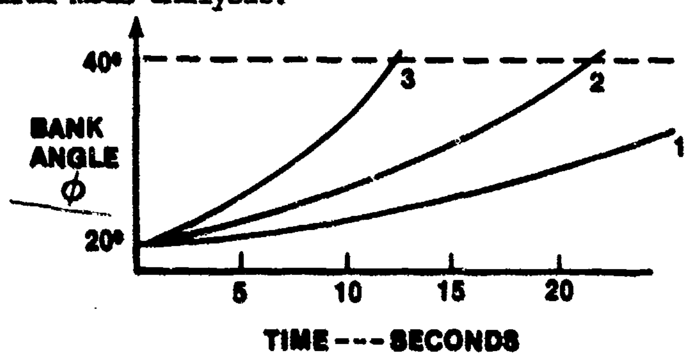


FIGURE 8.38. SPIRAL MODE ANALYSIS

8.8.6.4 Spiral Mode Mil Spec Requirement. Spiral Stability is specified in MIL-F-8785C in Table VIII. This table established minimum times to double amplitude when the aircraft is put into a bank up to 20°, and the controls are freed.

8.8.7 Roll Mode

The roll mode is the primary method that the pilot uses in controlling the lateral attitude of an aircraft. The roll mode represents an aperiodic (nonoscillatory) response to a pilot's lateral stick input which involves almost a pure roll about the x-axis.

Of primary concern to all pilots is the roll performance involving the time required for the aircraft to accelerate to and reach a steady state roll rate in response to a pilot's lateral input. The roll performance parameter

is useful in describing the roll response of an airplane in roll mode time constant, τ_R. Physically, τ_R, is that time for which the airplane has reached 63% of its steady-state roll rate following a step input of the ailerons. The roll mode time constant directly influences the pilot's opinion of the maneuvering capabilities of an airplane. In addition, τ_R can affect the piloting technique used in bank angle control tasks.

8.8.7.1 Roll Mode Flight Test Technique. Trim the aircraft for hands-off flight. Roll into a 45° bank in one direction and stabilize the aircraft. Abruptly apply a small step aileron input and hold throughout 90° of bank angle change. The size of the step aileron input should be sufficiently small to allow the aircraft to achieve steady-state roll rate prior to the 90° bank angle change; however, sufficiently large to measure the roll rate with the instrumentation system onboard the aircraft.

8.8.7.2 Roll Mode Data Required. Record aircraft configuration, weight, cg position, lateral fuel loading, attitude and airspeed. The test variables are bank, roll rate, control position and control surface position.

8.8.7.3 Roll Mode Data Reduction. The roll mode time constant, τ_R, can be measured from a time history trace of the roll rate response (Figure 8.39B) or bank angle response (Figure 8.39A) to a step aileron input. From the roll rate, p, trace the time constant, τ_R, can be measured as the time for the roll rate to achieve 63.2% of the steady-state roll rate response (Figure 8.39A) τ_R can also be determined from the bank angle, ϕ, trace as the time for the extension of the linear slope of the ϕ trace to intersect the initial ϕ axis as represented in Figure 8.39A. It is important to note that the roll mode time constant is independent of the size of the step aileron input.

8.8.7.4 Roll Mode Mil Spec Requirements. The roll mode requirements are specified in MIL-F-8785C, Table VII. This table establishes limits on the maximum allowable time for the roll mode time constant.

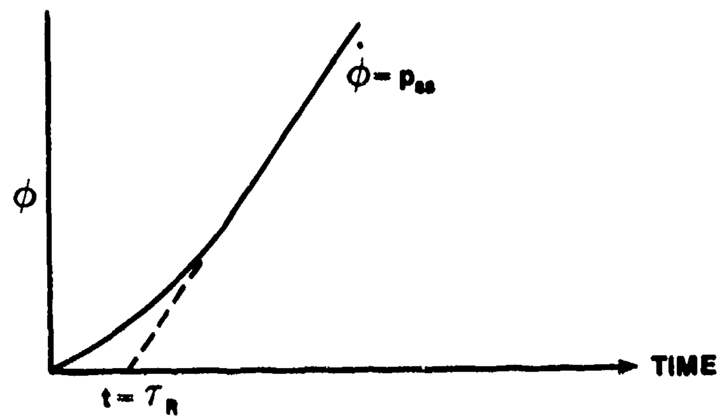


FIGURE 8.39A. BANK ANGLE TRACE

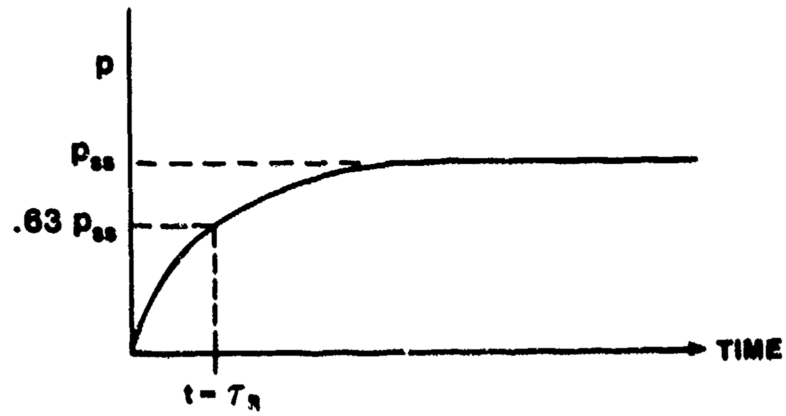


FIGURE 8.39B. ROLL RATE TRACE

8.8.8 Roll-Sideslip Coupling

In contrast to most other requirements which specify desired response to control inputs, roll-sideslip coupling produces unwanted responses. The Dutch roll mode of motion can be seen in the p and β traces. How these parameters are phased with each other will highlight closed loop problems. These unwanted responses detract from precision of control and can contribute to PIO tendencies.

Roll-sideslip coupling is manifested in at least three ways depending on the Dutch roll ϕ/β ratio.

Low ϕ/β Ratio (less than 1.5):

More sideslip than roll motion. In this case, if roll rate or aileron control excite sideslip, the flying qualities can be degraded by such motion

as an oscillation of the nose on the horizon during a turn or a lag or initial reversal in yaw rate during a turn entry or by pilot difficulty in quickly and precisely acquiring a given heading (ILS or GCA). In addition, the pilot has great difficulty damping Dutch roll with aileron only.

Large ϕ/β Ratio (1.5 to 6):

The coupling of β with p and ϕ becomes important, causing oscillations in roll rate and ratcheting of bank angle. Here, the pilot may have difficulty in precisely controlling roll rate or in acquiring a given bank angle without overshoots.

Very Large ϕ/β Ratio (>6):

Sensitivity of roll to rudder pedals or response to atmospheric disturbances may be so great that the aircraft response is never considered good.

In addition to the different problems caused by the magnitude of the ϕ/β ratio, the degree of difficulty in controlling these unwanted motions is very important. If the airplane is easy to coordinate during turn entries, then the pilot may tolerate relatively large unwanted motions during rudder - pedal - free turn entries since he can control these unwanted motions if desired. On the other hand, when coordination is difficult, the pilot will tolerate very small unwanted motions, since he must either accept these motions or may even aggravate them if he tries to coordinate. The parameter "ψ_β" was introduced as the most precise measure of this very nebulous, but important factor - difficulty of coordination. The use of "ψ_β" is primarily important when looking at small lateral control inputs and the resulting aircraft response in either roll or yaw.

8.8.8.1 <u>Roll Rate Oscillations (Paragraph 3.3.2.2.)</u>. This paragraph and the small input paragraph are primarily looking at aircraft with a Dutch roll ϕ/β ratio between 1.5 to 6.0 (moderate to large). This particular paragraph is specified for large inputs (at least 90° of roll). In general, any large oscillations after a step aileron input (rudder free) are not wanted. In the table, the percentage values are given for different levels and categories which should not be exceeded. If there is a large change, the pilot will see

ratcheting and won't like the aircraft's characteristics.

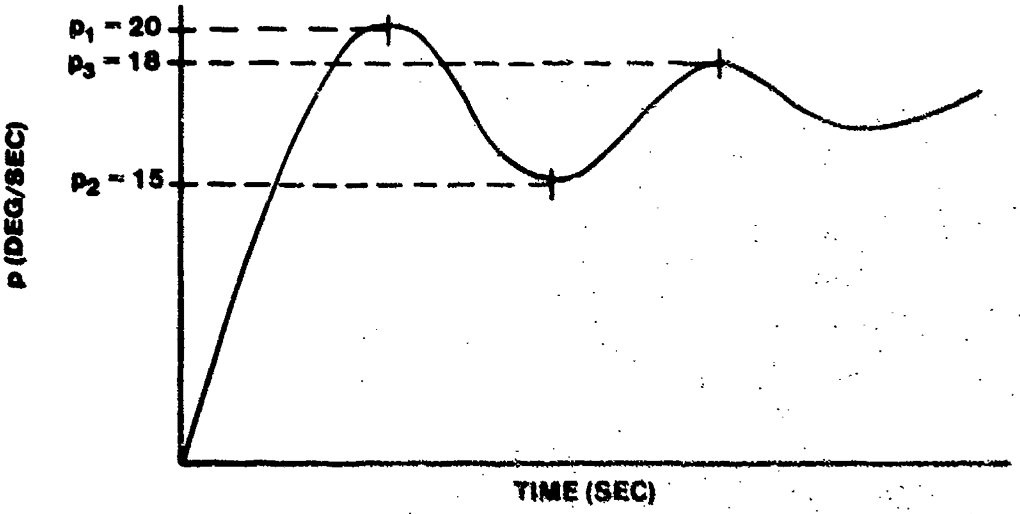


FIGURE 8.40. ROLL RATE OSCILLATIONS

p_1 is the first peak in the roll rate trace, and p_2 is the first minimum. The ratio of p_2/p_1 shall not exceed the values in the table. If p_2 crossed the time axis into negative territory, the sign of p has changed and an automatic failure should be given.

8.8.9 Roll Rate Requirements For Small Inputs (Paragraph 3.2.2.2.1)

This paragraph addresses precision of control with small aileron inputs. The degree of oscillation - p_{osc}/p_{av} - are unwanted motions that pilots, who are flying an ILS or some other task, perceive as a performance difficulty. The larger the oscillation or the greater the difficulty in damping the oscillation, the greater the pilot's workload and ability to accomplish the task. In other words, the degradation in flying qualities is proportional to the amount of roll rate oscillation, p_{osc}, about some mean value of roll rate, p_{av}. The term "ψ_β" has been referred to as the "difficulty of coordination" parameter. This is based on the fact that an aircraft should develop adverse yaw with aileron inputs so that the pilot can normally coordinate the turn. For example, if a right roll is initiated with aileron, then the aircraft will yaw left (adverse), causing the pilot to use right rudder to bring the adverse yaw to zero. Thus, he needs right rudder for right aileron inputs. If proverse yaw resulted from the input, then the pilot would have to cross control rudder and aileron to coordinate the roll. With this background, an

examination of Figure 8.40 from Paragraph 3.2.2.2.1 can now be done.

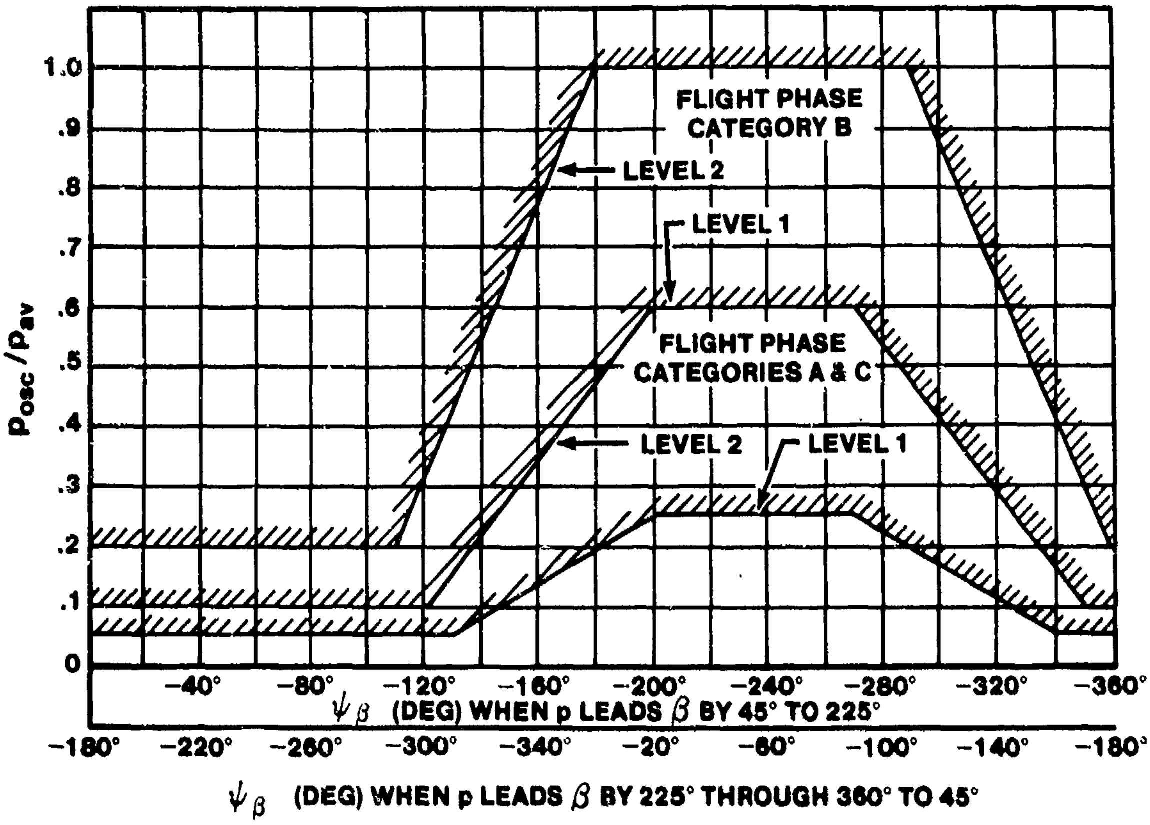


FIGURE 8.41. ROLL RATE OSCILLATION REQUIREMENTS

From this figure it can be seen that the ratio of roll rate oscillation to steady state roll rate can be greater for some values of ψ_β than for others. The assumption that p leads β and that the aircraft has positive dihedral, will be made for the following discussion. Specifically, the specified values of p_{osc}/p_{av} for $0^{\circ} \geq \psi_\beta \geq -90^{\circ}$ are far more stringent than for $-180 \leq \psi_\beta$ -270°. There are at least three reasons for this.

First, aileron inputs proportional to bank angle errors generate yaw acceleration that tend to damp the Dutch roll oscillations, when $-180^{\circ} \geq \psi_\beta \geq -270^{\circ}$. Thus, the Dutch roll oscillation damps out more quickly with pilot roll inputs. Conversely, if ψ_β is between 0° and 90°, the aileron input tends to excite Dutch roll and can even cause lateral PIO. The latter case causes a pilot's tolerance of p_{osc}/p_{av} to be reduced.

Secondly, the requirements of p_{osc}/p_{av} vary considerably due to the difficulty of coordination previously mentioned. For $-180^{\circ} \geq \psi_\beta \geq -270^{\circ}$, normal coordination may be effected, that is, right rudder pedal is required

for right rolls. Thus, even if roll oscillations do occur, the pilot can manage the oscillations by using rudders. On the other, for $0^{\circ} \geq \psi_{\beta} \geq -90^{\circ}$, it is necessary to cross control to effect coordination that is, left rudder for right aileron. Since pilots do not normally cross control, and if they must, they have great difficulty in doing so, they either let the oscillations go unchecked or make them worse.

The final reason for the significant variation in p_{osc}/p_{av} with ψ_{β} is that the average roll rate, p_{av}, for a given input varies significantly with ψ_{β}. For positive dihedral, adverse yaw due-to-aileron ($\omega_{\beta} \cong 180^{\circ}$) tends to decrease average roll rate, whereas proverse yaw-due-to-aileron ($\psi_{\beta} \cong 0^{\circ}$) tends to increase roll rate. As a matter of fact, proverse yaw-due-to-aileron is sometimes referred to as "complementary yaw" because of this augmentation of roll effectiveness. Thus, for a given amplitude of p_{osc}, p_{osc}/p_{av} will be greater for $\psi_{\beta} \cong 180^{\circ}$ than it will be at $\psi_{\beta} \cong 0^{\circ}$.

p_{osc}/p_{av} and ψ_{β} are calculated using the following equations where p_{osc}/p_{av} depends on the value of ζ_D.

$$\zeta_D \leq 0.2 \quad \frac{p_{osc}}{p_{av}} = \frac{p_1 + p_3 - 2p_2}{p_1 + p_3 + 2p_2}$$

$$\zeta_D > 0.2 \quad \frac{p_{osc}}{p_{av}} = \frac{p_1 - p_2}{p + p_2}$$

where p_1, p_2, and p_3 are roll rates at the first, second and third peaks respectively. See Figure 8.42.

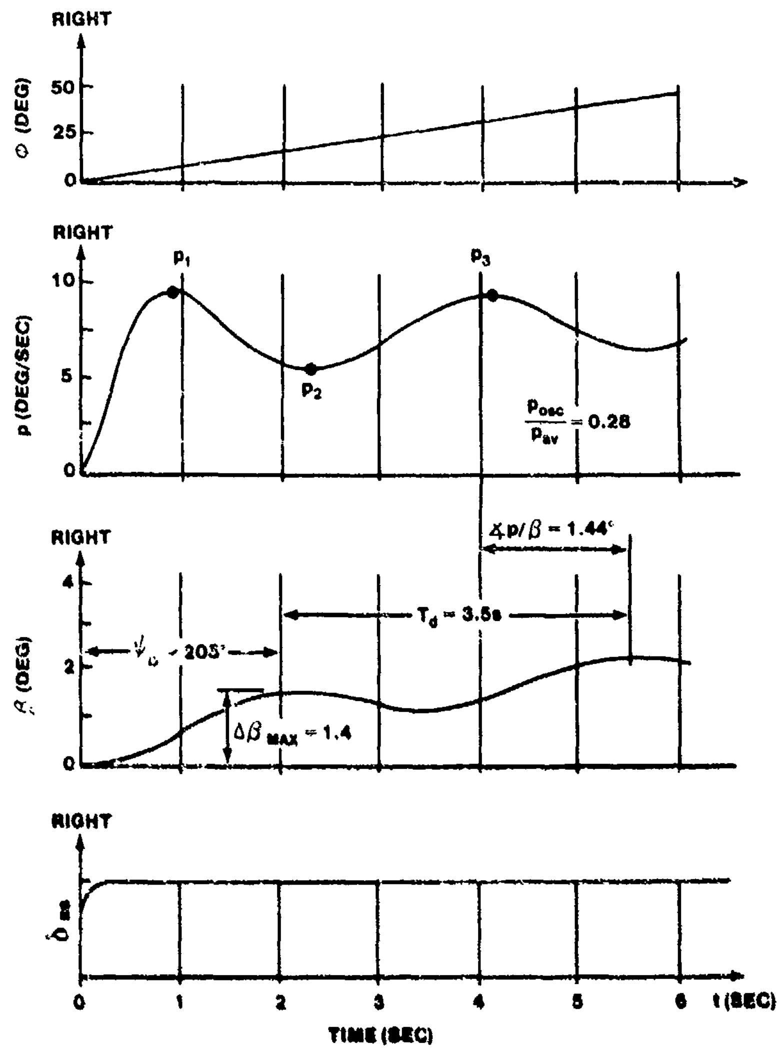


FIGURE 8.42. ROLL RATE OSCILLATION DETERMINATION

In order to calculate ψ_β, a sign convention must be assumed. First assume positive dihedral aircraft (p leading β will verify this). Second, use upper set of ψ_β numbers in all paragraphs showing two sets. (This can be verified by calculating the angle between the p and β trace maximums.) Third, if the roll is made to the right, look for the 1st local maximum on the β trace, and if the roll is made to the left, look for the 1st local minimum. Finally, use the formula

$$\psi_\beta = -\frac{360}{T_d} t_{n_\beta} + (n-1)\ 360 \text{ deg}$$

$$n = \text{nth Local Maximum}$$

where:

T_D is by definition the Dutch roll period (shown in Figure 8.42); t_{n_B} is the time on the β trace for the 1st local maximum (right roll) or 1st local minimum (left roll). See Figure 8.42 for a right roll. If p/β is calculated, just compare the time difference between the p and β trace and divide by the T_d and multiply by 360°.

The only other requirement is to make sure that the input is small. It should take at least 1.7 times T_d for a 60° bank angle change.

8.8.10 <u>Bank Angle Oscillations (Paragraph 3.3.2.3)</u>.

In order to extend the roll-sideslip coupling requirement to larger control deflections and to account for some flight control non-linearities, this paragraph specifies similar calculations as (Paragraph 3.3.2.2.1), except that the input is an impulse. This input should be at the maximum rate and at the largest deflection possible. The resulting motion after the input will be bank angle oscillations around zero degrees. The bank angle should be applied after being stabilized at approximately 15° of bank. The difference in the shifting of curve in Figure 8.42 is due to ψ_β from a pulse being 90° more positive than for a step.

To calculate ϕ_{osc}/ϕ_{av} identify bank angles $\phi_1, \phi_2, + \phi_3$ as was done with p in finding p_{osc}/p_{av}.

$$\zeta \leq 0.2 \quad \frac{\phi_{osc}}{\phi_{av}} = \frac{\phi_1 + \phi_3 - 2\phi_2}{\phi_1 + \phi_2 + 2\phi_2}$$

$$\zeta > 0.2 \quad \frac{\phi_{osc}}{\phi_{av}} = \frac{\phi_1 - \phi_2}{\phi_1 + \phi_2}$$

Next calculate ψ_β as was discussed previously on the β trace.

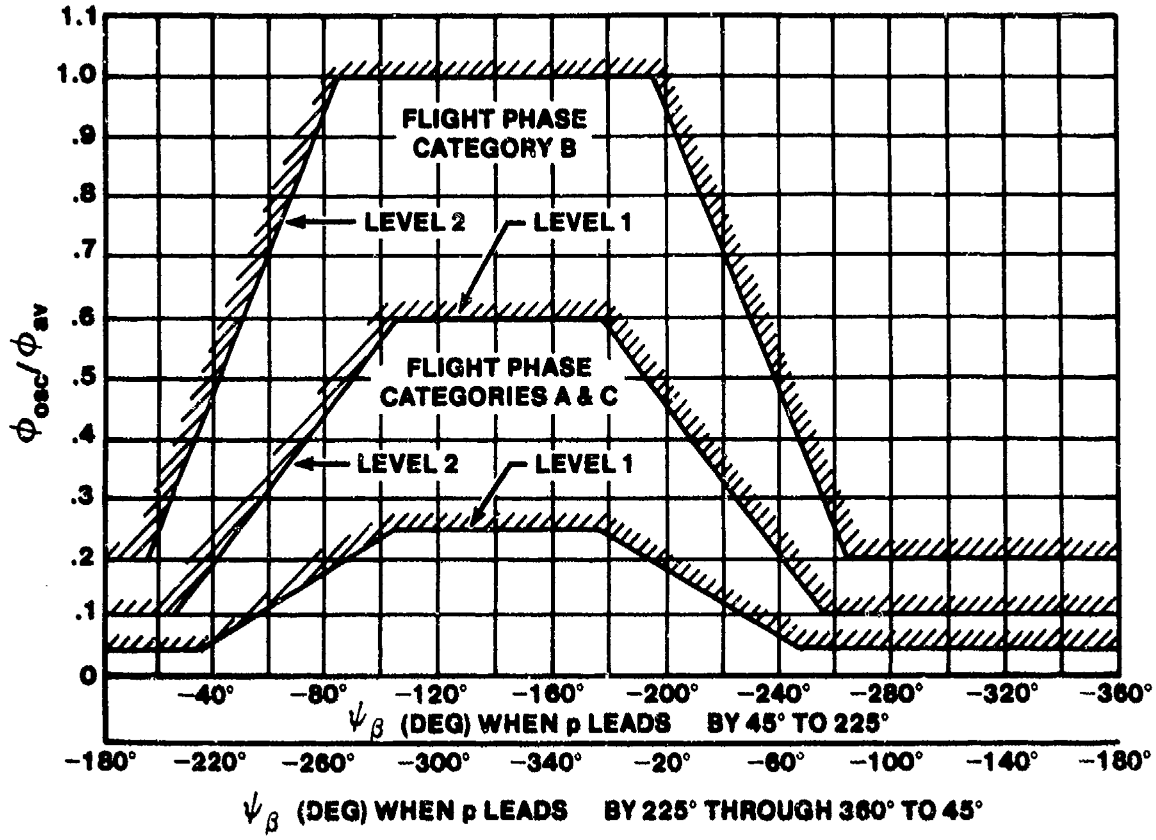


FIGURE 8.43. BANK ANGLE OSCILLATION REQUIREMENTS

8.8.11 Sideslip Excursions (Paragraph 3.3.2.4)

This paragraph and the next one for small inputs are for low Dutch roll ϕ_β (< 1.5) and are associated with sideslip rather than roll or bank angle tracking. The basis for the paragraph is research indicating a maximum amount of sideslip generated that can be tolerated by the pilot, whether the sideslip is adverse or proverse and the phase relationship between the sideslip and the roll in the Dutch roll are the overriding factors. The coordination of control with proverse yaw is very difficult and unnatural so the levels specified are much lower than those for adverse yaw. Another factor to be considered is the side-force and side acceleration caused by sideslip angles at high speed. Research has concluded that if such acceleration is very high (> 0.2 g's), then the resulting motions cause interference with normal pilot duties.

In order to calculate the required parameters for this paragraph, refer to Figure 8.41 for β and k. The Dutch roll period is determined as before, and half its value is compared with two seconds. Whichever is greater, the β

excursion proverse yaw and adverse yaw needs to be used during that time after the rudder pedal free aileron input. From Figure 8.41 there is no proverse yaw, but 1.4° of adverse yaw during two seconds of right roll is present. Many tests will have a small proverse yaw excursion before adverse yaw builds up, which would require a calculation of $\Delta\beta$ proverse. Next, calculate the "k" factor. It is known as the "severity of input" parameter. "k" is the ratio of what roll rate was achieved during the flight test, (ϕ_{τ}) command, versus what roll rate is required (ϕ_{τ}) by 3.3.4 roll paragraphs for the particular aircraft Class and Flight Phase.

$$k = \frac{(\phi_{\tau}) \text{ commanded (what you did)}}{(\phi_{\tau}) \text{ required (MIL SPEC Req)}}$$

Next calculate $\Delta\beta/k$ proverse and $\Delta\beta/k$ adverse and compare them to the requirements of the paragraph. This test must occur with the aileron step fixed for at least 90° of bank angle change.

8.8.11.1 <u>Sideslip Excursion Requirement For Small Inputs (Paragraph 3.3.2.4.1)</u>. The requirements for this paragraph are similar to those seen in the roll rate paragraph for small inputs. Even though the roll paragraph applied to problems with roll as opposed to sideslip, the pilot opinions were similar when coordination is required with adverse or proverse yaw. The difference in the sketch of this paragraph is almost totally due to the difference in ability to coordinate during turn entries and exits. As ψ_{β} varies from 0° to -360°, it indicates the coordination problem discussed previously. When adverse yaw is present $-180^{\circ} \geq \psi_{\beta} \geq -270^{\circ}$, coordination is easy and oscillations can be readily minimized. As more proverse yaw is seen $-360^{\circ} \geq \psi_{\beta} \geq -90^{\circ}$, cross controlling is required and the oscillations go unchecked or are amplified by pilot's efforts to coordinate with rudder pedal.

Only one new parameter is needed to calculate for this paragraph- β_{max}. It is the total algebraic change of β during half the Dutch roll period or two seconds, whichever is greatest. Next calculate k and ψ_{β} as discussed before. Make sure that the sign convention for ψ_{β} remains the same (Figure 8.43). The only other requirement is that dictated for the size of the input. It should be small enough that a 60° bank angle change takes more than the Dutch roll

period or two seconds, whichever is longer.

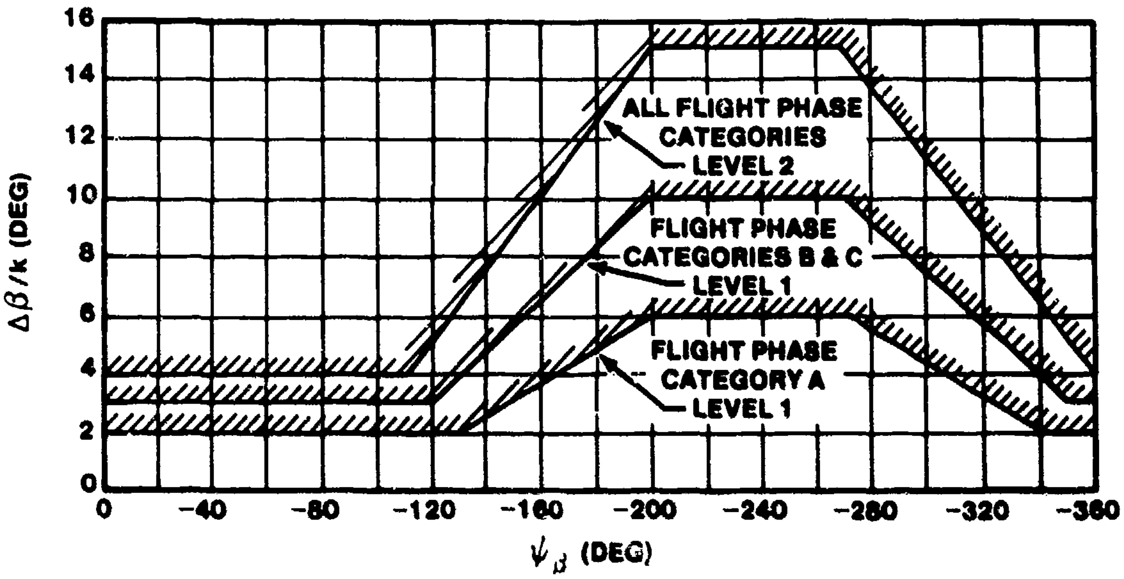


FIGURE 8.44. SIDESLIP EXCURSIONS FOR SMALL INPUTS

8.9 SUMMARY

The dynamic stability flight test methods discussed in this chapter can be used to investigate the five major modes of an aircraft's free response motion. These dynamic stability and contro' investigations, when coupled with pilot in the loop task analysis, will determine the acceptability of an aircraft's dynamic response characteristics.

PROBLEMS

8.1. Define Static Stability

8.2. Define:

a. Positive Static Stability
b. Neutral Static Stability
c. Negative Static Stability

8.3. Define Dynamic Stability

8.4. Define:

a. Positive Dynamic Stability
b. Neutral Dynamic Stability
c. Negative Dynamic Stability

8.5. List two assumptions for using the LaPlace Transformation to find the characteristic equation of a given system.

8.6. Given the following expression, determine the steady state roll rate due to an aileron step input of 10^{o}.

$$0.3\dot{\phi}(t) + 0.5\phi(t) = 0.7\delta_A(t) \quad ; \quad P(S) = \phi\,(S)$$

Units → rad/sec

8.7. For Problem 8.6, what is the time constant (τ).

8.8. An aircraft is in the design stage and the following set of equations predict one of the modes of motion about the longitudinal axis (short period):

$$13.78\,\dot{\alpha}(t) + 4.5\,\alpha(t) - 13.78\,\dot{\theta}(t) = -0.25\,\delta(t)$$
$$0.055\,\dot{\alpha}(t) + 0.619\,\alpha(t) + 0.514\,\ddot{\theta}(t) + 0.19\,\dot{\theta}(t) = -0.71\,\delta_e(t)$$

a. Determine the characteristic equation.

b. Determine the transfer function θ/δ_e.

c. Find the following:

(1) Undamped natural frequency (ω_n).

(2) Damping Ratio (ζ).

(3) Damped Frequency (ω).

8.9. For the longitudinal modes of motion, Short Period and Phugoid, list the variables of interest.

8.10. From the T-38 wind tunnel data (Attachment 1), the following longitudinal equations of motion were derived:

$$1.565\,\dot{u} + .00452\,u + .060500\,\theta - .042\,\alpha = 0$$

$$.236\,u - 3.15\,\dot{\theta} + 3.13\,\dot{\alpha} + 5.026\,\alpha = 0$$

$$.0489\,\ddot{\theta} + .039\,\dot{\theta} + .16\,\alpha = .13\,\delta_e$$

FIND:

a. Laplace Transform of the equations.

b. 4th Order characteristic equation.

8.11. The 4th Order characteristic equation of Problem 8.10 was factored into the following 2nd order equations:

$$(s) = (s^2 + .00214s + .00208)\,(s^2 + 2.408s + 4.595) = 0$$

FIND:

a. Roots of the characteristic equation for Short Period and Phugoid modes.

b. Phugoid: ω_n, ζ, ω_d, τ, $t_{1/2}$, Period -

c. Short Period: ω_n, ζ, ω_d, τ, $t_{1/2}$, Period -

8.12. The short period mode of a fighter aircraft was flight tested at 30,000 ft and .80 Mach with 13,500 lb of fuel. Results were

$$\omega_{n_{SP}} = 4 \text{ radians/sec}$$

$$\zeta_{SP} = .2$$

a. The short period is to be flight tested at 30,000 ft and .80 Mach with 500 lb of fuel. (Fuel in this aircraft is distributed lengthwise along the fuselage, and the CG location for 500 lb of fuel is the same as it was for 13,500 lb of fuel.) Discuss why and how you predict ζ_{SP} and $\omega_{n_{SP}}$ will change at this new fuel weight.

b. The short period is to be flight tested at 5000 ft and .80 Mach with 13,500 lb of fuel. Predict the changes, if any, expected in ζ_{SP} and $\omega_{n_{SP}}$ at this low altitude point.

c. Predict the changes, if any, expected in $\omega_{n_{SP}}$ when the aircraft accelerates to M = 1.2 ($C_{m_\alpha} \uparrow$ due to rearward movement of the aerodynamic center.)

8.13. The undamped period of the T-38 short period mode is found to be 1.9 seconds at Mach = .09 at 30,000 ft. What would you predict the period to be at 50,000 ft at Mach 0.9?

8.14. During a cruise performance test of the X-75B DYN, the following performance parameters were recorded:

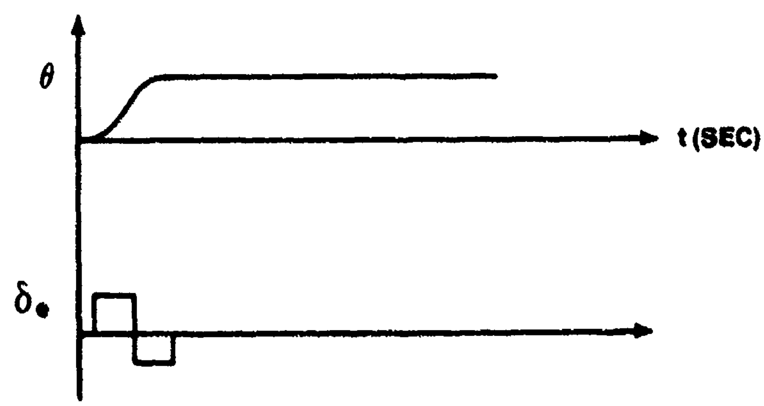
Altitude = 20,000 ft PA
Airspeed = 360 KTAS
ENG RPM = 62%
Thrust = 1,200 lbf
Gross Weight = 12,000 lb

a. Estimate the phugoid damping ratio for the given flight conditions.

b. Estimate the phugoid frequency (damped).

c. Moving the wing from full aft to full forward position should cause the short period frequency to <u>increase/decrease</u> and cause the damping ratio to <u>increase/decrease</u>?

8.15. Given the following time history traces:



What is the probable static margin, value of C_{m_α} and static stability?

8.16. The approximation equations for the Phugoid mode of motion for an aircraft are:

$$\dot{u} + 0.04u + 40\theta = -6\delta_e(t)$$
$$0.001u - \dot{\theta} = -2\delta_e(t)$$

a. Determine the characteristic equation.

b. Calculate ζ and ω_n.

c. Is this a dynamically stable or unstable mode?

d. Calculate the time to half amplitude ($t_{1/2}$) or double amplitude (t_2).

8.17. During reentry from orbital flight the space shuttle airplane will fly at hypersonic speeds and at a constant dynamic pressure to manage heating and airloads.

Wind tunnel tests show that the hypersonic stability derivatives are independent of Mach at these speeds. How does the short period frequency and damping vary during the hypersonic descent at constant dynamic pressure?

8.18. For the lateral directional modes of motion, spiral, roll, and Dutch roll, list the variables of interest.

8.19. From the T-38 wind tunnel data (Attachment 1), the Lateral-Directional equations of motion were evaluated and resulted in the following 4th order characteristic equation:

$$\Delta(s) = s^4 + 7.692\, s^3 + 24.41\, s^2 + 125.8\, s + 1.193 = 0$$

Factored Form:

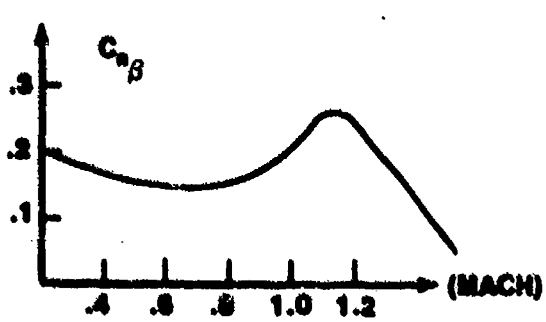
$$\Delta(s) = (s + .00955)\ (s + 6.822)\ (s^2 + .87\, s + 18.4) = 0$$

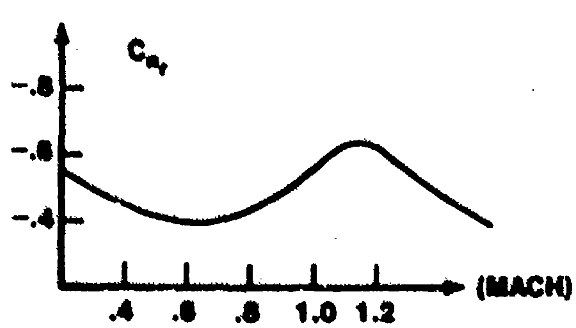
FIND:

a. Roots of the characteristic equation
b. Dutch Roll: ω_n, ζ, ω_d, $t_{1/2}$, Period
c. Roll Mode: time constant τ
d. Spiral Mode: time constant τ

8.20. Right after takeoff you experience an emergency and jettison your wingtip tanks (2). Which roll mode parameter(s) is/are affected and how will the roll mode be affected?

8.21. Given the following wind tunnel data for the X-75B DYN:





a. Does the Dutch Roll damping increase/decrease as Mach increases from 0.6 to 1.2?

b. Does the directional stability increase/decrease when decelerating from Mach 1.2 to .6?

8.22. You are flying a KC-135 and you transfer fuel from the outboard to the inboard wing tanks.

a. Dutch Roll damping will increase/decrease.

b. Roll mode time constant will increase/decrease.

8.23. What is the stability derivative that determines the ϕ/β ratio in Dutch Roll if C_{n_β} is unchanged?

8.24. The approximation equation for an aircraft's roll mode is:

$$\dot{p} + .25p = 5.5\ \delta_A\ (t)$$

a. Determine the steady state roll rate for a step aileron input of 10°.

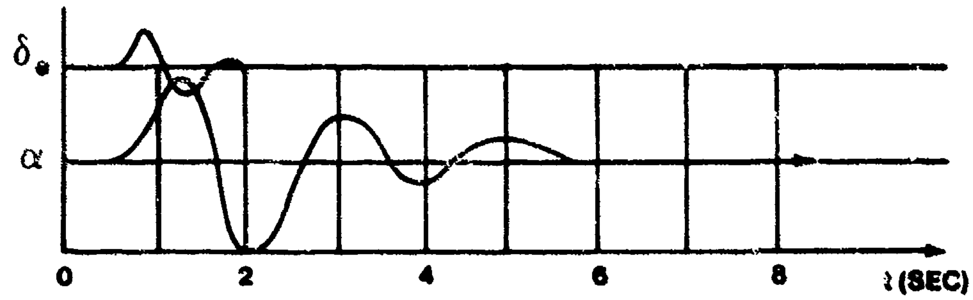
b. Determine the magnitude of roll rate after an elapsed time of:

(1) $t = 1$ time constant (1τ)

(2) $t = 5$ time constants (5τ)

8.25. For the longitudinal and lateral directional modes of motion, list the approximate period for the periodic modes and the method(s) of exciting the five modes of motion.

8.26. Given the following time history trace:



a. Type of dynamic mode represented?

b. Is this a stable dynamic mode?

c. Estimate using flight test relationships:

(1) Damping Ratio (ζ_{SP})

(2) Period of Oscillation (T)

(3) Frequency of Oscillation (ω)

(4) Undamped Natural Frequency (ω_n)

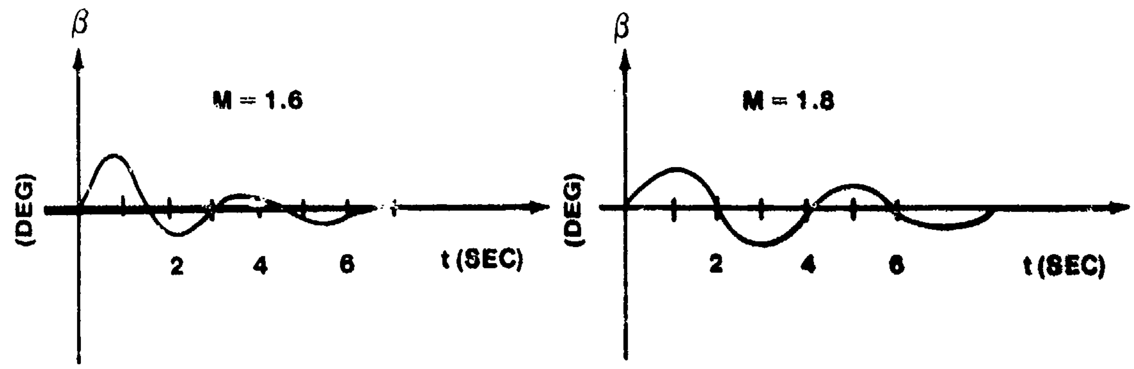
(5) Time Constant (τ)

8.27. You are a consulting engineer with the job of monitoring flight test data and giving GO/NO-GO advice on a real time basis. Today's mission is to investigate the Dutch roll mode at Mach of 1.6, 1.8, and 2.0.

Predicted values of ζ_{DR} and $\omega_{n_{DR}}$

Mach	ω_n (rad/sec)	ζ_{DR}
1.6	3.29	.30
1.8	2.23	.35
2.0	No data available	

The sideslip angle free response flight data plots for the first two test points are:



a. For both M = 1.6 and M = 1.8, estimate ζ_{DR} and f_d (cycles per second).

b. Based on the estimated values, calculate $\omega_{n_{DR}}$ for Mach = 1.6 and 1.8.

c. In light of the results observed from M = 1.6 and 1.8 points, should the M = 2.0 point be flown today? Backup your recommendation.

ATTACHMENT 1

BACKGROUND INFORMATION

FOR T-33

Flight Conditions

Altitude = 20,000 ft MSL   Density = .001267

Mach = .8   True Airspeed = 830 ft/sec

Aircraft Dimensions:

Wing Area = 170 ft^2 Wing Span = 25.25 ft   MAC = 7.73 ft

I_y = 28,166 slug-ft^2 I_x = 1,479 slug-ft_2 I_z = 29,047 slug-ft^2

Stability Derivatives

(Wind Tunnel)

(Dimensions - per/rad)

$C_{m_\alpha} = -.16$

$C_{m_q} = -8.4$

$C_{m_{\delta_e}} = .13$

$C_{z_\alpha} = -C_{L_\alpha} = -5.026$

$C_{x_\alpha} = .084$

$C_{D_\alpha} = .036$

$C_{z_q} = -C_{L_q} = -4.25$

$C_{T_u} = 1.09 \times 10^{-5}$ sec/ft

$C_L = .12$ $C_D = .0040$

$C_{y_\beta} = -1.25$

$C_{y_p} = -.10$

$C_{y_r} = .92$

$C_{\ell_\beta} = .0745$

$C_{\ell_p} = -.35$

$C_{\ell_r} = .055$

$C_{n_\beta} = .28$

$C_{n_r} = -.54$

$C_{n_p} = .08$

ANSWERS

8.6 14.0 deg/sec

8.7 $\tau = 0.6$ sec

8.8 a) $s^2 + .804s + 1.325$

b) $\frac{\theta(s)}{\delta_e(s)} = -\frac{1.383\ (s + 0.343)}{s(s^2 + 0.804s + 1.325)}$

c) 1. $\omega_n = 1.15$ rad/sec

2. $\zeta = 0.35$

3. $\omega = \omega_d = 1.08$ rad/sec

8.10 b) $s^4 + 2.408\ s^3 + 4.555\ s^2 + 0.015\ s + 0.095 = 0$

8.11 a) $s_{1,2} = -1.204 \pm 1.77j$ (short period)

$s_{3,4} = -.0011 \pm 0.05j$ (phugoid)

b) $\omega_n = 0.05$ rad/sec, $\zeta = 0.02$, $\omega_d = 0.045$ rad/sec,

$\tau = 934.6$ sec, $t_{1/2} = 644.9$ sec, T = 137.8 sec

c) $\omega_n = 2.14$ rad/sec, $\zeta = 0.56$, $\omega_d = 1.77$ rad/sec,

$\tau = 0.83$ sec, $t_{1/2} = 0.57$ sec, T = 3.5 sec

8.12 a) ζ_{SP} increases, $\omega_{n_{sp}}$ increases

b) ζ_{SP} increases, $\omega_{n_{sp}}$ increases

c) $\omega_{n_{sp}}$ increases

8.13 $T_{50K} = 2.98$ sec

8.14 a) $\zeta = 0.07$

b) $\omega_n = 0.074$ rad/sec

c) ω_n decreases, ζ increases

8.15 zero, zero, neutral static stability

8.16 a) $s^2 + 0.04s + 0.04$

b) $\zeta = 0.11$, $\omega_n = 0.2$ rad/sec

c) stable

d) $t_{1/2} = 34.5$ sec

8.17 ω_n constant, ζ increases

8.19 a) $s = -0.0096$ (spiral)

$s = -6.82$ (roll), $s = -0.43 \pm 4.27j$ (Dutch roll)

b) $\omega_n = 4.29$ rad/sec, $\zeta = 0.1$, $\omega_d = 4.26$ rad/sec,

$t_{1/2} = 1.58$ sec, $T = 1.47$ sec

c) $\tau = 0.15$ sec

d) $\tau = 104.7$ sec

8.20 I_{xx} decrease, C_{ℓ_p} change slightly, τ decrease

8.21 a) increase

b) decrease

8.22 a) increase

b) decrease

8.23 C_{ℓ_β}

8.24 a) p_{ss} = 220 deg/sec

b) 1. p = 138.6 deg/sec

2. p = 218.5 deg/sec

8.26 a) short period

b) stable

c) 1. ζ = 0.3

2. T = 2 sec

3. $\omega = \omega_d$ = 3.14 rad/sec

4. ω_n = 3.29 rad/sec

5. τ = 1.01 sec

8.27 a) M = 1.6; ζ = 0.3, f_d = 0.5 cycles/sec

M = 1.8; ζ = 0.4, f_d = 0.25 cycles/sec

b) M = 1.6; ω_n = 3.29 rad/sec

M = 1.8; ω_n = 1.71 rad/sec

c) Yes, be careful of large sideslip angle excursions

BIBLIOGRAPHY

8.1 Harper, Robert P., and Cooper, George E. A Revised Pilot Rating Scale for the Evaluation of Handling Qualities. CAL Report No. 153, Cornell Aeronautical Laboratory, Inc., 1966.

8.2 Blakelock, John H. Automatic Control of Aircraft and Missiles. New York: John Wiley and Sons, 1965.

8.3 Newell, F.D. Ground Simulator Evaluations of Coupled Roll Spiral Mode Effects on Aircraft Handling Qualities. AFFDL-TR-65-39, Air Force Dynamics Laboratory, Wright-Patterson AFB, Ohio, March 1965.

8.4 Anon., Military Specification. Flying Qualities of Piloted Airplanes. MIL-F-8785C, 5 Nov 80, UNCLASSIFIED.